Optimisation and Game theoretical aspects in Mathematical Ecology

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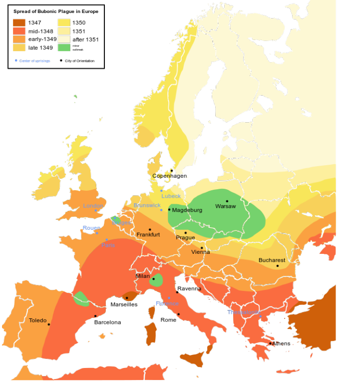
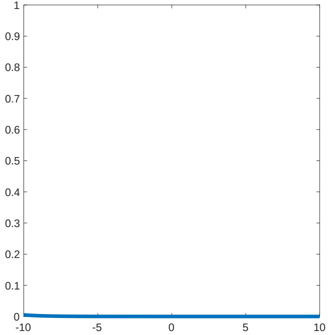
Benasque, IX partial differential equations, optimal design and numerics
Joint work with **Idriss Mazari**

Reaction-diffusion equations

$$\partial_t u = \underbrace{f(u)}_{\text{Reaction}} + \underbrace{\mu \Delta u}_{\text{Diffusion}}$$





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[1] Fisher Kolmogorov Petrovsky and Piskunov

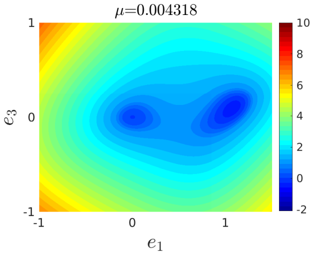
[2] Fife -McLeod

Reaction-diffusion equations

•Gradient structure

$$\partial_t u = -\nabla_u J(u)$$

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \int_0^{u(x)} f(s)ds \right) dx$$



•Convergence to a steady state

$$\begin{cases} -\mu\Delta u = f(u) & x \in \Omega \\ +\text{Boundary conditions} \end{cases}$$

Part I: Optimization in Mathematical Ecology

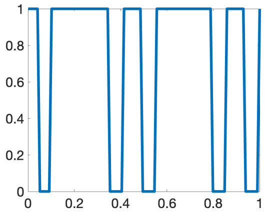
Fires Left These Wallabies Nothing to Eat. Help Arrived From Above, The New York Times, 18 March 2020.

We are interested on maximizing the total population

$$\max_{m\in\mathcal{M}} \int_\Omega u dx, \qquad \mathcal{M} := \left\{ m \in L^\infty(\Omega; [0,\kappa]), \quad \frac{1}{|\Omega|}\int_\Omega m = m_0. \right\}$$

where u follows

$$\begin{cases} -\mu \Delta u = u \left(\underbrace{m}_{\text{Resources}} - u \right) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases}$$



Maximizers of the problem are **Bang-Bang**[3] [4]

Question

How is μ related to the optimal resource distribution?

[3] I.Mazari, G. Nadin, Y. Privat, 2020, & I.Mazari, G. Nadin, Y. Privat, 2021

[4] K. Nagahara and E. Yanagida, 2018, Y. Lou, 2008

Fragmentation

Theorem (Mazari-RB 2020)

$$\|m_\mu^*\|_{BV((0,1)^d)} \underset{\mu \to 0^+}{\to} +\infty$$

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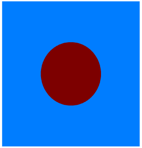


$$\begin{cases} -\mu\Delta u = u(m-u) & \text{in } (0,1)^d \\ \partial_\nu u = 0 & \text{on } \{0,1\}^d \end{cases}$$

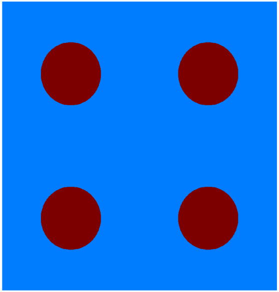
$$\int_{(0,1)^d} u = P$$

[5] I Mazari, D Ruiz-Balet, A fragmentation phenomenon for a nonenergetic optimal control problem: Optimization of the total population size in logistic diffusive models, SIAP 2020

