On optimality of surface structures

Centro de Ciencias de Benasque Pedro Pascual

Workshop: **IX Partial differential equations, optimal design and numerics**, Aug 21st – Sep 2nd 2022

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Selected Topics in Optimal Design

Michell Problem (1904)
(mathematical framework: G. Bouchitté & G. Buttazzo (2001))

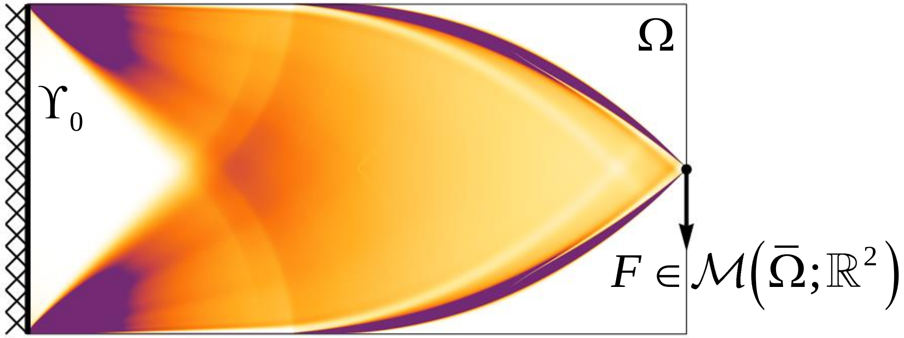
$$\min\left\{\int_{\bar\Omega}\rho^0(\varsigma)\ :\ \varsigma\in\mathcal{M}\big(\bar\Omega;\mathcal{S}^{2\times2}\big),\ -\mathrm{Div}\,\varsigma=F\ \text{ in }\ \mathbb{R}^2\setminus\Upsilon_0\right\}\quad(\mathcal{P})$$

$$=\sup\left\{\int_{\bar\Omega}\langle u,F\rangle\ :\ u\in C_0^1(\Omega\setminus\Upsilon_0)^2,\ \ \rho\big(e(u)\big)\le 1\ \text{ in }\ \bar\Omega\right\}\quad(\mathcal{P}^*)$$

$$e(u)=\tfrac12\left(\nabla u+(\nabla u)^{\mathrm{T}}\right)$$

Michell potential

$$\rho(\xi)=\max_i|\lambda_i(\xi)|\qquad\qquad \rho^0(\sigma)=\sum_i|\lambda_i(\sigma)|$$



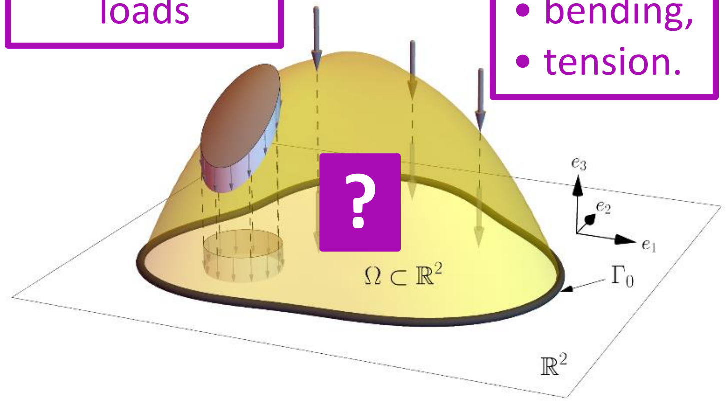
Different energy potentials

- $\rho(\xi)=\|\xi\|_2,\ \ \rho^0(\sigma)=\|\sigma\|_2$
- $\rho(\xi)=\big(\langle H\xi,\xi\rangle\big)^{1/2},$
  $\rho^0(\sigma)=\big(\langle H^{-1}\sigma,\sigma\rangle\big)^{1/2}$

Optimal Vault Problem
(Rozvany & Prager (1979))

Transmissible loads

Avoiding:
- bending,
- tension.



The design variables:

- $z:\bar\Omega\to\mathbb{R},\quad z=0\ \text{ on }\ \partial\Omega$
- $\underline{\varsigma}:\ G_z\ \to\ \mathcal{S}^{3\times3}_-$

Optimal Vault Problem – relaxed formulation

Data: **(2D)** $\Omega \subset \mathbb{R}^2$,	$f \in \mathcal{M}(\bar{\Omega}; \mathbb{R})$
$\underline{\Omega} := \Omega \times \mathbb{R} \subset \mathbb{R}^3$, **3D**	$\Upsilon_0 := \partial\Omega \times \{0\}$

COMPACT Set of loads attainable by transmistion of f

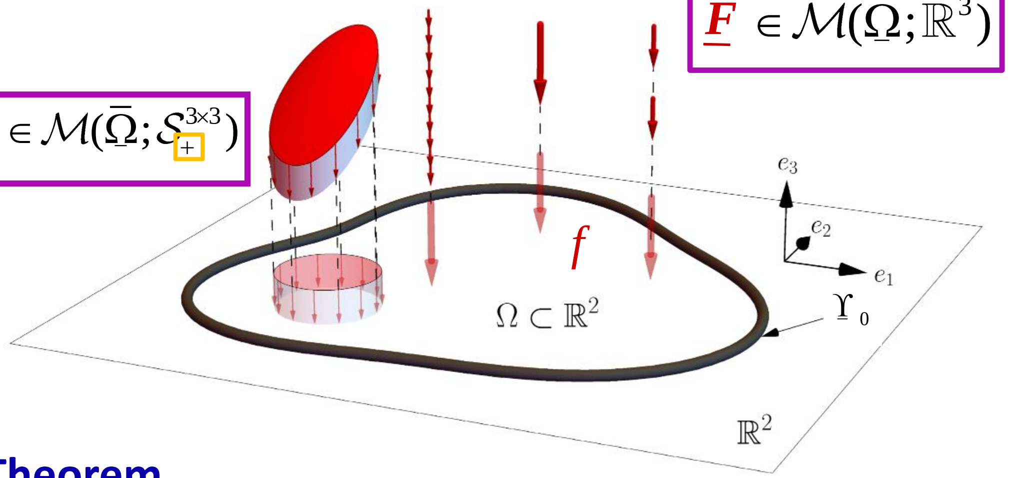
$$\mathcal{T}(\underline{\Omega}, f) := \left\{ \underline{F} = \mathrm{e}_3 \underline{f} \ : \ \underline{f} \in \mathcal{M}_+(\bar{\underline{\Omega}}), \quad \pi_{\mathbb{R}^2} \# \underline{f} = f \right\}$$

Optimal Vault Problem (pure tension setting)

$$\inf_{\substack{\underline{\varsigma} \in \mathcal{M}(\bar{\underline{\Omega}}; \mathcal{S}^{3\times 3}_+) \\ \underline{F} \in \mathcal{M}(\bar{\underline{\Omega}}; \mathbb{R}^3)}} \left\{ \int_{\bar{\underline{\Omega}}} \rho^0(\underline{\varsigma}) \ : \ \underline{F} \in \mathcal{T}(\underline{\Omega}, f), \ -\mathrm{Div}\, \underline{\varsigma} = \underline{F} \ \text{ in } \mathbb{R}^3 \setminus \Upsilon_0 \right\} \tag{OVP}$$

$\underline{\varsigma} \in \mathcal{M}(\bar{\underline{\Omega}}; \mathcal{S}^{3\times3}_+)$

$\underline{F} \in \mathcal{M}(\bar{\underline{\Omega}}; \mathbb{R}^3)$

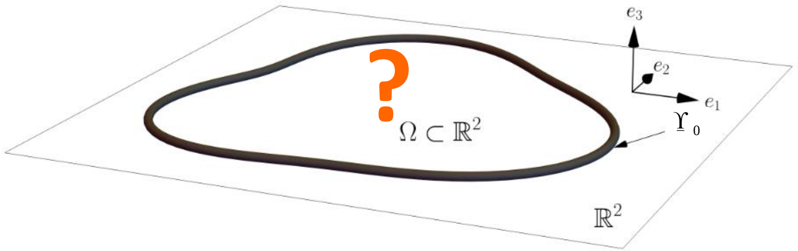


Theorem

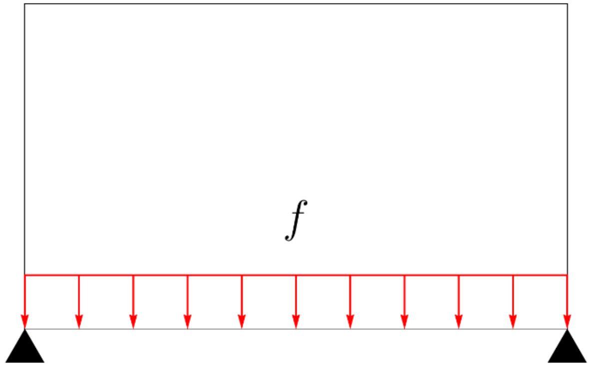
Solution $\underline{\varsigma}$ of (OVP) always exists.

Is $\underline{\varsigma}$ a vault? Namely, is there $z \in C_0(\Omega)$ such that

$\mathrm{spt}\, \underline{\varsigma} \subset G_z$?



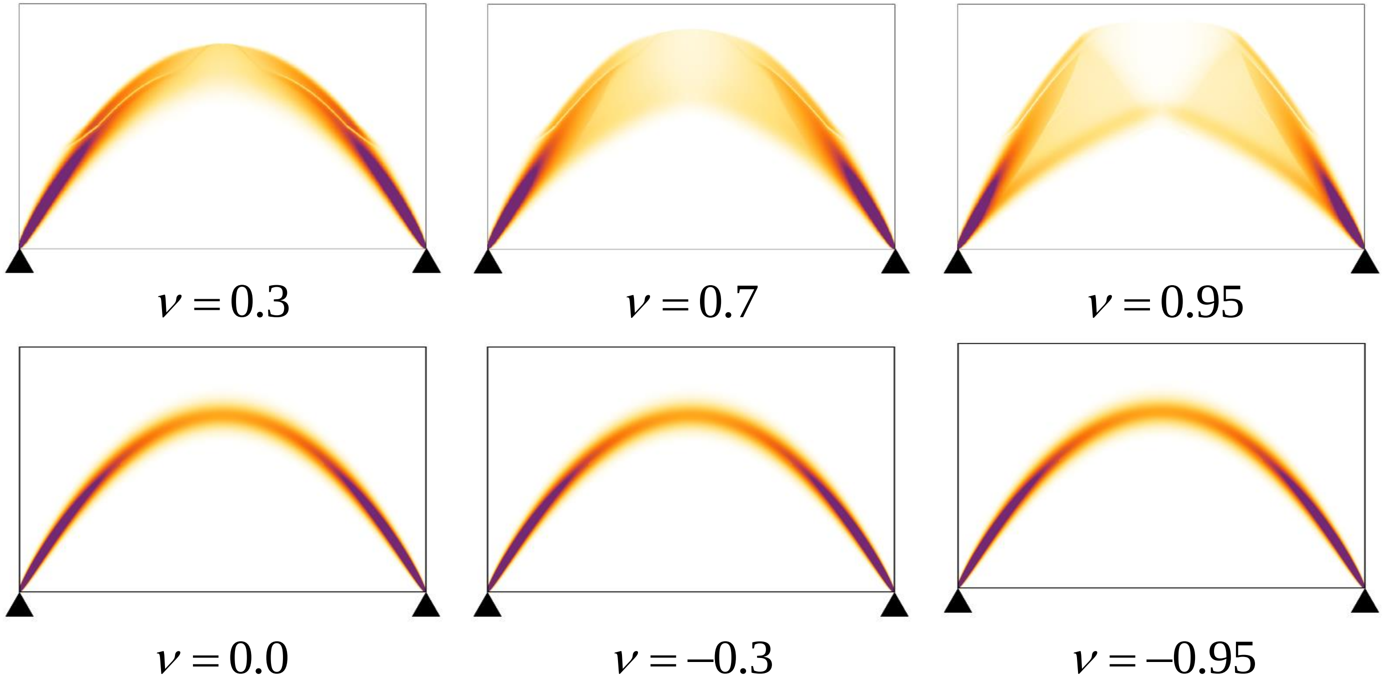
Optimal Arch Problem – relaxed formulation

