Optimal design problems involving the coefficients and the domain where the equation is posed.

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Optimal design problem

Assume: $\Omega \subset \mathbb{R}^N$ open, $F: \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$F(.,s) \text{ measurable in } \Omega, \quad \forall s \in \mathbb{R},$$
$$F(x,.) \text{ continuous in } \mathbb{R}, \quad \text{a.e } x \in \Omega,$$

$\exists r \in L^1(\Omega), \gamma \geq 0$ such that

$$|F(x,s)| \leq r(x) + \gamma |s|^2, \quad \forall s \in \mathbb{R}, \text{ a.e } x \in \Omega.$$

$f \in H^{-1}(\Omega)$, $0<\alpha < \beta$, $\kappa_\alpha, \kappa_\beta > 0$,

$$\min \int_\Omega F(x,u)dx$$

$$\begin{cases} -\text{div}\left(\left(\alpha \chi_{\omega_\alpha} + \beta \chi_{\omega_\beta}\right) \nabla u\right) = f \text{ in } \omega_\alpha \cup \omega_\beta, \\ u = 0 \text{ on } \partial(\omega_\alpha \cup \omega_\beta), \\ \omega_\alpha, \omega_\beta \subset \Omega \text{ measurable}, \quad \omega_\alpha \cup \omega_\beta \quad \text{open} \\ \omega_\alpha \cap \omega_\beta = \emptyset, \qquad |\omega_\alpha| \leq \kappa_\alpha, \quad |\omega_\beta| \leq \kappa_\beta. \end{cases}$$

The problem has been widely studied when

Control coefficients problem:

$$\omega_\alpha \cup \omega_\beta = \Omega.$$

The open set where the equation is posed is known. The most classical results are due to F. Murat, L.Tartar (Allaire, Cherkaev, Conca, Kohn, Lipton, Lurie…).

Control problem in the domain:

$$\alpha = \beta.$$

The control variable is the set where the equation is posed. The most classical results are due to G. Buttazzo, G. Dal Maso.

In both cases is know that the problem has no solution in general. A relaxation is needed.

Namely: Let us try to use the direct method of the Calculus of Variations.

Take $\omega_\alpha^n, \omega_\beta^n$ satisfying the constrains such that

$$\int_\Omega F(x, u_n)dx \to \text{Infimum of the problem,}$$

u_n solution of

$$\begin{cases} -\mathrm{div}\left(\left(\alpha\chi_{\omega_\alpha^n} + \beta\chi_{\omega_\beta^n}\right)\nabla u_n\right) = f \quad \text{in } \omega_\alpha^n \cup \omega_\beta^n \\ u_n = 0 \quad \text{on } \partial\left(\omega_\alpha^n \cup \omega_\beta^n\right). \end{cases}$$

Extending u_n by zero outside $\omega_\alpha^n \cup \omega_\beta^n$,

$$\chi_{\omega_\alpha^n} \rightharpoonup \theta_\alpha \quad \text{in } L^\infty(\Omega) - *$$
$$\chi_{\omega_\beta^n} \rightharpoonup \theta_\beta \quad \text{in } L^\infty(\Omega) - *$$
$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega)$$
$$\left(\alpha\chi_{\omega_\alpha^n} + \beta\chi_{\omega_\beta^n}\right)\nabla u_n \rightharpoonup \sigma \quad \text{in } L^2(\Omega)^{\mathrm{N}}$$
$$\int_\Omega F(x, u_n)dx \to \int_\Omega F(x, u)dx$$

with $\theta_\alpha, \theta_\beta \in L^\infty(\Omega; [0,1])$.