Fragmentation

Theorem (Mazari-RB 2020)

$$\|m_\mu^*\|_{BV((0,1)^d)} \underset{\mu\to 0^+}{\longrightarrow} +\infty$$



Periodize the state and the control

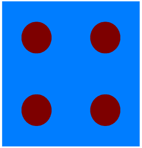
$$\begin{cases} -\mu\Delta v = v(m-v) & \text{in } (-1,1)^d \\ \partial_\nu v = 0 & \text{on } \{-1,1\}^d \end{cases}$$

$$\int_{(-1,1)^d} v = 4P$$

Fragmentation

Theorem (Mazari-RB 2020)

$$\|m_\mu^*\|_{BV((0,1)^d)} \underset{\mu\to 0^+}{\to} +\infty$$



Rescale back to $(0,1)^d$

$$\begin{cases} -\frac{\mu}{4}\Delta\tilde{u} = \tilde{u}(\tilde{m}-\tilde{u}) & \text{in } (0,1)^d \\ \partial_\nu \tilde{u} = 0 & \text{on } \{0,1\}^d \end{cases}$$

$$\int_{(0,1)^d} \tilde{u} = P$$

Same population but smaller diffusivity

Fragmentation

Theorem (Mazari-RB 2020)

$$\|m_\mu^*\|_{BV((0,1)^d)} \underset{\mu\to 0^+}{\to} +\infty$$

- With the **periodization** one can obtain a **lower bound** for the maximum population
- If the maximizers (for any μ) were **uniformly bounded in BV**, then we can make use of a uniform convergence to prove an **upper bound** for the maximum population.

Fragmentation

Theorem (Mazari-RB 2020)

$$\|m_\mu^*\|_{BV((0,1)^d)} \underset{\mu\to 0^+}{\longrightarrow} +\infty$$

Perspectives

- Is there a limit profile when $\mu \to 0$?
- Are there fractal structures when $\mu \to 0$?
- Is there "*some sort*" of recurrence relationship between maximizers of different μ?

Part II: Turnpike without running cost

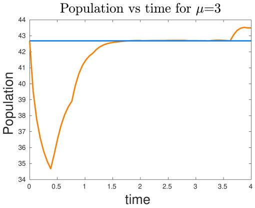
Can we come back to the evolution problem?

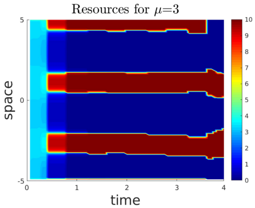
Consider

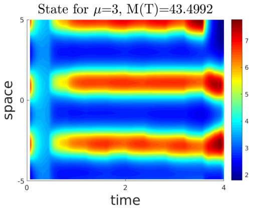
$$\max_{m \in \mathcal{M}_T} \int_\Omega u(T)dx \qquad \begin{cases} \partial_t u - \Delta u = m(x,t)u - u^2 & (x,t) \in \Omega \times (0,T) \\ \partial_\nu u = 0 & (x,t) \in \partial\Omega \times (0,T) \\ u(0) = u_0 > 0 \end{cases}$$

$$\mathcal{M}_T := \left\{ m \in L^\infty(\Omega \times (0,T); [0,\kappa]), \quad \frac{1}{|\Omega|} \int_\Omega m(x,t)dx = m_0 \quad t \in (0,T) \right\}$$

Are the optimal solutions "close" to the elliptic optimisers?



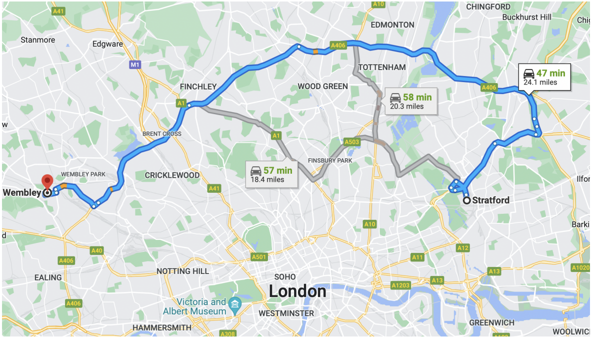




What is Turnpike

The Turnpike principle

*In **large time** horizons the optimal solutions are **near** an optimal solution of an **associated infinite horizon problem***



John Von Neuman

Paul Samuelson

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$$J(u) = \int_0^{T} \|x - x_r\|^2 + \|u\|^2 dt + \|x(T) - x_f\|^2$$

$$\begin{cases} x' = Ax + Bu \\ x(0) = x_0 \end{cases}$$

$$J_e(u) = \|x - x_r\|^2 + \|u\|^2$$

$$\begin{cases} 0 = Ax + Bu \\ x(0) = x_0 \end{cases}$$

Let (x^*, u^*) be optima for the time evolution problem and (x_e, u_e) be optima for the static case.

$$\|x^*(t) - x_e\|^2 + \|u^*(t) - u_e\|^2 \leq C\left(e^{-t} + e^{-(T-t)}\right)$$

See **Geshkovski, Zuazua 2022** for a recent review.

Linear models first!