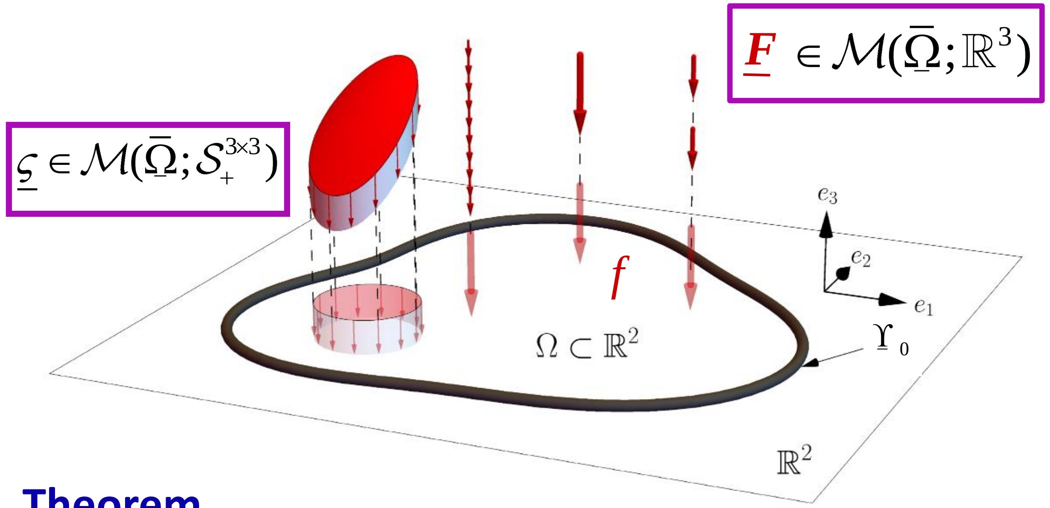


$$\rho(\xi) = \left(\langle H\xi, \xi\rangle\right)^{1/2},$$

$$\rho^0(\xi) = \left(\langle H^{-1}\sigma, \sigma\rangle\right)^{1/2}$$

H depends on Poisson ratio ν





Theorem

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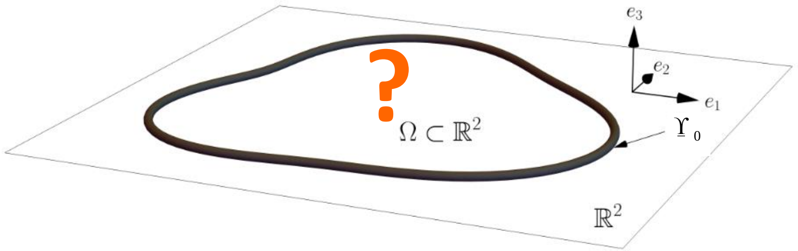


Optimal Vault Problem – relaxed formulation

We choose the Michell energy potential:

$$\rho(\underline{\xi}) = \max_i \left|\lambda_i(\underline{\xi})\right| \quad \longleftrightarrow \quad \rho^0(\underline{\sigma}) = \sum_i \left|\lambda_i(\underline{\sigma})\right|$$

$$\downarrow$$

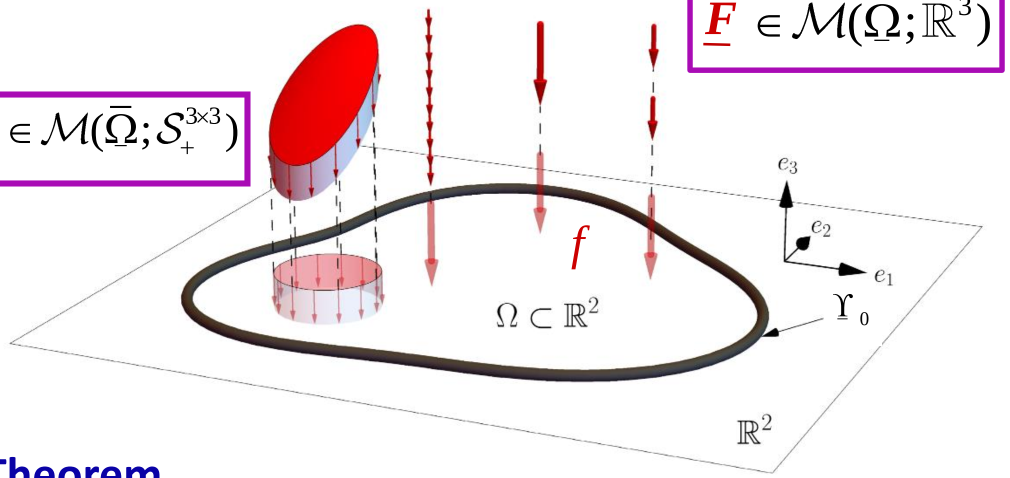
$$\rho^0(\underline{\sigma}) = \mathrm{Tr}\,\underline{\sigma}, \quad \underline{\sigma} \in \mathcal{S}^{3\times 3}_{+}$$

$$\downarrow$$

$$\rho_{+}(\underline{\xi}) = \max_i \lambda_i(\underline{\xi}_{+}) \quad \longleftrightarrow \quad \rho^0_{+}(\underline{\sigma}) := \begin{cases} \mathrm{Tr}\,\underline{\sigma} & \text{if } \underline{\sigma} \in \mathcal{S}^{3\times 3}_{+} \\ +\infty & \text{if } \underline{\sigma} \notin \mathcal{S}^{3\times 3}_{+} \end{cases}$$

$\underline{\varsigma} \in \mathcal{M}(\bar{\Omega}; \mathcal{S}^{3\times 3}_{+})$

$\underline{F} \in \mathcal{M}(\bar{\Omega}; \mathbb{R}^3)$



Theorem

Solution $\underline{\varsigma}$ of (OVP) always exists.

Is $\underline{\varsigma}$ a vault? Namely, is there $z \in C_0(\Omega)$ such that

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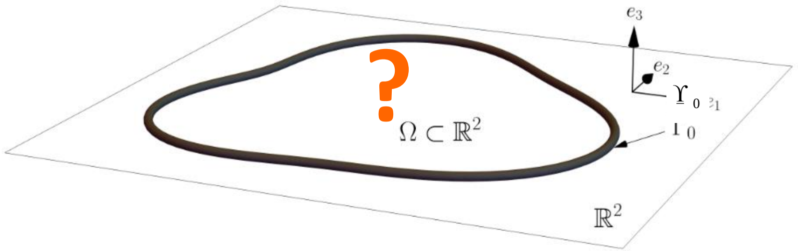
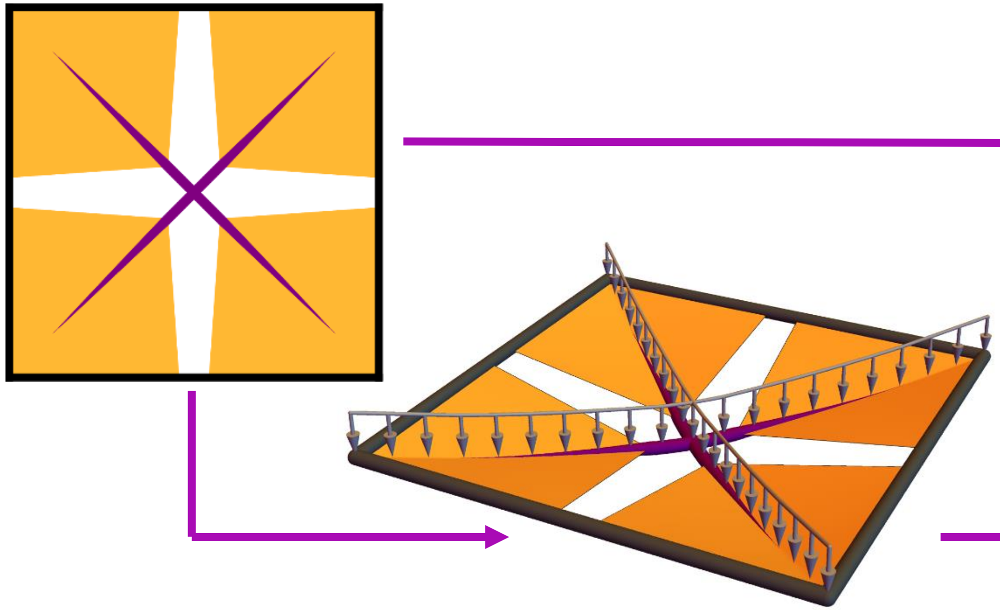


Recasting optimal 3D vault from optimal 2D membrane

2D Optimal membrane problem

KB, G. Bouchitté (2022)
Arch. Ration. Mech. Anal.