In general $\nexists \omega_\alpha, \omega_\beta$ with

$$\theta_\alpha = \chi_{\omega_\alpha}, \quad \theta_\beta = \chi_{\omega_\beta},$$

Moreover

$$\chi_{\omega_\alpha^n} \rightharpoonup \theta_\alpha \quad \text{in } L^\infty(\Omega) - *, \qquad \chi_{\omega_\beta^n} \rightharpoonup \theta_\beta \quad \text{in } L^\infty(\Omega) - *$$

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega)$$

$$\left(\alpha\chi_{\omega_\alpha^n} + \beta\chi_{\omega_\beta^n}\right)\nabla u_n \rightharpoonup \sigma \quad \text{in } L^2(\Omega)^{\mathrm{N}},$$

does not imply

$$\sigma = \left(\alpha\theta_\alpha + \beta\theta_\beta\right)\nabla u.$$

$$\text{If } \omega_\alpha^n \cup \omega_\beta^n = \Omega \Rightarrow -\mathrm{div}\sigma = f \quad \text{in } \Omega$$

Consequence: The relaxation process is linked to the passage to the limit in a sequence of PDE problems. This is the goal of the homogenization theory.

Classical results:

Theorem (S. Spagnolo 1968, Extensions F. Murat, L. Tartar 1974)

$A_n \in L^\infty(\Omega)^{N\times N}$ symmetric, $\mathrm{Eig}(A_n) \subset [\alpha, \beta]$. Then, for a subsequence,
$\exists A \in L^\infty(\Omega)^{N\times N}$ symmetric, $\mathrm{Eig}(A) \subset [\alpha, \beta]$, such that

$\forall f \in H^{-1}(\Omega)$ the solution u_n of

$$\begin{cases} -\mathrm{div}(A_n \nabla u_n) = f \ \text{ in } \Omega \\ u_n = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

satisfies

$$u_n \rightharpoonup u \ in \ H_0^1(\Omega),$$

u solution of

$$\begin{cases} -\mathrm{div}(A \nabla u) = f \ \text{ in } \Omega \\ u = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

We write

$$A_n \xrightarrow{H} A.$$

Theorem (L. Tartar 1985, K.A. Lurie, A.V. Cherkaev 1986, $N = 2$)

If $A_n = \alpha\chi_{\omega_n} + \beta\chi_{\Omega\backslash\omega_n}$, $\chi_{\omega_n} \rightharpoonup \theta$ $in\ L^\infty(\Omega) -*$, $\quad \theta \in L^\infty(\Omega; [0,1])$.

Then the eigenvalues $\lambda_1, \dots, \lambda_N$ of A satisfy $\mu^-(\theta) \le \lambda_i \le \mu^+(\theta)$

$$\sum_{i=1}^{N} \frac{1}{\lambda_i - \alpha} \le \frac{1}{\mu^-(\theta) - \alpha} + \frac{N-1}{\mu^-(\theta) - \alpha}$$

$$\sum_{i=1}^{N} \frac{1}{\beta - \lambda_i} \le \frac{1}{\beta - \mu^-(\theta)} + \frac{N-1}{\beta - \mu^+(\theta)},$$

$$\text{with } \mu^-(\theta) = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1}, \quad \mu^+(\theta) = \theta\alpha + (1-\theta)\beta,$$

θ represents the proportion of material α used in the homogenized mixture.

Remark: For our purpose it is enough to know $\forall \xi \in \mathbb{R}^N$

$$\{A\xi : A \text{ homogenized matrix}\}$$

$$= B\left(\frac{\mu^+(\theta) + \mu^-(\theta)}{2}\xi, \frac{\mu^+(\theta) - \mu^-(\theta)}{2}|\xi|\right)$$

$$= \left\{A\xi : A \text{ symmetric}, \qquad \mathrm{Eig}(A) \subset [\mu^-(\Omega), \mu^+(\Omega)]\right\}.$$

The set

$$\{A\xi : A \text{ homogenized matrix}\}$$

is known for more general mixtures (L. Tartar 1997).

Definition:

$$\mathcal{M}(\Omega) = \{\nu \in \text{ Borel nonnegative measure, not necessarily Radon vanishing on the sets of null capacity}\}$$

$$\text{cap}(\omega) = \min\left\{\int_\Omega |\nabla u|^2 dx : u \in H_0^1(\Omega), u \geq 1 \text{ in a neighbourhood of } \omega\right\}.$$

Recall: A function of $H^1(\Omega)$ has a representative, well defined up to a set of null capacity.
The Hausdorff dimension of a set of null capacity is $N - 2$.

