THIRD-ORDER EIGENVALUE PROBLEMS

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BULK-BOUNDARY EIGENVALUE PROBLEMS

$$\boxed{E[u] = \frac{\int_\Omega |\Delta u|^2\,dx}{\int_\Omega u^2 + \gamma \int_{\partial\Omega} u^2}} \qquad 0 \le \gamma \le \infty$$

$$\mathcal{H} = \{u \in H^2 / \partial_\nu u = 0\}$$

$$\begin{cases} Lu := \Delta^2 u = \lambda u & \text{in } \Omega \\ Nu := \partial_\nu u = 0 & \text{on } \partial\Omega \\ Bu := -\partial_\nu \Delta u = \gamma \lambda u & \text{on } \partial\Omega \end{cases}$$

Historical remarks:

$\boxed{\gamma = \infty}$

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \\ -\partial_\nu \Delta u = \lambda u & \text{on } \partial\Omega \end{cases}$$

Kuttler-Sigillito 60's.
(Biharmonic Steklov eigenvalue problems)

$\boxed{0 < \gamma < \infty}$

Linearization of the Cahn-Hilliard equation with dynamic boundary conditions.
[Knopf-Signori, Knopf-Lam-Liu-Metzger, ...]

FUNCTIONAL ANALYSIS:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \\ -\partial_\nu \Delta u = g & \text{on } \partial\Omega \end{cases}$$

$\mathcal{H} = \{u \in H^2(\Omega) \,/\, \partial_\nu u = 0\}$

L^2_γ : $\|u\|^2 = \int_\Omega u^2 + \gamma \int_{\partial\Omega} u^2$

Fredholm condition: $\int_\Omega f + \gamma \int_{\partial\Omega} g = 0$.

- Self-adjoint (Agmon-Douglis-Nirenberg)
- Good elliptic estimates
- $\exists$ $0 = \lambda_1 \leq \lambda_2 \leq \dots \nearrow \infty$ eigenvalues

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CONVERGENCE TO THE LIMITING CASES

Theorem:

$\lambda_k(\gamma) \longrightarrow \lambda_k(0)$ as $\gamma \to 0$

$\gamma \cdot \lambda_k(\gamma) \longrightarrow \lambda_k(\infty)$ as $\gamma \to \infty$.

Eigenfunctions converge (up to rescaling).

Proof:

- monotonicity in γ
- Min-max principle

EIGENVALUE PROBLEM IN A BALL:

Theorem 4.1. *Fix $\gamma \geq 0$. The positve eigenvalues of*

$$(4.3) \qquad \begin{cases} \Delta^2 u = \lambda^4 u & \text{in } \mathbb{B}_1, \\ \partial_\nu u = 0 & \text{on } \partial\mathbb{B}_1, \\ \partial_\nu(\Delta u) = -\gamma\lambda^4 u & \text{on } \partial\mathbb{B}_1, \end{cases}$$

are all and only the solutions of the equation

$$(4.4) \qquad 2j_\ell'(\lambda_\ell)i_\ell'(\lambda_\ell) + \gamma\lambda_\ell(j_\ell'(\lambda_\ell)i_\ell(\lambda_\ell) - i_\ell'(\lambda_\ell)j_\ell(\lambda_\ell)) = 0,$$