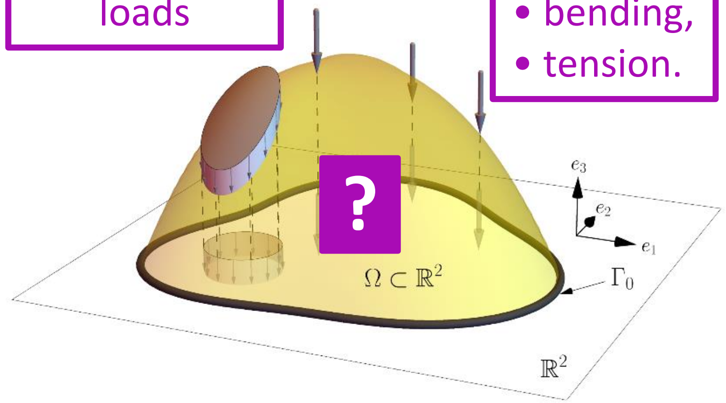


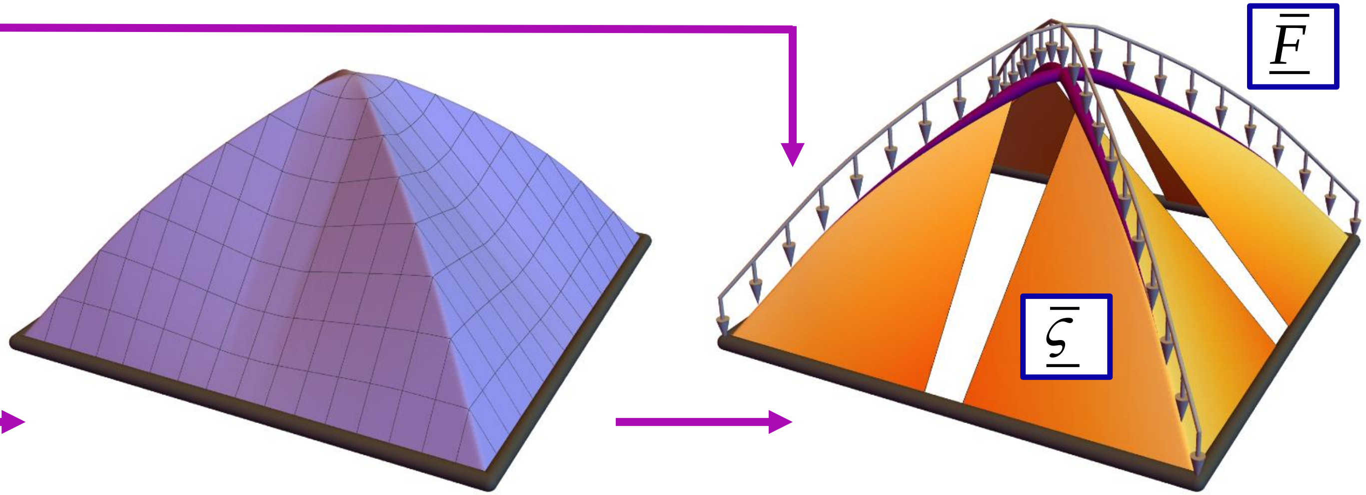
3D Optimal vault problem

KB (2022)
Comput. Math. Appl.

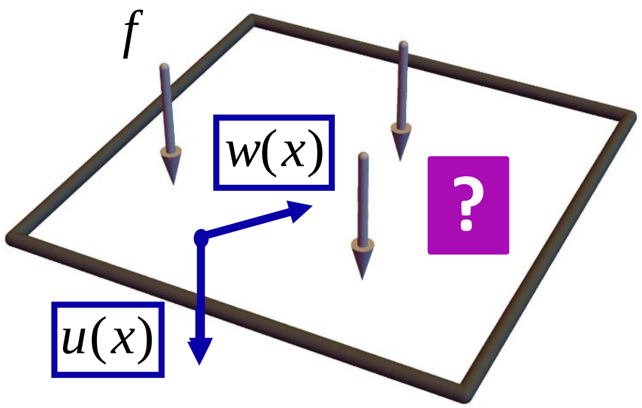
Transmissible loads

Avoiding:
- bending,
- tension.





Optimal Design of Membranes



Membrane Model

- moderate deflections:

$$\varepsilon = \mathcal{A}(u,w) = e(w) \;+\; \tfrac{1}{2}\nabla u \otimes \nabla u$$

- perfect susceptibility to wrinkling:

$$\rho(\xi) = \max_i \left|\lambda_i(\xi)\right| \quad \Rightarrow \quad \rho_{+}(\xi) = \max_i \lambda_i(\xi_{+})$$

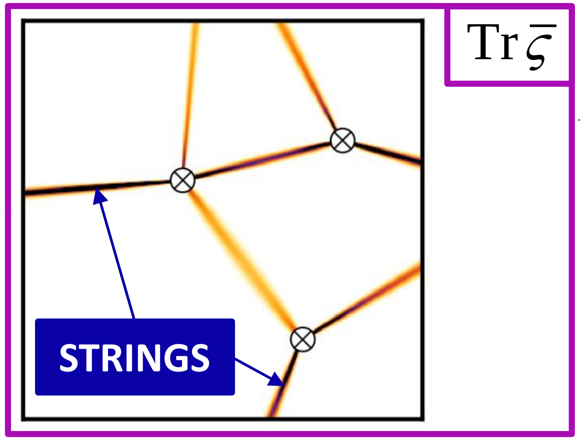
$$\mathcal{S}^{2\times 2} \times \mathbb{R}^2 \ni (\xi,\theta) \to \rho\left(\xi + \tfrac{1}{2}\theta \otimes \theta\right)$$

non-convex

$$\mathcal{S}^{2\times 2} \times \mathbb{R}^2 \ni (\xi,\theta) \to \rho_{+}\left(\xi + \tfrac{1}{2}\theta \otimes \theta\right)$$

convex

FEM simulation



The Pair of Mutually Dual Problems

$$\min_{(\varsigma,\vartheta)\in\mathcal{M}(\overline{\Omega};\mathcal{S}^{2\times 2}\times\mathbb{R}^2)} \left\{ \int_{\overline{\Omega}} \mathrm{Tr}\,\varsigma + \int_{\overline{\Omega}} \tfrac{1}{2}\left\langle \varsigma^{-1}\vartheta,\vartheta\right\rangle \;:\; \varsigma \geq 0, \; \begin{array}{l} -\mathrm{div}\,\varsigma = 0 \\ -\mathrm{div}\,\vartheta = f \end{array} \right\} \qquad (\mathcal{P})$$

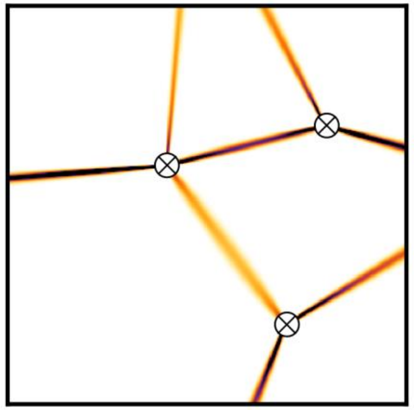
$$\sup_{(u,w)\in\mathcal{D}(\Omega)^3} \left\{ \int_{\overline{\Omega}} u\, f \;:\; \rho_{+}\left(e(w) + \tfrac{1}{2}\nabla u \otimes \nabla u\right) \leq 1 \quad \text{in } \overline{\Omega} \right\} \qquad (\mathcal{P}^*)$$

Relaxed solutions of $(\mathcal{P}^*)$ satisfy:

$$u \;\in\; C^{0,\frac{1}{2}} \cap W^{1,2}(\Omega)$$

$$w \;\in\; BV \cap L^{\infty}(\Omega;\mathbb{R}^2)$$

Optimal Design of Membranes – approx. via systems of strings



$\mathrm{Tr}\,\bar{\varsigma}$

$$\sup_{(u,w)\in\mathcal{D}(\Omega)^3} \left\{ \int_{\bar\Omega} u\, f \;:\; \max_i \lambda_i\big(\mathcal{A}(u,w)(x)\big) \le 1 \quad \forall x \in \bar\Omega \right\} \qquad (\mathcal{P}^*)$$

$$\mathcal{A}(u,w) = e(w) + \tfrac{1}{2}\nabla u \otimes \nabla u$$

$$\langle w(y)-w(x), y-x\rangle + \frac{1}{2}\big(u(y)-u(x)\big)^2 \le |y-x|^2 \qquad \forall (x,y) \in \bar\Omega\times\bar\Omega$$

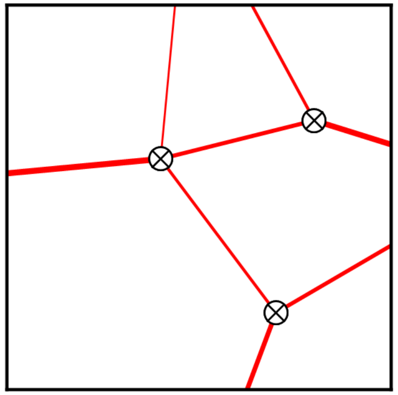
Approximation by systems of strings

$$\inf_{(\pi,\Pi)\in\mathcal{M}(\bar\Omega\times\bar\Omega;\mathbb{R}\times\mathbb{R}_+)} \left\{ \mathcal{J}(\pi,\Pi) \;:\; (\pi,\Pi)\in\Xi(\Omega, f) \right\} \qquad (\tilde{\mathcal{P}})$$

"Ground structure" approach

+

Conic quadratic programming



Maximizing the Monge-Kantorovich metric

In-plane displacement

$w : \bar\Omega \to \mathbb{R}^2$

Metric tensor

$g_w = \mathrm{I} - e(w)$

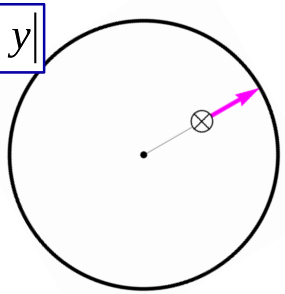
Distance

$d_w(x,y)$

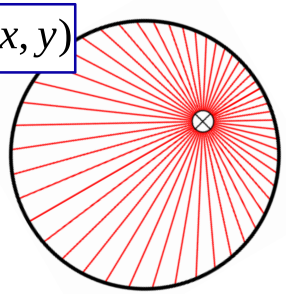
Theorem

$(\bar u, \bar w)$ solves $(\mathcal{P}^*)$ $\Rightarrow$ $d_{\bar w}$ maximizes the MK cost of transporting f to the boundary $\partial\Omega$.