We will start with a **linear model** with **bilinear control**

$$\max_{m \in \mathcal{M}_T} \int_\Omega f(u(T))dx$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+ \setminus \{0\}$

$$\begin{cases} \partial_t u - \Delta u = \boldsymbol{m(x,t)} u & (x,t) \in \Omega \times (0,T) \\ u = 0 & (x,t) \in \partial\Omega \times (0,T) \\ u(0) = u_0 > 0 \end{cases}$$

$$\mathcal{M}_T := \left\{ m \in L^\infty(\Omega \times (0,T); [0,\kappa]), \quad \frac{1}{|\Omega|} \int_\Omega m(x,t)dx = m_0 \quad t \in (0,T) \right\}$$

In general, we can't assume that there will be steady-states

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In general, we can't assume that there will be steady-states

$$\begin{cases} -\Delta u = m(x)u & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

ONLY IF u is eigenfunction with 0 eigenvalue

Associated infinite horizon problem

What could the candidate associated infinite horizon problem be?

The natural candidate would be an **Optimal Eigenvalue**

$$\min_{m\in\mathcal{M}} \lambda_1(m) = \min_{m\in\mathcal{M}} \inf_{v\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2 - m(x)v^2 dx}{\int_\Omega v^2 dx}$$

If the control were to be static $m(x,t) = m(x)$ for T large

$$u \sim Ce^{-\lambda_1(m)T} e_1(m)$$

INTUITION:

•The first eigenfunction is positive,

•we want the "fastest" growth

This implies that our turnpike candidate is a SUBSPACE

Optimising the Eigenvalue

$$\begin{cases} -\Delta e_1 - me_1 = \lambda_1 e_1 & x \in \Omega \\ e_1 = 0 & x \in \partial\Omega \end{cases}$$

Naming

$$\xi = \frac{\partial}{\partial m} e_1(m)[h], \quad \lambda_1' = \frac{\partial}{\partial m} \lambda_1(m)[h]$$

Differentiating

$$\begin{cases} -\Delta \xi - m\xi - \lambda_1 \xi = he_1 + \lambda_1' e_1 & x \in \Omega \\ \xi = 0 & x \in \partial\Omega \end{cases}$$

Fredholm alternative

$$he_1 + \lambda_1' e_1 \text{ orthogonal to } \mathrm{Ker}(-\Delta - m - \lambda_1) := \mathrm{span}\{e_1\}$$

Therefore

$$\lambda_1'(m)[h] = \int_\Omega h(x) e_1(m)^2 dx$$

Optimising the Eigenvalue

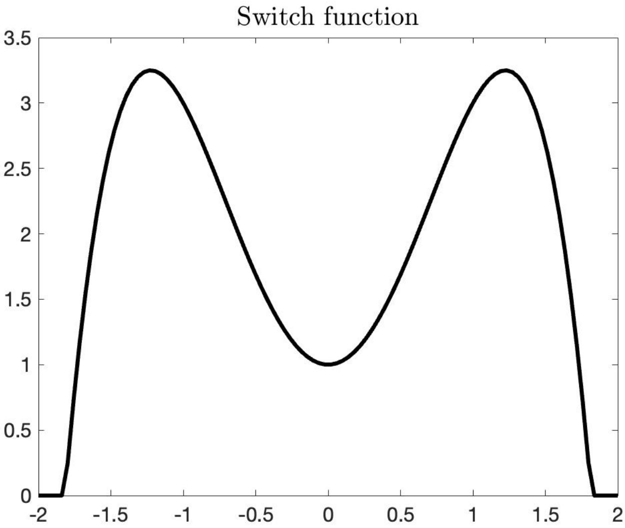
$$\lambda_1'(m)[h] = \int_\Omega h(x) e_1(m)^2 dx$$

$$m \in \mathcal{M} := \{m \in L^\infty(\Omega; [0,\kappa]), \quad \frac{1}{|\Omega|}\int_\Omega m dx = m_0\}$$

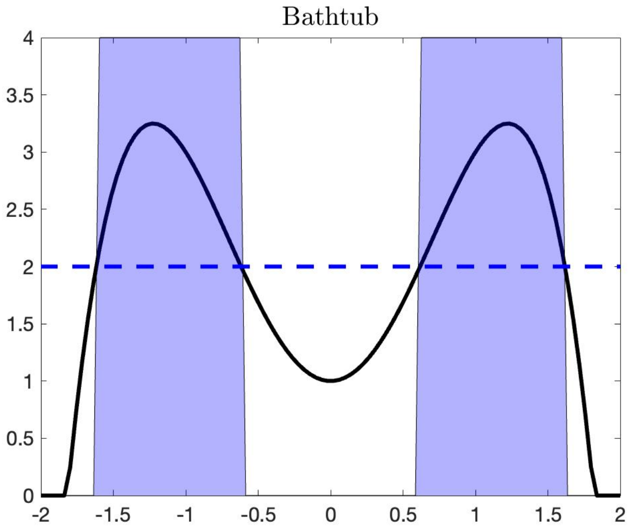
Maximising the derivative

$$\max_{h\in\mathcal{M}} \int_\Omega h(x) \underbrace{e_1(m)^2}_{\text{Switch function}} dx$$