If $\nu \in \mathcal{M}(\Omega)$ then, $\exists \mathcal{C}_\nu \subset \Omega$ quasi-closed

$$(\forall \varepsilon > 0,\ \exists F \subset \Omega \text{ closed},\ \ \text{cap}(F \Delta \mathcal{C}_\nu) < \varepsilon)$$

such that

$$\nu(B) = \infty,\ \ \forall B\ \ \text{Borel},\ \ \text{cap}(B \cap \mathcal{C}_\nu) > 0$$

$$\Omega \setminus \mathcal{C}_\nu\ \ \text{is. } \sigma\text{-finite for } \nu.$$

Theorem (G. Dal Maso, U. Mosco 1987,
previous result D. Cioranescu F. Murat 1982,
extensions G. Dal Maso, A. Garroni 1994)

$\omega_n \subset \Omega$, open sets, $A \in L^\infty(\Omega)^{N\times N}$ symmetric uniformly elliptic. Then, for a subsequence, $\exists \nu \in \mathcal{M}(\Omega)$ such that $\forall f \in H^{-1}(\Omega),\ u_n$ solution of

$$\begin{cases} -\mathrm{div}(A\nabla u_n) = f \ \text{ in } \omega_n \\ u_n = 0 \ \text{ on } \partial\omega_n, \end{cases}$$

satisfies $u_n \rightharpoonup u \ in \ H_0^1(\Omega)$, u solution of

$$\begin{cases} -\mathrm{div}(A\nabla u) + \nu u = f \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

in the variational sense (it is not a problem in the distributions sense)

$$\begin{cases} u \in H_0^1(\Omega) \cap L_\nu^2(\Omega) \\ \displaystyle\int_\Omega A\nabla u \cdot \nabla v dx + \int_\Omega uv d\nu = \langle f, v\rangle \\ \forall v \in H_0^1(\Omega) \cap L_\nu^2(\Omega). \end{cases}$$

Remark: Problem

$$\begin{cases} -\mathrm{div}(A\nabla u) = f \ \text{ in } \omega \\ u = 0 \ \text{ on } \partial\omega, \end{cases}$$

can be written as

$$\begin{cases} -\mathrm{div}(A\nabla u) + \nu u = f \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

defining

$$\nu(B) = \begin{cases} 0 & \text{if } cap(B \setminus \omega) = 0 \\ +\infty & \text{if } \mathrm{cap}(B \setminus \omega) > 0, \end{cases} \qquad \forall B \subset \Omega, \quad Borel.$$

(thus $\nu = \mathcal{C}_\nu$)

In this form the equation is stable by homogenization.

Theorem (G. Dal Maso, F. Murat, 2004,
extensions to nonlinear problems C. Calvo-Jurado, J. Casado-Díaz 2002).

$A_n \overset{H}{\rightarrow} A$ en $\Omega, \nu_n \in \mathcal{M}(\Omega)$, then, for a subsequence $\exists \nu \in \mathcal{M}(\Omega)$, such that $\forall f \in H^{-1}(\Omega)$ the solutions u_n of

$$\begin{cases} -\mathrm{div}(A_n \nabla u_n) + \nu_n u_n = f \ \text{ in } \Omega \\ u_n = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

converge weakly in $H_0^1(\Omega)$ to u solution of

$$\begin{cases} -\mathrm{div}(A \nabla u) + \nu u = f \ \text{ in } \Omega \\ u = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

Counterexample

$$\min \int_{\Omega} |u - u_d|^2 dx$$

$$\begin{cases} -\text{div}\left(\left(\alpha\chi_{\omega_\alpha} + \beta\chi_{\omega_\beta}\right)\nabla u\right) = f \quad \text{in } \omega_\alpha \cup \omega_\beta, \\ u = 0 \text{ on } \partial(\omega_\alpha \cup \omega_\beta), \\ \omega_\alpha, \omega_\beta \subset \Omega \text{ measurable} \\ \omega_\alpha \cap \omega_\beta = \emptyset, \qquad |\omega_\alpha| \leq \kappa_\alpha, \quad |\omega_\beta| \leq \kappa_\beta. \end{cases}$$

u_d solution of

$$\begin{cases} -\text{div}(A\nabla u_d) + \nu u_d = f \quad \text{in } \Omega \\ u_d = 0 \text{ on } \partial\Omega, \end{cases}$$

For A, f conveniently chosen.