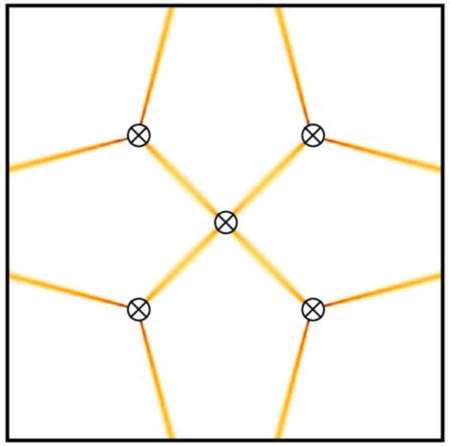
$|x-y|$

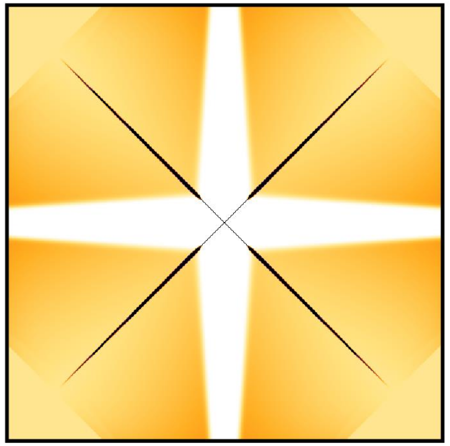


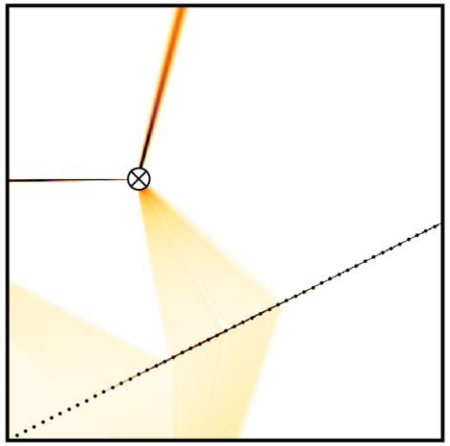
$d_{\bar w}(x,y)$

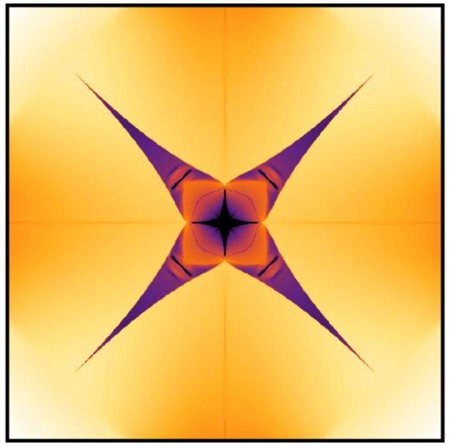


Optimal Membrane Problem – simulations







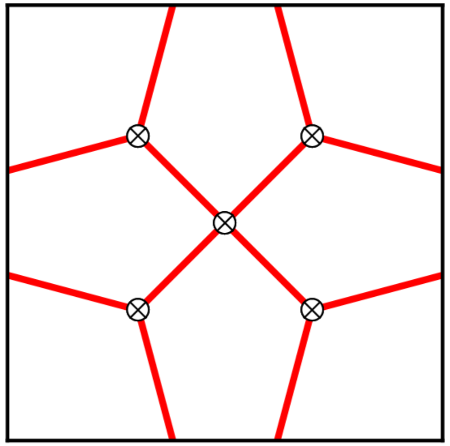


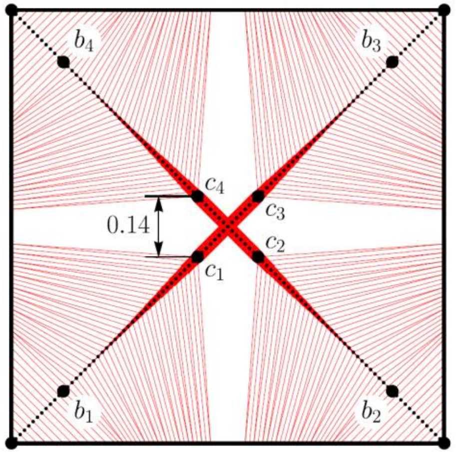
$$f = \sum_{i=0}^{4} \delta_{x_i}$$

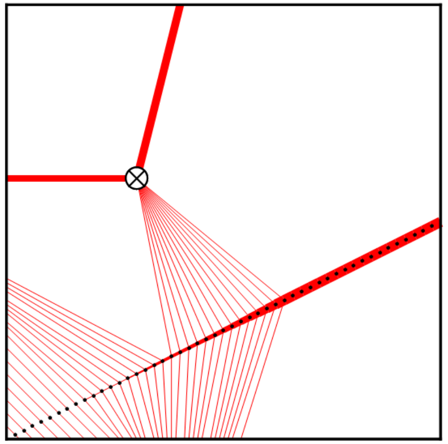
$$f = \sum_{i=1}^{2} \mathcal{H}^1 \llcorner [A_i, B_i]$$

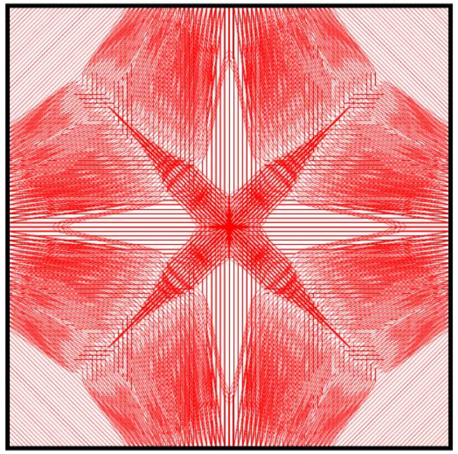
$$f = \delta_{x_0} + \mathcal{H}^1 \llcorner [A, B]$$

$$f = \mathcal{L}^2 \llcorner \Omega$$

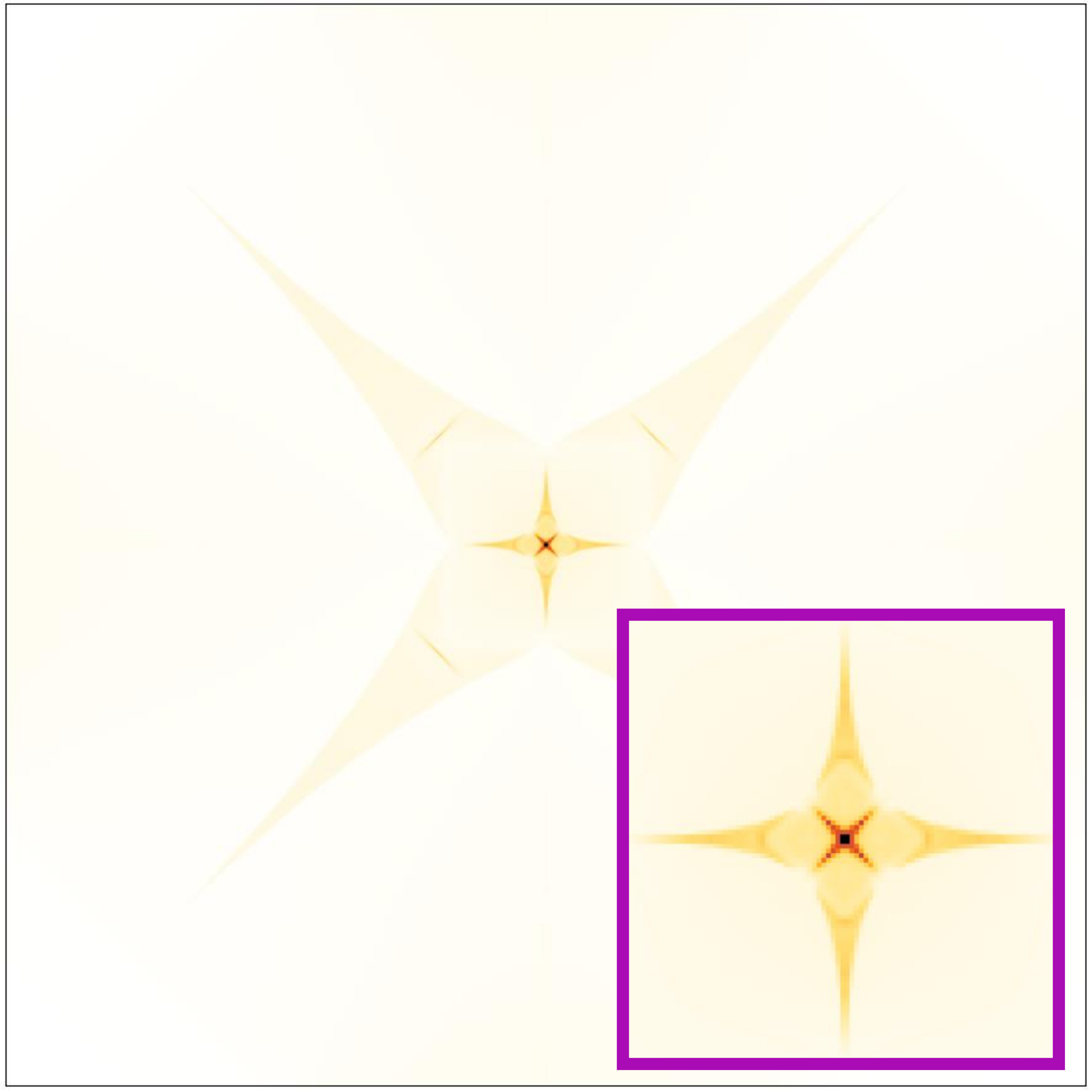


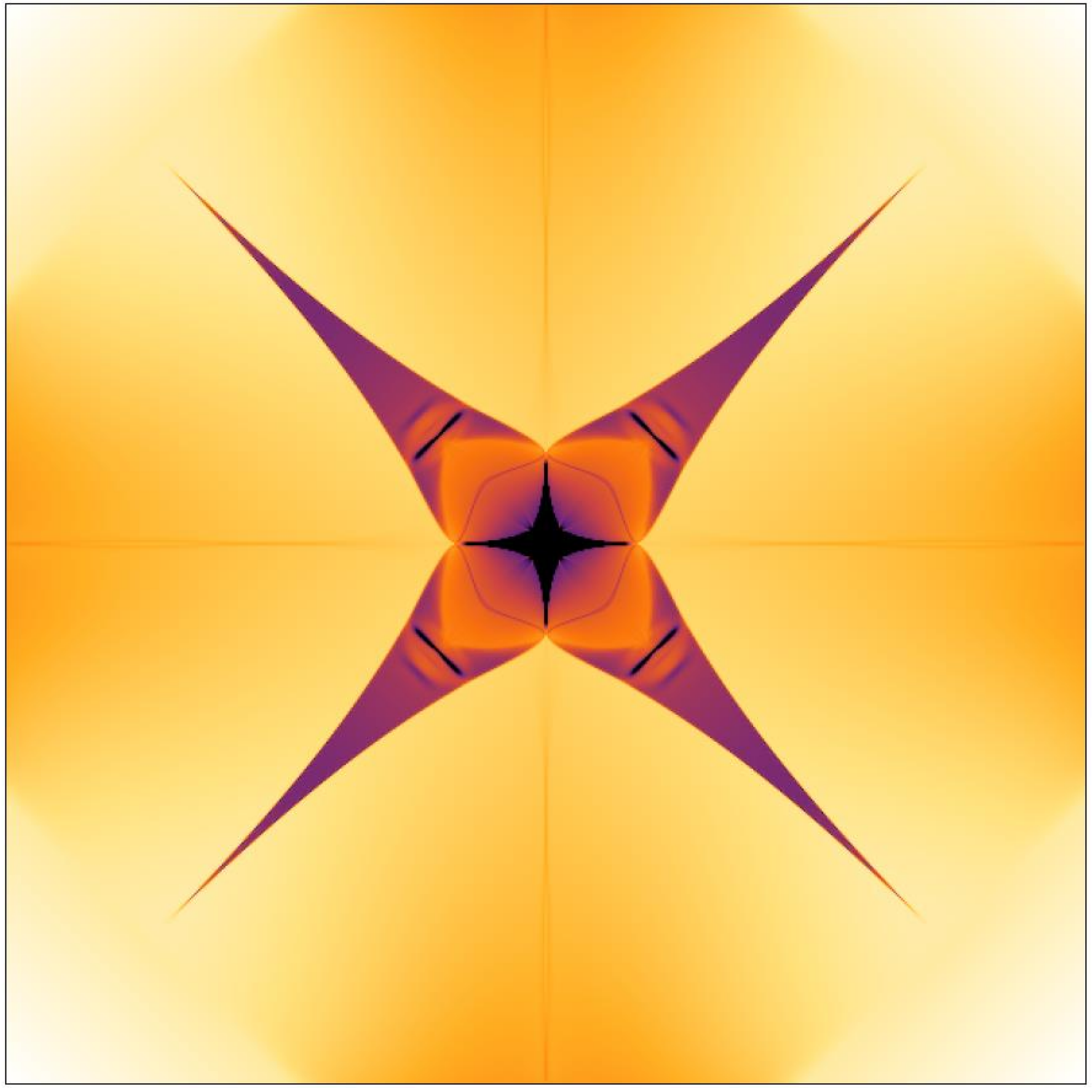






Optimal Membrane Problem – simulations





The link between: optimal 3D vault & optimal 2D membrane

$$\mathcal{Z} = \inf_{\underline{\varsigma},\, \underline{F}} \left\{ \int_{\bar{\Omega}} \operatorname{Tr} \underline{\varsigma} \;:\; \underline{F} \in \mathcal{T}(\Omega, f),\; -\operatorname{Div} \underline{\varsigma} = \underline{F} \quad \text{in } \mathbb{R}^3 \setminus \Upsilon_0 \right\}$$