Optimality conditions: Bathtub Principle



Optimality conditions: Bathtub Principle



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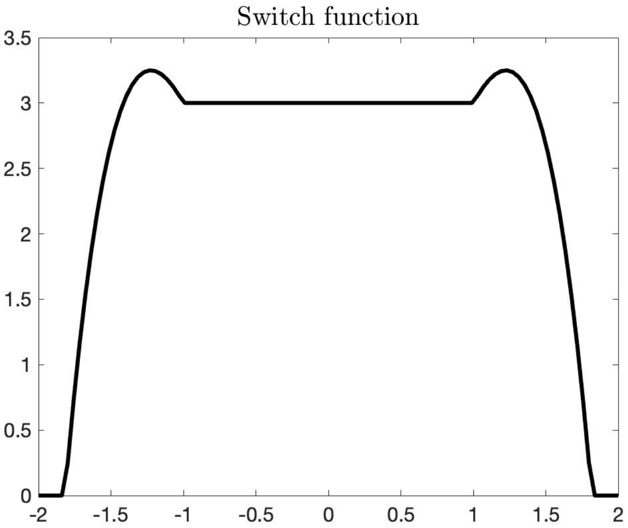

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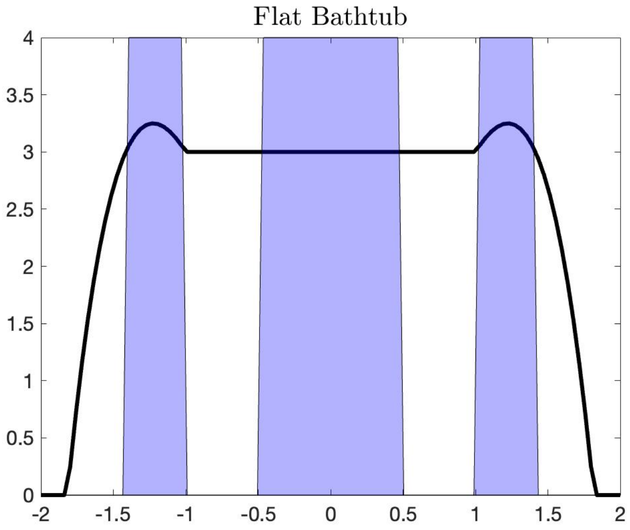

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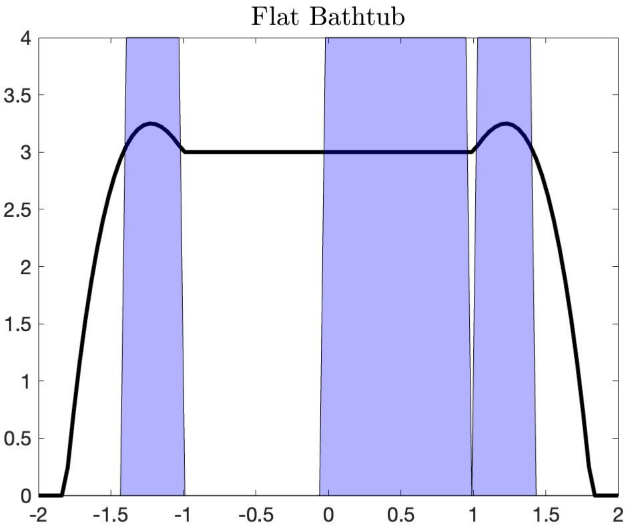

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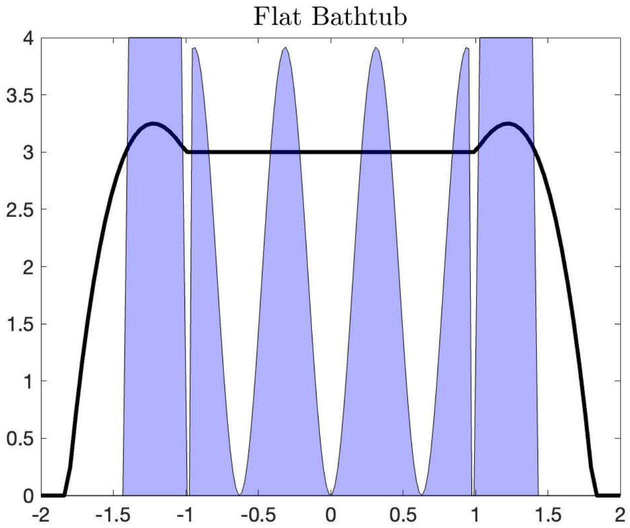


Optimality conditions

The derivative can be expressed as:

$$\max_{h \in \mathcal{M}} \int_{\Omega} h(x) \underbrace{e_1(m)^2}_{\text{Switch function}} \, dx$$