Theorem: A relaxed formulation of

$$\min \int_\Omega F(x,u)dx$$

$$\begin{cases} -\mathrm{div}\left(\left(\alpha\chi_{\omega_\alpha} + \beta\chi_{\omega_\beta}\right)\nabla u\right) = f \quad \text{in } \ \omega_\alpha \cup \omega_\beta \\ u = 0 \ \ on \ \ \partial\left(\omega_\alpha \cup \omega_\beta\right) \\ \omega_\alpha, \omega_\beta \subset \Omega \quad \text{measurable}, \quad \omega_\alpha \cup \omega_\beta \ \text{ open}, \ \omega_\alpha \cap \omega_\beta = \emptyset \\ |\omega_\alpha| \le \kappa_\alpha, \ \ |\omega_\beta| \le \kappa_\beta. \end{cases}$$

is given by

$$\min \int_\Omega F(x,u)dx$$

$$\begin{cases} -\mathrm{div}(A\nabla u) + \nu u = f \quad \text{in } \ \Omega \\ u = 0 \ \text{ on } \ \partial\Omega, \quad \theta_\alpha, \theta_\beta \in L^\infty(\Omega;[0,1]), \quad A \in L^\infty(\Omega)^{N\times N} \ \text{ symmetric}, \ \ \nu \in \mathcal{M}(\Omega) \\ \theta_\alpha + \theta_\beta \le 1, \quad \theta_\alpha + \theta_\beta = 1 \text{ in } \Omega \setminus \mathcal{C}_\nu, \quad \mathrm{Eig}(A) \in [\mu^-(\theta_\alpha), \mu^+(\theta_\alpha)] \ \text{ a.e. in } \ \Omega \setminus \mathcal{C}_\nu \\ \int_\Omega \theta_\alpha dx \le \kappa_\alpha, \ \int_\Omega \theta_\beta dx \le \kappa_\beta. \end{cases}$$

Remark: $\exists$ a solution $(\theta_\alpha, \theta_\beta, A, \nu, u)$ of the relaxed problem such that $(\theta_\alpha, A, \nu, u)$ solves

$$\min \int_\Omega F(x,u)dx$$

$$\begin{cases} -\text{div}(A\nabla u) + \nu u = \quad \text{in } \Omega \\ u = 0 \text{ on } \partial\Omega, \quad \theta_\alpha \in L^\infty(\Omega \setminus \mathcal{C}_\nu; [0,1]), \quad A \in L^\infty(\Omega \setminus \mathcal{C}_\nu)^{N\times N} \text{ symmetric}, \ \nu \in \mathcal{M}(\Omega) \\ \text{Eig}(A) \in [\mu^-(\theta_\alpha), \mu^+(\theta_\alpha)] \text{ a.e. in } \Omega \setminus \mathcal{C}_\nu \\ |\Omega| - \kappa_\beta \leq \int_{\Omega\setminus\mathcal{C}_\nu} \theta_\alpha dx \leq \kappa_\alpha, \end{cases}$$

$\theta_\beta = (1-\theta_\alpha)\chi_{\Omega\setminus\mathcal{C}_\nu}$.

Optimality Conditions

Assume $k \in L^2(\Omega)$, $\lambda > 0$.

$$F(.,0) \in L^1(\Omega)$$
$$F(x,.) \in C^1(\mathbb{R}) \text{ a.e. } x \in \Omega$$
$$|\partial_s F(x,s)| \leq k(x) + \lambda|s| \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

Theorem: *Assume* $(\theta_\alpha, A, \nu, u)$ solution. Define the adjoint state q by

$$\begin{cases} -\mathrm{div}(A\nabla q) + \nu q = \partial_s F(x,u) \text{ in. } \Omega \\ q = 0 \text{ on } \partial\Omega. \end{cases}$$