$$Z = \sup_{(u,w)\in \mathcal{D}(\Omega)^3} \left\{ \int_{\bar{\Omega}} u\, f \;:\; \max_i \lambda_i \left(e(w) + \tfrac{1}{2} \nabla u \otimes \nabla u \right) \leq 1 \quad \text{in } \bar{\Omega} \right\}$$

$$\mathcal{Z} = \inf_{\underline{F} \in \mathcal{T}(\Omega, f)} \inf_{\underline{\varsigma} \geq 0} \left\{ \int_{\bar{\Omega}} \operatorname{Tr} \underline{\varsigma} \;:\; -\operatorname{Div} \underline{\varsigma} = \underline{F} \quad \text{in } \mathbb{R}^3 \setminus \Upsilon_0 \right\} = \inf_{\underline{F} \in \mathcal{T}(\Omega, f)} \sup_{\underline{v} \in \mathcal{D}(\mathbb{R}^3 \setminus \Upsilon_0)^3} \left\{ \int_{\bar{\Omega}} \langle \underline{v}, \underline{F} \rangle \;:\; \max_i \lambda_i \left(e(\underline{v}) \right) \leq 1 \quad \text{in } \underline{\bar{\Omega}} \right\}$$

$$= \sup_{\underline{v} \in \mathcal{D}(\mathbb{R}^3 \setminus \Upsilon_0)^3} \inf_{\underline{F} \in \mathcal{T}(\Omega, f)} \left\{ \int_{\bar{\Omega}} \langle \underline{v}, \underline{F} \rangle \;:\; \max_i \lambda_i \left(e(\underline{v}) \right) \leq 1 \quad \text{in } \underline{\bar{\Omega}} \right\}$$

$$\inf_{\underline{F} \in \mathcal{T}(\Omega, f)} \int_{\bar{\Omega}} \langle \underline{v}, \underline{F} \rangle \;\geq\; \begin{cases} \sqrt{2} \int_{\bar{\Omega}} u\, f & \text{if } \underline{v}(\underline{x}) = w(x) + \sqrt{2}\, u(x) \mathrm{e}_3 \\ -\infty & \text{otherwise} \end{cases}$$

$$\geq \sup_{(u,w)\in \mathcal{D}(\Omega)^3} \left\{ \sqrt{2} \int_{\bar{\Omega}} u\, f \;:\; \max_i \lambda_i \left(e(\underline{v}) \right) \leq 1 \quad \text{in } \underline{\bar{\Omega}},\; \underline{v}(\underline{x}) = w(x) + \sqrt{2}\, u(x) \mathrm{e}_3 \right\}$$

$$= \sqrt{2} \sup_{(u,w)\in \mathcal{D}(\Omega)^3} \left\{ \int_{\bar{\Omega}} u\, f \;:\; \max_i \lambda_i \left(\begin{bmatrix} e(w) & \frac{1}{\sqrt{2}} \nabla u \\ \frac{1}{\sqrt{2}} (\nabla u)^{\mathrm{T}} & 0 \end{bmatrix} \right) \leq 1 \quad \text{in } \bar{\Omega} \right\}$$

$$= \sqrt{2}\, Z$$

Proposition (property of Michell potential)

For any $\xi \in \mathcal{S}^{2\times 2}$ and $\theta \in \mathbb{R}^2$

$$\max_i \lambda_i \left(\begin{bmatrix} \xi & \frac{1}{\sqrt{2}} \theta \\ \frac{1}{\sqrt{2}} \theta^{\mathrm{T}} & 0 \end{bmatrix} \right) \leq 1 \quad \Leftrightarrow \quad \max_i \lambda_i \left(\xi + \tfrac{1}{2} \theta \otimes \theta \right) \leq 1$$

Lemma

$$\mathcal{Z} \;\geq\; \sqrt{2}\, Z$$

Construction of the Optimal Vault

$$\min_{(\varsigma,\vartheta)} \left\{ \int_{\bar\Omega} \mathrm{Tr}\,\varsigma \;+\; \int_{\bar\Omega} \tfrac{1}{2}\left\langle \varsigma^{-1}\vartheta,\vartheta \right\rangle \;:\; \varsigma \geq 0,\; \begin{array}{l} -\mathrm{Div}\,\varsigma = 0 \\ -\mathrm{div}\,\vartheta = f \end{array} \right\} \qquad (\mathcal{P})$$

$$\sup_{(u,w)\in\mathcal{D}(\Omega)^3} \left\{ \int_{\bar\Omega} u\, f \;:\; \max_i \lambda_i\left(e(w) + \tfrac{1}{2}\nabla u \otimes \nabla u\right) \leq 1 \;\; \text{in } \bar\Omega \right\} \qquad (\mathcal{P}^*)$$

Defining 2D objects:

$$\mu := \mathrm{Tr}\,\varsigma + \tfrac{1}{2}\left\langle \varsigma^{-1}\vartheta,\vartheta \right\rangle \in \mathcal{M}_+\left(\bar\Omega\right)$$

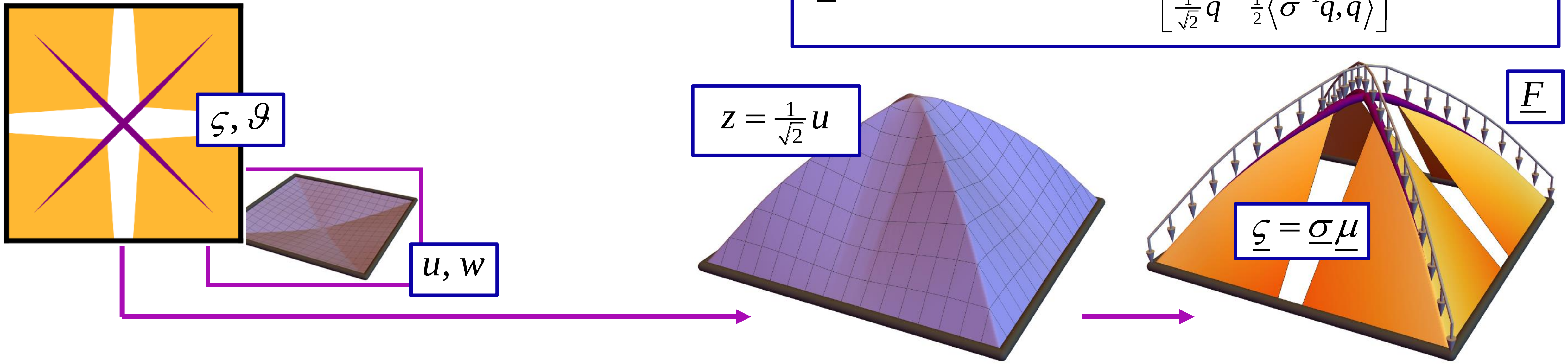
$$\sigma = \frac{d\varsigma}{d\mu} \in L^\infty_\mu(\bar\Omega;\mathcal{S}^{2\times 2}_+), \qquad q = \frac{d\vartheta}{d\mu} \in L^2_\mu(\bar\Omega;\mathbb{R}^2)$$

Defining the elevation function, elevating the load:

$$z := \tfrac{1}{\sqrt{2}}\,u, \qquad \underline{F} := (id,z)\#(\mathrm{e}_3\, f)$$

Constructing 3D vault:

$$\underline{\mu} := \sqrt{2}\,(id,z)\#\mu, \qquad \underline{\sigma} := \begin{bmatrix} \sigma & \frac{1}{\sqrt{2}}q \\ \frac{1}{\sqrt{2}}q & \frac{1}{2}\left\langle \sigma^{-1}q,q\right\rangle \end{bmatrix} \circ (id,z)^{-1}$$



Construction of the Optimal Vault

- $\underline{F} \in \mathcal{T}(\Omega, f)$
- $\int_{\bar{\Omega}} \mathrm{Tr}\, \underline{\varsigma} = \sqrt{2}\, Z$
- $-\mathrm{Div}\, \underline{\varsigma} = \underline{F} \quad \text{in } \mathbb{R}^3 \setminus \Upsilon_0$ (by optimality conditions for OMP)

Assumption: solutions u, w are Lipschitz continuous.