The optimal m is a level set of the first eigenfunction

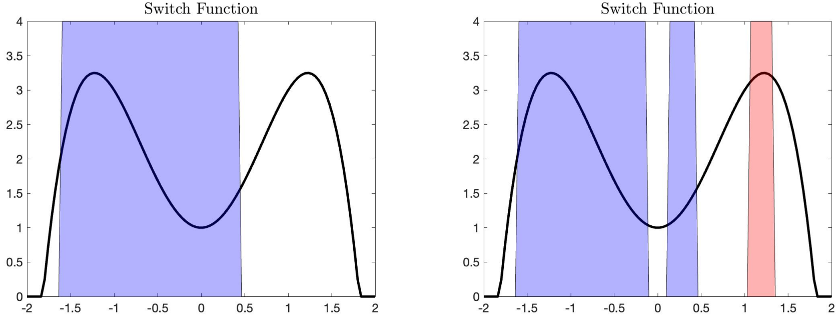
$$\{m = k\} = \{e_1(m)^2 \geq c\}$$
$$|\{m = k\}| = |\{e_1(m)^2 \geq c\}| = m_0$$

Numerical approach

The parameter ϵ will play the role of the **step size**

$$\max_{h\in\mathcal{M}_{\epsilon}} \int_{\{m=0\}} h(x)e_1(m)^2 + \max_{h\in\mathcal{M}_{m_0-\epsilon}} \int_{\{m=\kappa\}} h(x)e_1(m)^2 dx$$

$$\mathcal{M}_{m_0} := \{m \in L^{\infty}(\Omega;[0,\kappa]), \quad \frac{1}{|\Omega|}\int_{\Omega} m dx = m_0\}$$



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$$u \sim Ce^{-\lambda_1(m)T} e_1(m)$$

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Turnpike without running cost

Turnpike Theorem Mazari-RB 2021

There exist $M > 0$ such that, for any $T > 0$ and any solution m_T^* of the parabolic optimisation problem, there holds

$$\int_0^T \inf_{m^* \in I^*} \|m_T^*(t) - m^*\|_{L^1}^2 dt \leq M$$

where I^* is the **optimal spectral set**

Turnpike to the span of the optimal eigenfunction, not to a steady-state!

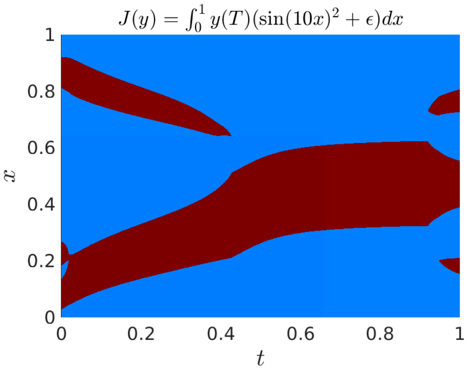
Key:

•Quantitative inequalities

$$\forall m \in \mathcal{M}, \quad \lambda_1(m) - \lambda_1(m^*) \geq C \inf_{m^* \in I^*} \|m - m^*\|_{L^1}^2$$

•Integral Turnpike, not exponential

Simulation



Remarks

- Turnpike helps to accelerate the time-dependant optimisation: **Preconditioning**
- There can be **Turnpike to a subspace**
- We can have **Turnpike without running cost**

Part III: Harvesting problems and Game Theory

SCIENCE | NEWS

The sea is running out of fish, despite nations' pledges to stop it

Major countries that are promising to curtail funding for fisheries are nevertheless increasing handouts for their seafood industries.

BY **TODD WOODY**

PUBLISHED OCTOBER 8, 2019 • 7 MIN READ

As global fish stocks that feed hundreds of millions of people dwindle, nations are scrambling to finalize by year's end an international agreement to ban government subsidies that fuel overfishing.

The sea is running out of fish, despite nations' pledges to stop it, The National Geographic, 2019

The simplest model

Consider

$$\begin{cases} -\mu\Delta u = u(k(x)-u) - \underbrace{\alpha(x)u}_{\text{Harvesting}} & x \in \Omega \\ \partial_\nu u = 0 & x \in \partial\Omega \end{cases}$$

Class of functions

$$\mathcal{M} := \left\{ \alpha \in L^\infty(\Omega; [0,\kappa]), \quad \frac{1}{|\Omega|}\int_\Omega \alpha(x)dx \leq V_0 \right\}$$

- Resources $k(x)$ fixed
- Limited number of fishermen in one point
- Limited total number of fishermen

Maximize harvesting

$$\max_{\alpha\in\mathcal{M}} \int_\Omega \alpha(x)udx$$

Is there a way to understand overfishing through these models?