Then $uq \leq 0$ q.e. in Ω, $uq = 0$ ν-a.e. in Ω

$$\begin{cases} A\nabla u = \dfrac{\mu^+(\theta_\alpha) + \mu^-(\theta_\alpha)}{2} \nabla u + \dfrac{\mu^+(\theta_\alpha) - \mu^-(\theta_\alpha)}{2} \dfrac{|\nabla \mathrm{u}|}{|\nabla q|} \nabla q \text{ a.e. in } \{\nabla q \neq 0\} \\ \mathrm{A}\nabla q = \dfrac{\mu^+(\theta_\alpha) + \mu^-(\theta_\alpha)}{2} \nabla q + \dfrac{\mu^+(\theta_\alpha) - \mu^-(\theta_\alpha)}{2} \dfrac{|\nabla \mathrm{q}|}{|\nabla u|} \nabla u \text{ a.e. in } \{\nabla u \neq 0\} \end{cases}$$

Defining $E^+ = \dfrac{|\nabla u||\nabla q| + \nabla u \cdot \nabla q}{2}$, $E^- = \dfrac{|\nabla u||\nabla q| - \nabla u \cdot \nabla q}{2}$,

$$\exists \tau \geq 0 \text{ such that } \theta_\alpha = \begin{cases} 0 & \text{if } \dfrac{\beta}{\alpha} < E^+ + \tau \\ \dfrac{1}{\beta - \alpha} \sqrt{\dfrac{\alpha\beta E^-}{E^+ + \tau}} & \text{if } \dfrac{\alpha}{\beta} \leq E^+ + \tau \leq \dfrac{\beta}{\alpha} \\ 1 & \text{if } E^+ + \tau < \dfrac{\alpha}{\beta}. \end{cases}$$

Consequence: If $F(x,s)$ is concave in s, there exists a solution $(\theta_\alpha, A, \nu, u)$ with $\nu = \mathcal{C}_\nu$.

Numerical Resolution

For $n \in \mathbb{N}$, we replace the relaxed problem by

$$\min \int_\Omega F(x,u)\,dx$$

$$\begin{cases} -\mathrm{div}(A\nabla u) + \nu u = f \quad \text{in } \Omega \\ u = 0 \ \text{ on } \partial\Omega, \quad \theta_\alpha \in L^\infty(\Omega;[0,1]), \quad A \in L^\infty(\Omega)^{N\times N} \text{ symmetric}, \ \ \nu \in L^\infty(\Omega;[0,n]) \\ \mathrm{Eig}(A) \in [\mu^-(\theta_\alpha), \mu^+(\theta_\alpha)] \ \text{ a.e. in } \Omega \\ \displaystyle\int_\Omega \left(\theta_\alpha - \frac{\nu}{n}\right)^+ dx \leq \kappa_\alpha, \ \int_\Omega \left(1 - \theta_\alpha - \frac{\nu}{n}\right)^+ dx \leq \kappa_\beta. \end{cases}$$

Theorem: The approximate solution has a solution $(\theta_\alpha^n, A_n, \nu_n, u_n)$ of the approximate problem. For a subsequence

$$u_n \rightharpoonup u \ \text{ in } \ H_0^1(\Omega)$$

$$\left(\theta_\alpha^n - \frac{\nu_n}{\alpha}\right)^+ \rightharpoonup \theta_\alpha, \quad \left(1 - \theta_\beta^n - \frac{\nu_n}{\beta}\right)^+ \rightharpoonup \theta_\beta \ \text{ in } \ L^\infty(\Omega)\text{-} *$$

$$A_n \nabla u_n \rightharpoonup A \nabla u \ \text{ in } \ L^2(\Omega)^N$$

$$\int_\Omega F(x, u_n)dx \to \int_\Omega F(x, u)dx$$

with $(\theta_\alpha, \theta_\beta, A, \nu, u)$ solution of the relaxed problem.

The relaxed problem can be solved using a gradient algorithm taking into account the convexity of the set of controls of the approximate problem.