Defining 2D objects:

$$\mu := \mathrm{Tr}\,\varsigma + \tfrac{1}{2}\left\langle \varsigma^{-1}\vartheta, \vartheta \right\rangle \in \mathcal{M}_+\left(\bar{\Omega}\right)$$

$$\sigma = \frac{d\varsigma}{d\mu} \in L^\infty_\mu(\bar{\Omega}; \mathcal{S}^{2\times 2}_+), \qquad q = \frac{d\vartheta}{d\mu} \in L^2_\mu(\bar{\Omega}; \mathbb{R}^2)$$

Defining the elevation function, elevating the load:

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Constructing 3D vault:

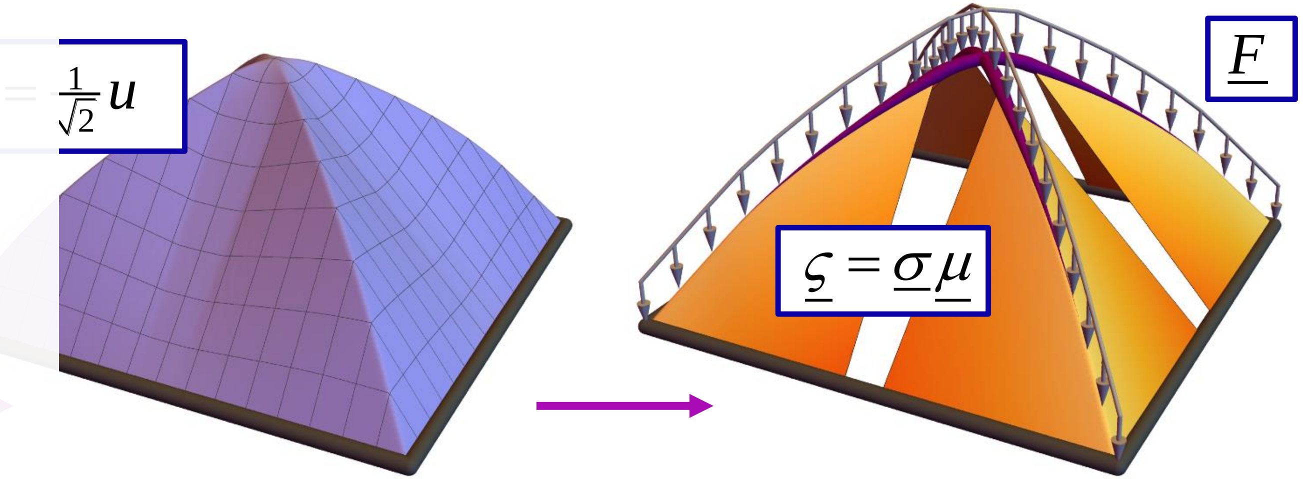
$$\underline{\mu} := \sqrt{2}\,(id, z)\#\mu, \qquad \underline{\sigma} := \begin{bmatrix} \sigma & \frac{1}{\sqrt{2}} q \\ \frac{1}{\sqrt{2}} q & \frac{1}{2}\left\langle \sigma^{-1} q, q \right\rangle \end{bmatrix} \circ (id, z)^{-1}$$

Lemma

$$\mathcal{Z} = \inf_{\underline{\varsigma}, \underline{F}} \left\{ \int_{\bar{\Omega}} \mathrm{Tr}\, \underline{\varsigma} \, : \, \underline{F} \in \mathcal{T}(\Omega, f), \; -\mathrm{Div}\, \underline{\varsigma} = \underline{F} \quad \text{in } \mathbb{R}^3 \setminus \Upsilon_0 \right\}$$

$$\mathcal{Z} \geq \sqrt{2}\, Z$$

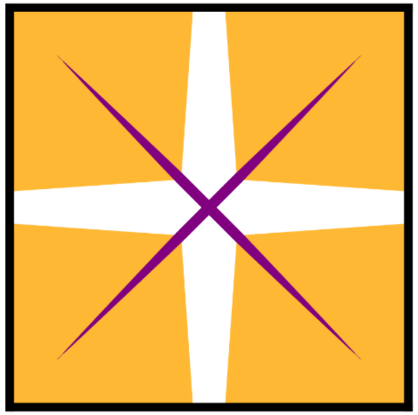
where $Z = \min \mathcal{P} = \sup \mathcal{P}^*$.



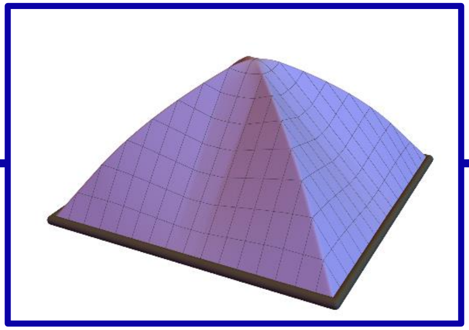
Numerical Approximation of Optimal Vaults

Optimal membrane

2D



Unprojection



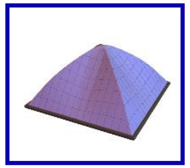
Optimal form

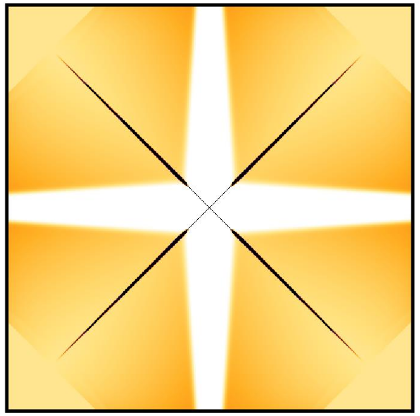
3D

NUMERICAL

Approximation of opt. memb.