Some facts

Solutions of

$$\begin{cases} -\mu\Delta u = u(k(x)-u) - \underbrace{\alpha(x)u}_{\text{Harvesting}} & x \in \Omega \\ \partial_\nu u = 0 & x \in \partial\Omega \\ u > 0 & x \in \Omega, \end{cases}$$

do not exist if **zero is linearly stable**

If we force $\|\alpha\|_{L^1} = V_0$ and define

$$H(V_0) = \max_{\|\alpha\|_{L^1}=V_0, \alpha\in\mathcal{M}} \int_\Omega \alpha u dx$$

$\exists K > 0$, such that $H(0) = 0$, $H(K) = 0$ and $H(s) > 0$ if $s \in (0, K)$

Some facts

If we limit ourselves in the case in which $k(x)$ is constant

$$\begin{cases} -\mu\Delta u = u(1-u) - \alpha(x)u & x \in \Omega \\ \partial_\nu u = 0 & x \in \partial\Omega \end{cases}$$

The harvested fish is equal to

$$\int \alpha u_\alpha dx = \int \mu\Delta u_\alpha + u_\alpha(1-u_\alpha)dx = \int u_\alpha(1-u_\alpha)dx$$

Boils down to maximize an integral with a **concave integrand**!

$$\max_{\alpha\in L^\infty(\Omega;[0,\kappa])} \int u_\alpha(1-u_\alpha) \text{ implies } \alpha(x) = \frac{1}{2}$$

$$H(1/2) = \frac{1}{4}$$

Simulations of the Harvesting problem

Differentiating

$$\begin{cases} -\mu\Delta u = u(k(x) - u) - \alpha(x)u & x \in \Omega \\ \partial_\nu u = 0 & x \in \partial\Omega \\ u > 0 & x \in \Omega, \end{cases}$$

with respect to α and introducing the adjoint

$$\begin{cases} -\mu\Delta p - p(k(x) - 2u) + \alpha(x)p = \alpha & x \in \Omega \\ \partial_\nu p = 0 & x \in \partial\Omega \end{cases}$$

One can write

$$DJ(\alpha)[h] = \int_\Omega h(x) \underbrace{u(1-p)}_{\text{Switch function}} dx$$