Example: $\Omega = (0,1)^2, \alpha = 1, \beta = 2$

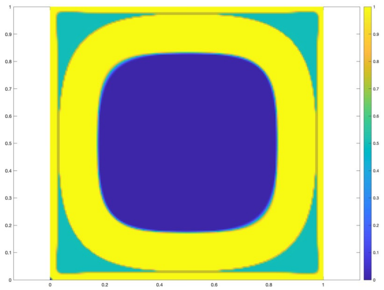
$$\max \int_\Omega u\,dx$$

$$\begin{cases} -\operatorname{div}(A\nabla u) + \nu u = f \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega, \quad \theta_\alpha \in L^\infty(\Omega \setminus \mathcal{C}_\nu; [0,1]), \quad A \in L^\infty(\Omega \setminus \mathcal{C}_\nu)^{N\times N} \text{ symmetric}, \ \nu \in \mathcal{M}(\Omega) \\ \operatorname{Eig}(A) \in [\mu^-(\theta_\alpha), \mu^+(\theta_\alpha)] \ \text{ a.e. in } \Omega \setminus \mathcal{C}_\nu \\ |\Omega| - \frac{\pi}{8} \leq \int_{\Omega\setminus\mathcal{C}_\nu} \theta_\alpha dx \leq \frac{\pi}{8}. \end{cases}$$

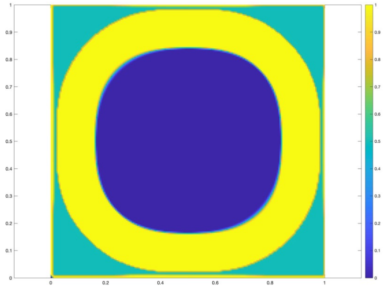
It is a classical problem. It provides the optimal arrangement of two materials in the cross section of a beam in order to minimize the torsion. The form of the cross section is unknown.

With the above parameters there exists a unique solution

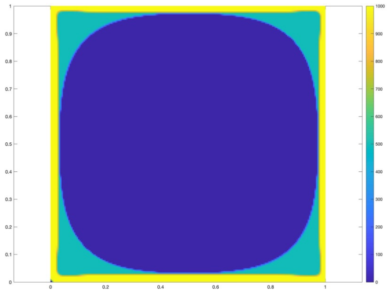
$$\nu = \infty_{\left\{\left|x-\left(\frac{1}{2},\frac{1}{2}\right)\right|>\frac{1}{2}\right\}}, \quad \theta_\alpha = \chi_{\left\{\frac{1}{2\sqrt{2}}<\left|x-\left(\frac{1}{2},\frac{1}{2}\right)\right|<\frac{1}{2}\right\}}, \quad A = \mu^-(\theta_\alpha) I.$$



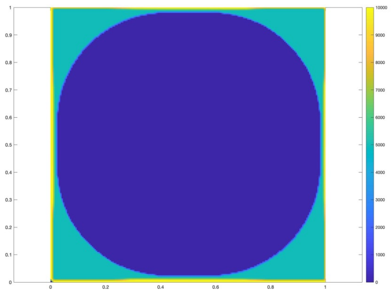
function θ_α, $n = 1000$,



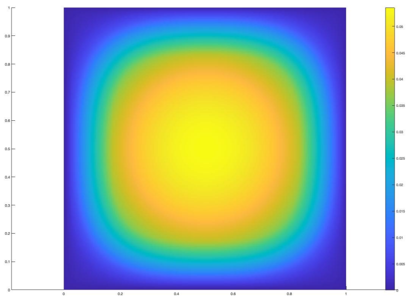
function θ_α, $n = 10000$,



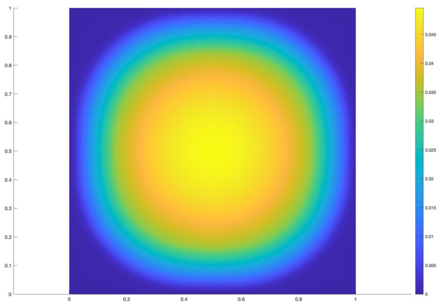
function ν, $n = 1000$,



function ν, $n = 10000$,



function u, $n = 1000$,



function u, $n = 10000$,