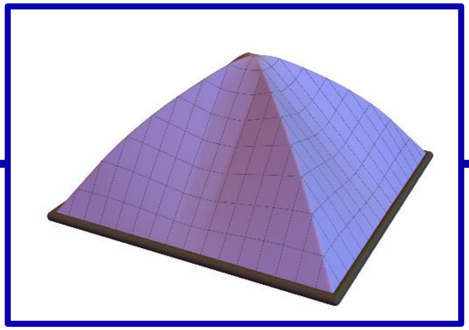
2D





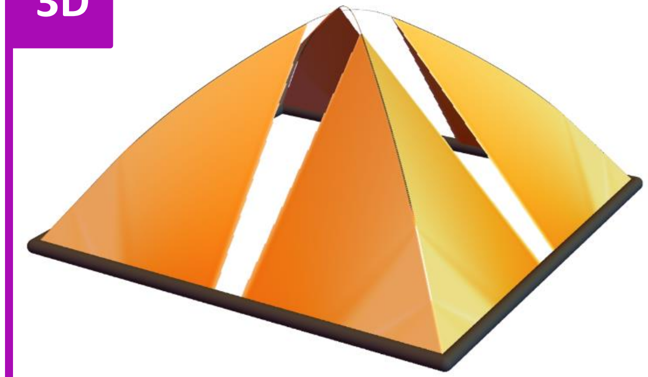
ANALYTICAL

Unprojection

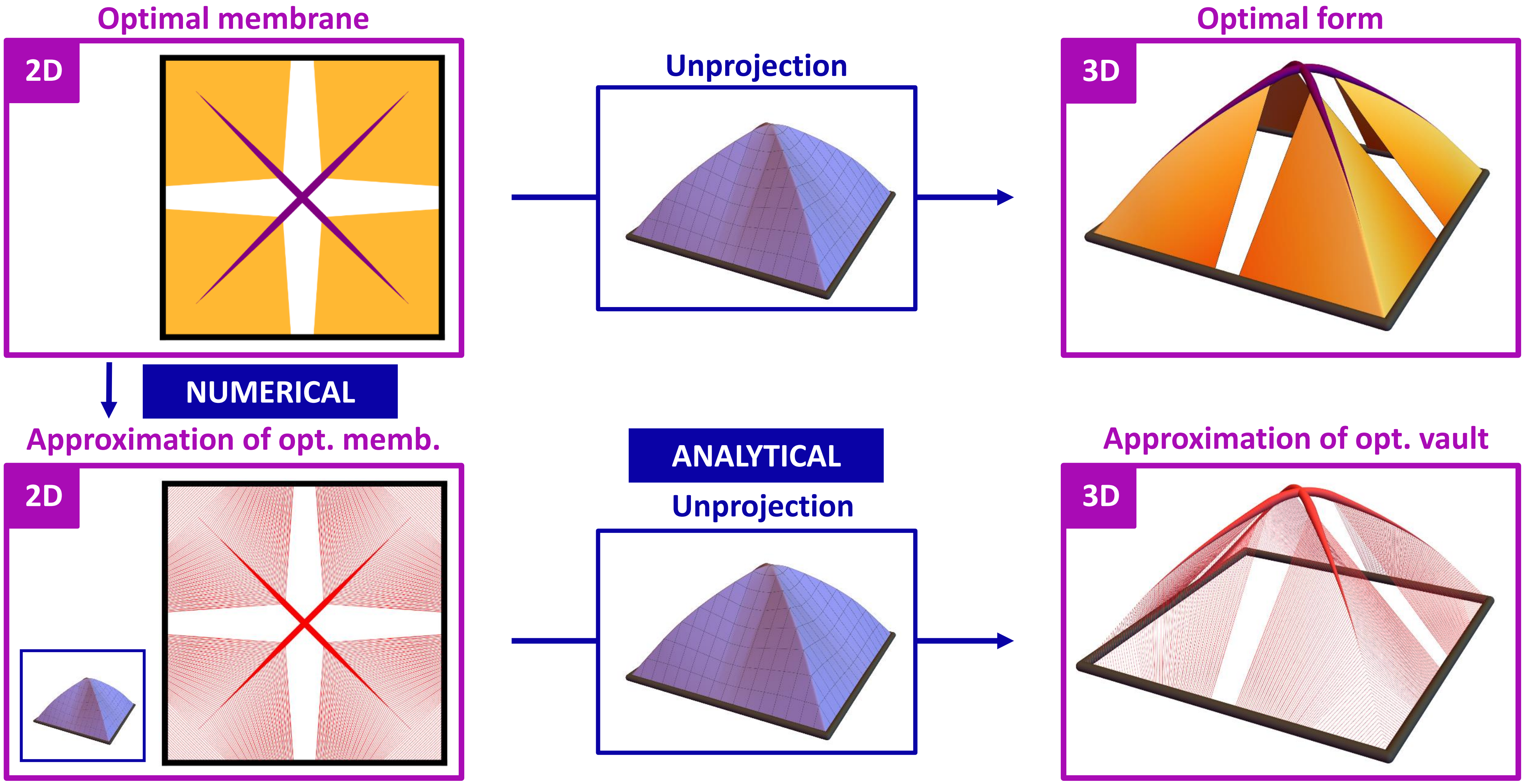


Approximation of opt. vault

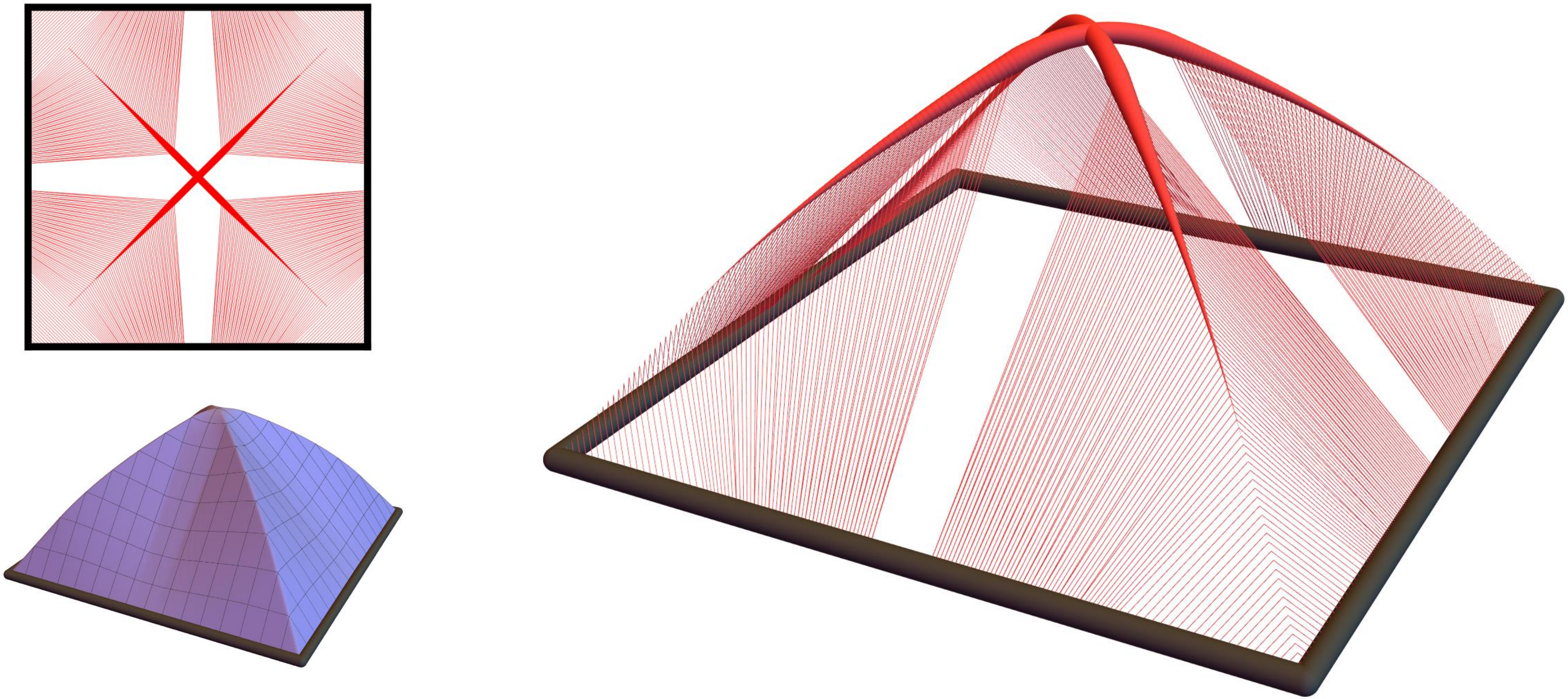
3D



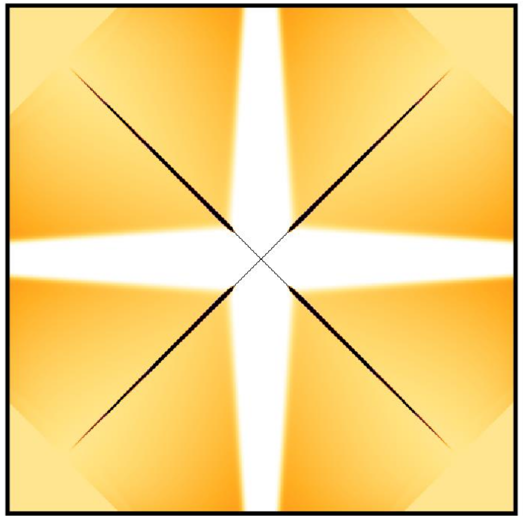
Numerical Approximation of Optimal Vaults

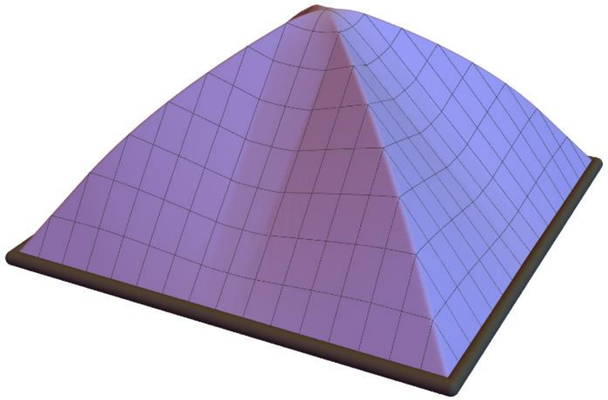


Approximiation of Optimal Vaults – simulations

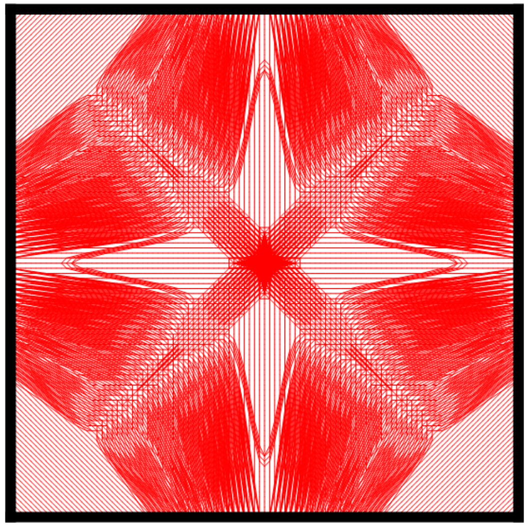


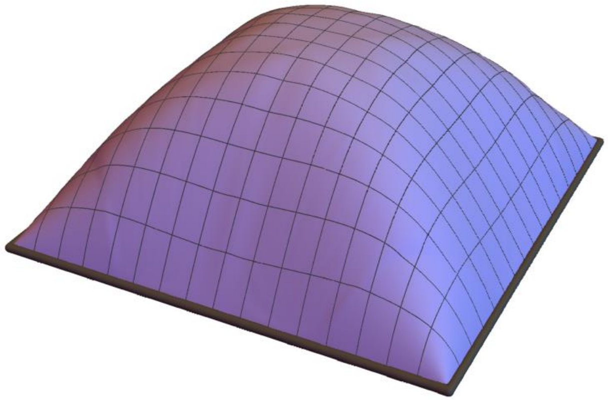
Approximiation of Optimal Vaults – simulations



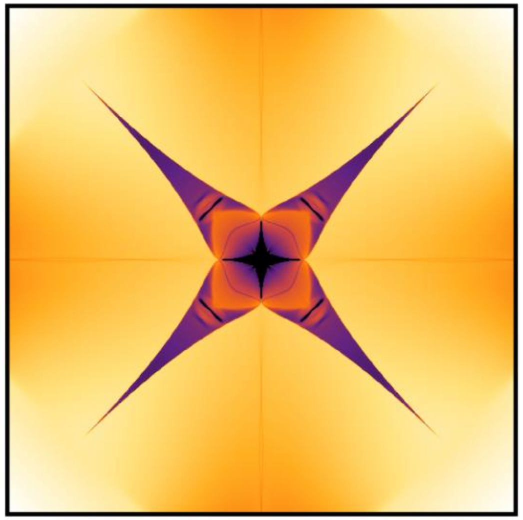


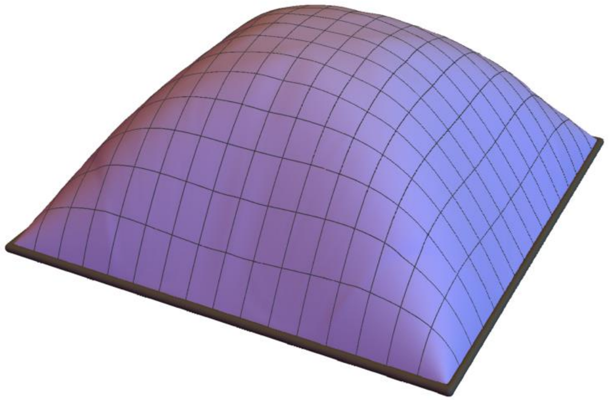
Approximiation of Optimal Vaults – simulations

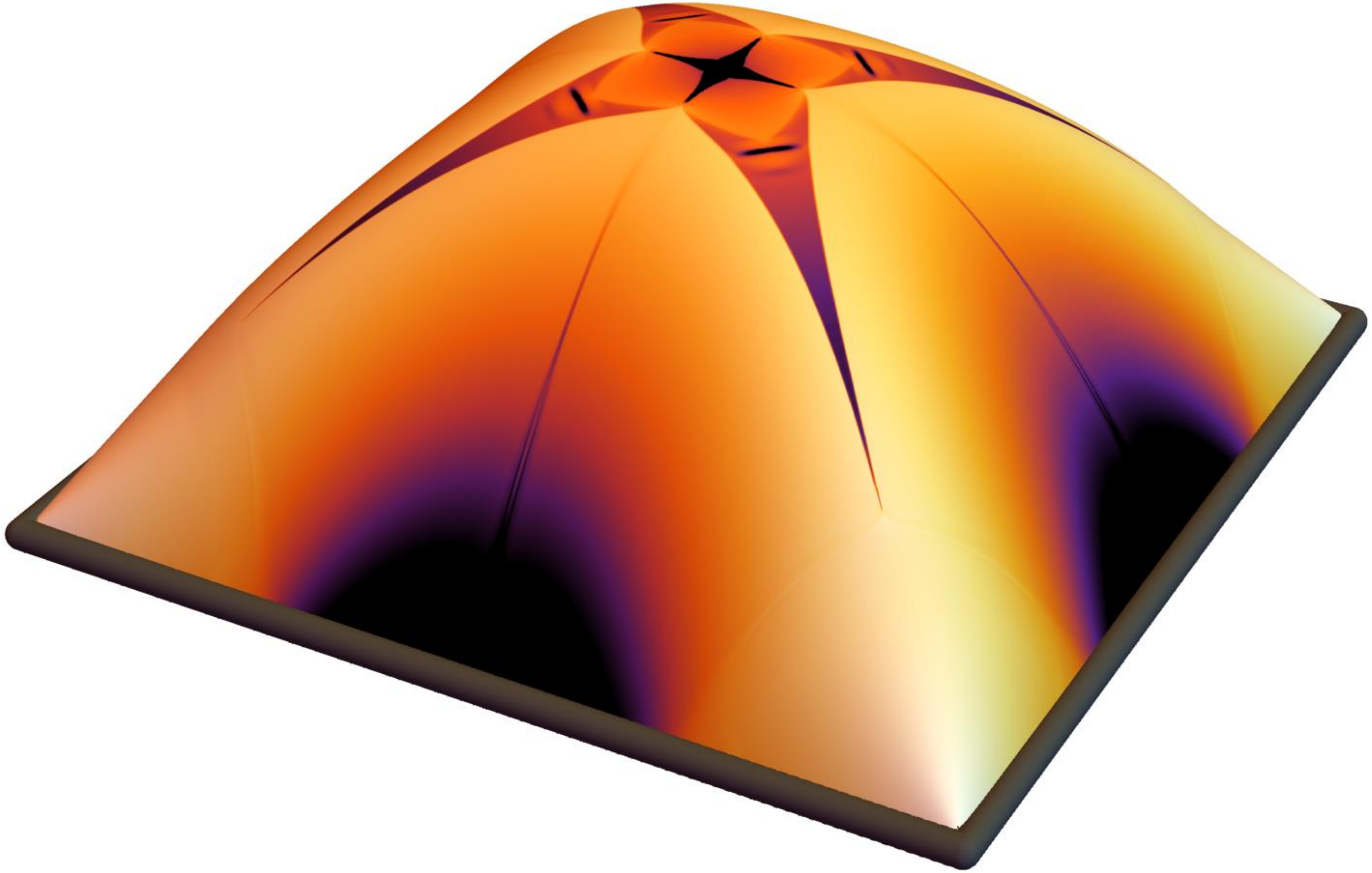




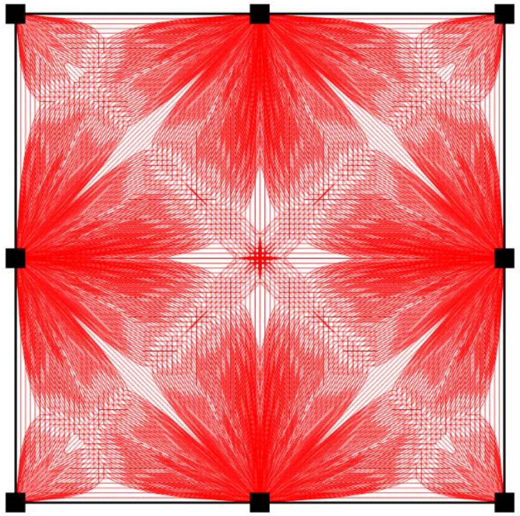
Approximiation of Optimal Vaults – simulations