Simulations of the Harvesting problem

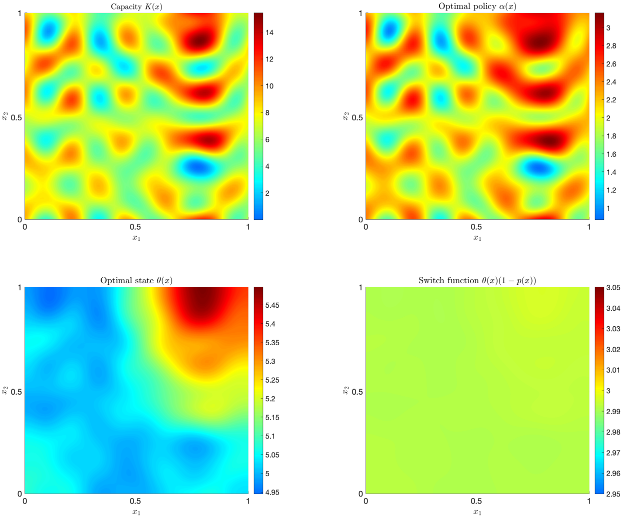


Figure: $V_0 = 0.3$ and $\kappa = 7$

Simulations of the Harvesting problem

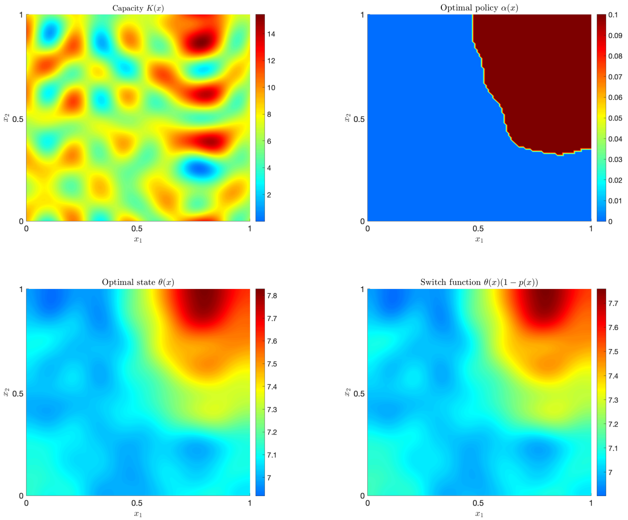


Figure: $V_0 = 0.3$ and $\kappa = 0.1$

Simulations of the Harvesting problem

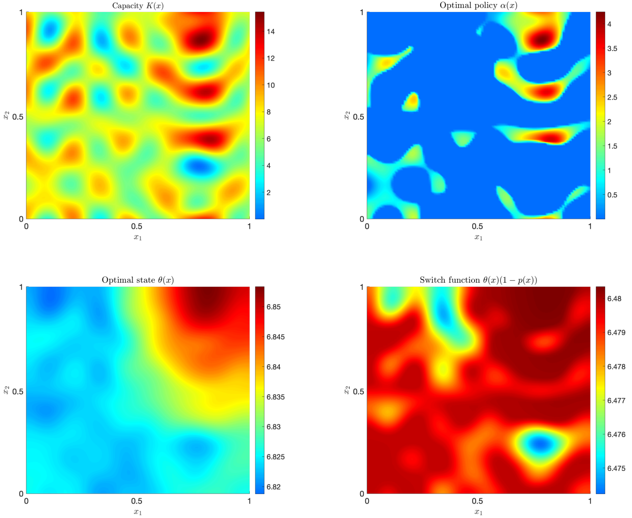


Figure: $V_0 = 0.05$ and $\kappa = 7$.

The model

$$\begin{cases} -\mu\Delta u = u\,(k(x) - u) - \underbrace{\boldsymbol{\alpha_1(x)}u}_{\text{Player 1}} - \underbrace{\boldsymbol{\alpha_2(x)}u}_{\text{Player 2}} & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases}$$

$$\max_{\alpha_1 \in \mathcal{M}} J_1(\alpha_1, \alpha_2) = \max_{\alpha_1 \in \mathcal{M}} \int_\Omega \alpha_1 u_{\alpha_1,\alpha_2} dx$$

$$\max_{\alpha_2 \in \mathcal{M}} J_2(\alpha_1, \alpha_2) = \max_{\alpha_2 \in \mathcal{M}} \int_\Omega \alpha_2 u_{\alpha_1,\alpha_2} dx$$

Nash equilibria

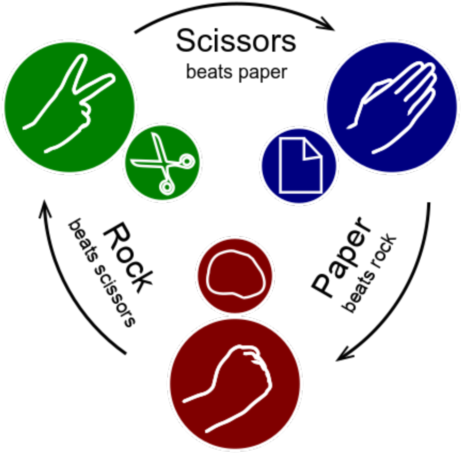
Nash equilibrium for a 2-player game

A Nash equilibrium is a pair of strategies (α_1^*, α_2^*) that satisfies

$$J_1(\alpha_1^*, \alpha_2^*) \geq J_1(\alpha, \alpha_2^*) \quad \forall \alpha \in \mathcal{M}$$

$$J_2(\alpha_1^*, \alpha_2^*) \geq J_2(\alpha_1^*, \alpha) \quad \forall \alpha \in \mathcal{M}$$

Non-existence of Nash equilibria



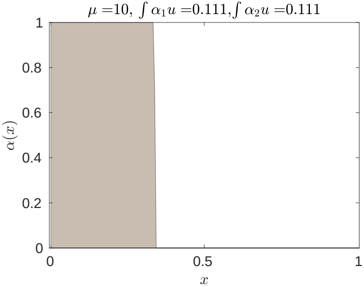
A temptative simulation

We run a **fixed point algorithm**,

$$\alpha_1^{(n+1)} \leftarrow \max_{\alpha \in \mathcal{M}} J_1(\alpha, \alpha_2^{(n)})$$

$$\alpha_2^{(n+1)} \leftarrow \max_{\alpha \in \mathcal{M}} J_2(\alpha_1^{(n)}, \alpha)$$

and **hope** for convergence



- **Convergence $\Longrightarrow$ existence of Nash eq.**
- The total harvested is equal to $0.222 < \frac{1}{4}$
- $k(x) = 1$ but the **Nash eq is bang-bang**
- For **potential games** the algo converges
- **No numerical guarantees**

Tragedy of the commons

Definition (≈ Wikipedia)

The **tragedy of the commons** is a situation in which individual users, who have **open access to a resource**, act independently according to their **own self-interest** and **cause depletion of the resource** through their uncoordinated action.

Effects on unregulating land in Ireland

William Forster Lloyd

Tragedy of the commons

Do there exist Nash equilibria that show a **depletion** of the fishery?

Consider

$$\begin{cases} -\mu\Delta u_{\vec{\alpha}} = u_{\vec{\alpha}}(1 - u_{\vec{\alpha}}) - \left(\sum_{i=1}^{N} \alpha_i(x)\right) u_{\vec{\alpha}} & x \in \Omega \\ \partial_\nu u_{\vec{\alpha}} = 0 & x \in \partial\Omega \end{cases}$$

Where each player is optimizing

$$J_i(\vec{\alpha}) = \int_\Omega \alpha_i u_{\vec{\alpha}} dx$$

Tragedy of the commons

There exist a sequence of Nash equilibria $\vec{\alpha}_N^* \in \mathcal{M}^N$, $N \in \mathbb{N}$ such that

$$\frac{1}{4} = \max_{\vec{\alpha} \in \mathcal{M}^N} \left(\sum_{i=1}^{N} J_i(\vec{\alpha}) \right) > \sum_{i=1}^{N} J_i(\vec{\alpha}_N^*) \underbrace{\longrightarrow}_{N \to +\infty} 0$$

High-diffusivity limit

Assymptotic expansion on μ

Take $|\Omega| = 1$

The functional expands like

$$J_\mu(\alpha) = J^0(\alpha) + \frac{1}{\mu} J_1(\alpha) + O\left(\frac{1}{\mu^2}\right)$$

$$J^0 : \alpha \mapsto \left(\int_\Omega \alpha\right)\left(K_0 - \int_\Omega \alpha\right)$$

$$J^1 : \alpha \mapsto \int_\Omega \alpha v_\alpha$$

The state expands like

$$u_{\alpha,\mu} = \underbrace{\left(K_0 - \int_\Omega \alpha\right)}_{=:M_\alpha} + \frac{v_\alpha}{\mu} + \underset{\mu\to\infty}{O}\left(\frac{1}{\mu^2}\right) \text{ where } \begin{cases} -\Delta v_\alpha - M_\alpha\left(K - \alpha - M_\alpha\right) = 0 & \text{in } \Omega \\ \frac{\partial v_\alpha}{\partial \nu} = 0 \\ \int_\Omega v_\alpha = \frac{1}{M_\alpha^2} \int_\Omega |\nabla v_\alpha|^2. \end{cases}$$

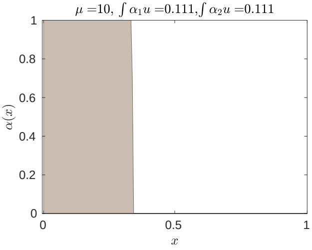
One result

Asymptotic Nash Theorem Mazari-RB 2022

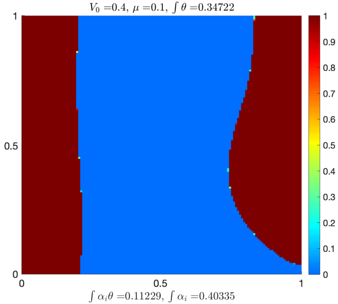
Assume V_1 , $V_2 > \frac{K_0}{4}$, assume K is constant and let

$$\alpha_i^* = \kappa_i \chi_{[0;l_i]} \text{ with } \kappa_i l_i = V_i \ (i = 1, 2).$$

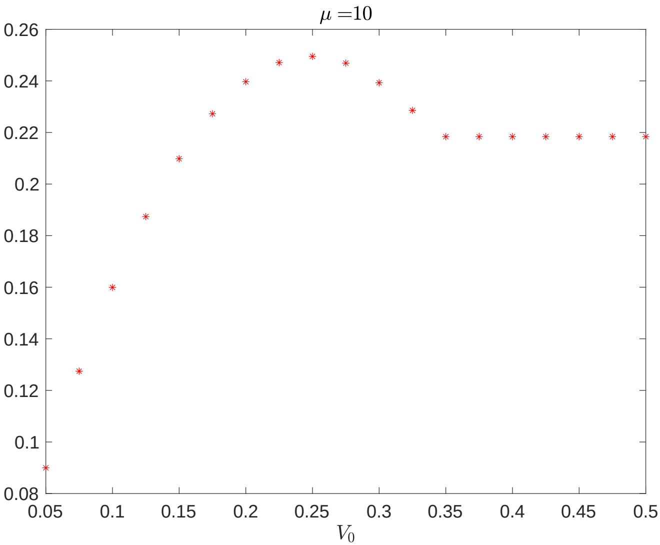
(α_1^*, α_2^*) is a Nash equilibrium in the asymptotic regime



Fragmentation of Nash equilibria?



Optimal regulation?



Perspectives

- How can we **come back to the original problem** without asymptotic regimes?
- How can we **prove the fragmentation of Nash** equilibria?
- Can we find a numerical algorithm that guarantees that an **ϵ-Nash is close to a Nash** equilibria?
- Other game variations and the **incorporation of time**. Can we find new phenomenology?
- Can we quantify the **price of Anarchy**?
- How can we **justify the PDE** from the practical perspective? **Statistics and prediction**, Can a spatio-temporal (PDE) approach enhance **new phenomenology** relevant for ecology not present in simpler models?

Thank you for your attention!

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