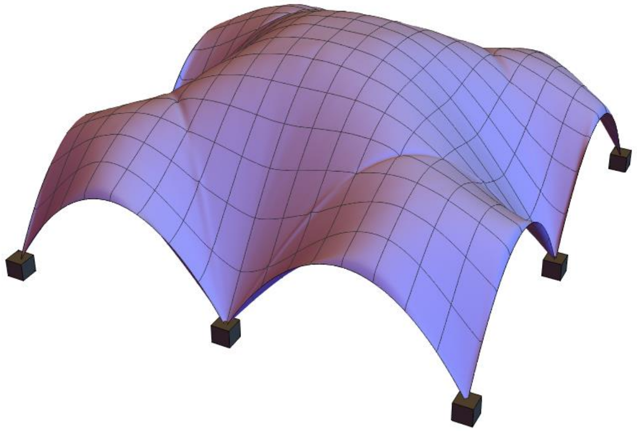




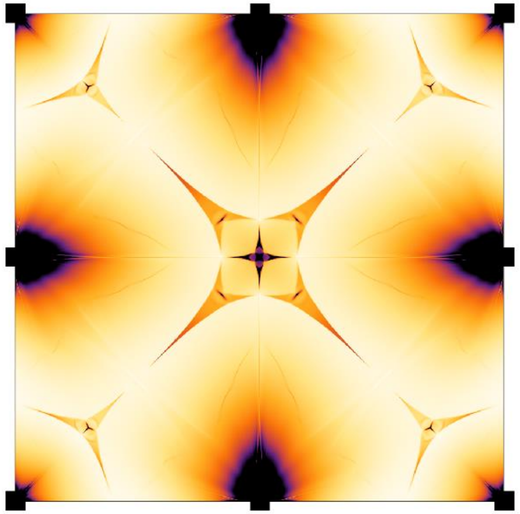


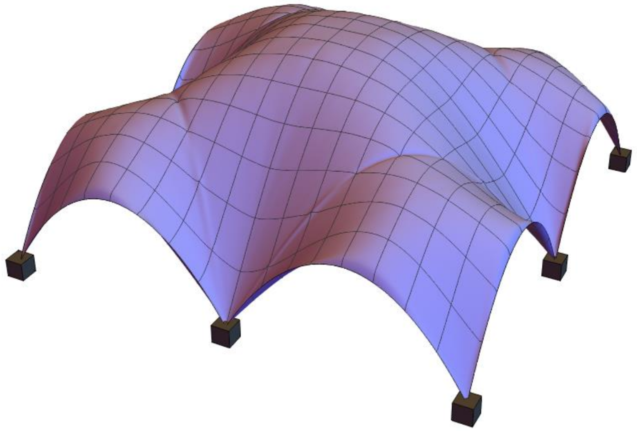
Approximiation of Optimal Vaults – simulations





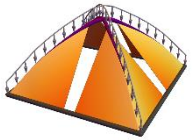
Approximiation of Optimal Vaults – simulations





Open problems

#1 How to prove the vault's optimality when (u,w) are not Lipschitz continuous?

Construction of $(\underline{\varsigma},\underline{F})$  ✓

$\int_{\bar{\Omega}} \mathrm{Tr}\,\underline{\varsigma} = \sqrt{2}\, Z$ ✓

$\underline{F} \in \mathcal{T}(\Omega, f)$ ✓

$-\mathrm{Div}\,\underline{\varsigma} = \underline{F}$ in $\mathbb{R}^3 \setminus \Upsilon_0$?

Currently $\nabla_\mu u, e_\mu(w)$ needed.

#2 Are there solutions $(\underline{\varsigma},\underline{F})$ of the relaxed problem that do not constitute vaults?

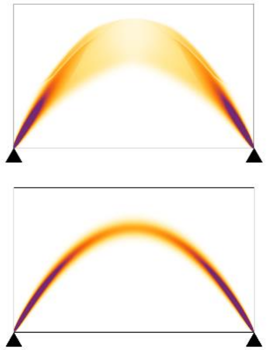
#3 What about potentials different than $\rho(\underline{\xi}) = \max_i \left|\lambda_i(\underline{\xi})\right|$?

2D

$\rho(\xi) = \left(\langle H\xi,\xi\rangle\right)^{1/2}$

$\nu > 0$

$\nu \leq 0$



3D

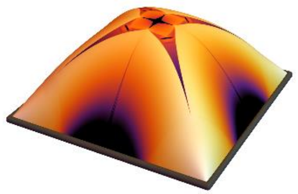
$\rho(\xi) = \left(\langle H\xi,\xi\rangle\right)^{1/2}$

$\nu > -1$ → ?

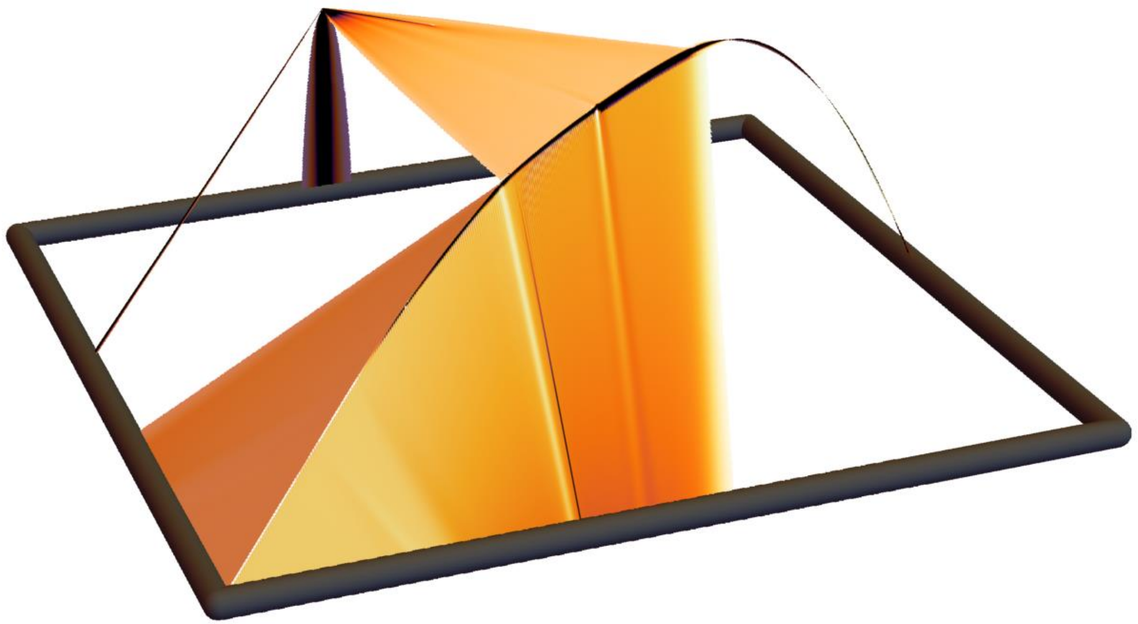
(no link to the membrane problem)

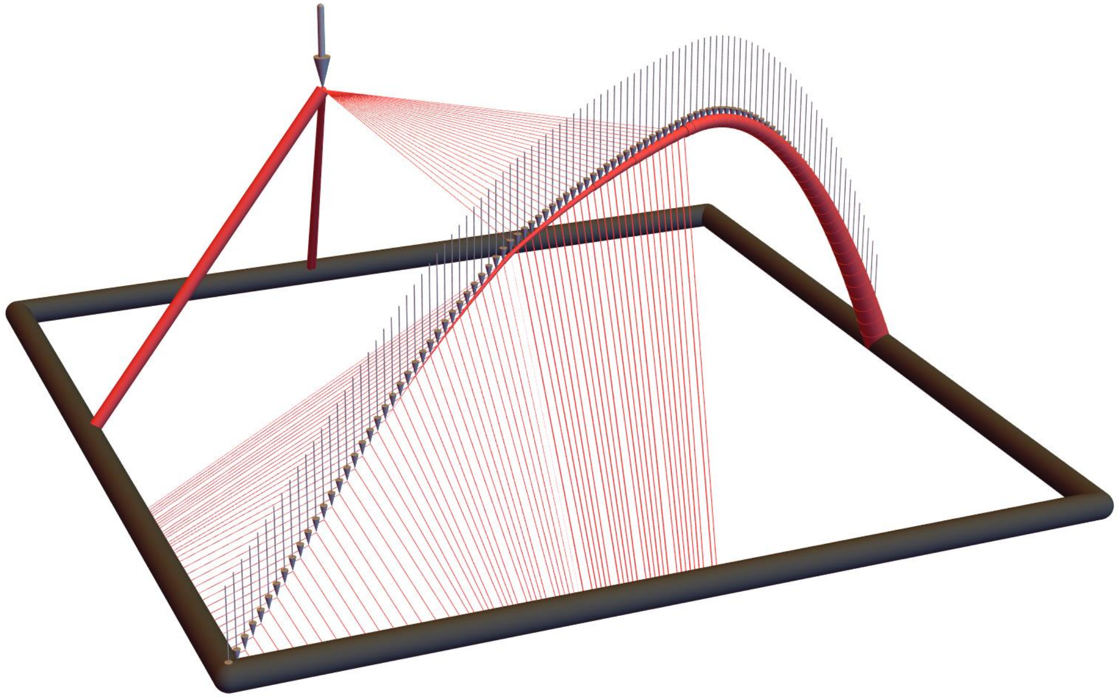
$\nu = -1$

$\Leftrightarrow \rho(\xi) = \max_i \left|\lambda_i(\underline{\xi})\right|$



#4, #5, #6,… Open problems concerning the Optimal Membrane Problem itself.





THANK YOU
FOR YOUR ATTENTION