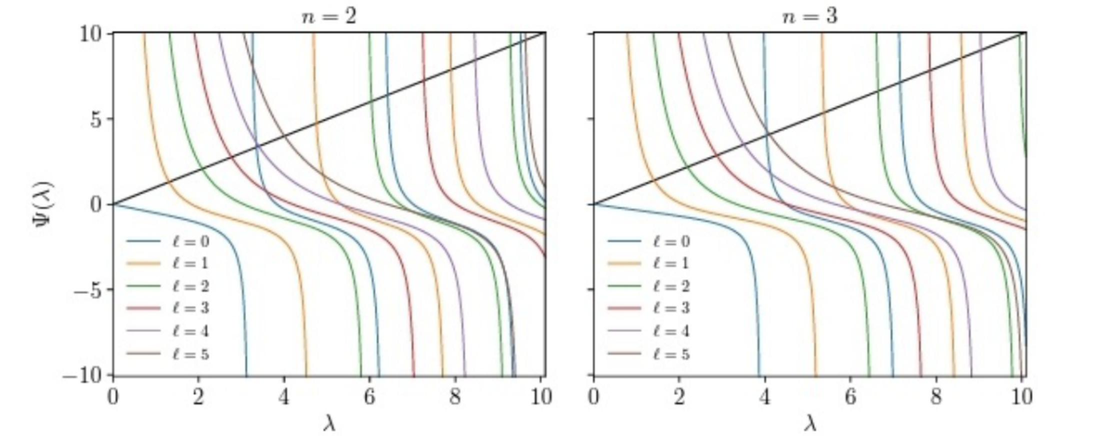
for some $\ell \in \mathbb{N}$. For any such solution, the associated eigenfunctions take the form

$$u_\ell(r)\mathcal{Y}_\ell(\theta)$$

for some spherical harmonic $\mathcal{Y}_\ell$ of order ℓ, where

$$(4.5) \qquad u_\ell(r) = i_\ell'(\lambda_\ell)j_\ell(\lambda_\ell r) - j_\ell'(\lambda_\ell)i_\ell(\lambda_\ell r).$$

The fundamental mode in the ball:



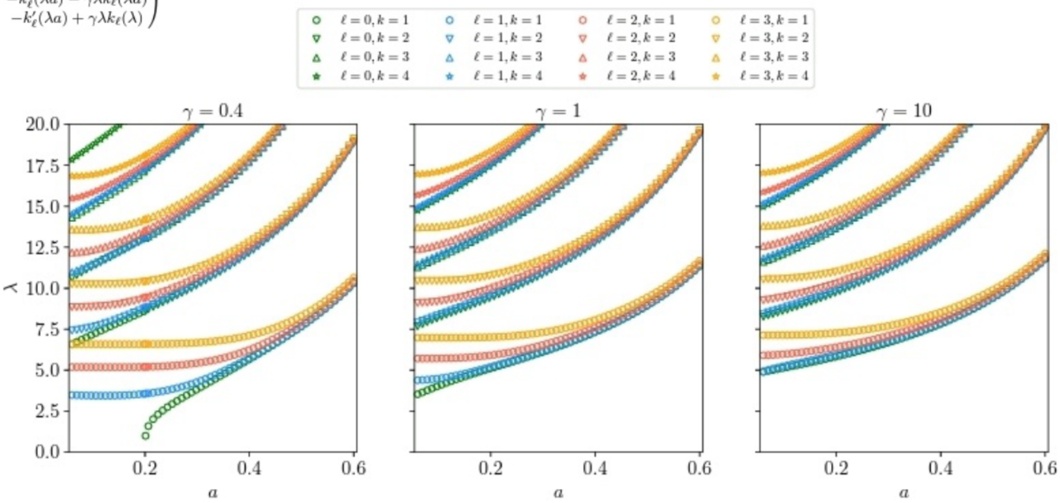
Theorem: The smallest non-zero eigenvalue corresponds to the mode $\ell = 1$.

Proof: Inspired by [Chasman] on bilaplacian eigenvalues.

The eigenvalue problem in the annulus: $\Omega = \{a < r < 1\}$

$$\begin{pmatrix} j'_\ell(\lambda a) & y'_\ell(\lambda a) & i'_\ell(\lambda a) & k'_\ell(\lambda a) \\ j'_\ell(\lambda) & y'_\ell(\lambda) & i'_\ell(\lambda) & k'_\ell(\lambda) \\ j'_\ell(\lambda a) - \gamma\lambda j_\ell(\lambda a) & y'_\ell(\lambda a) - \gamma\lambda y_\ell(\lambda a) & -i'_\ell(\lambda a) - \gamma\lambda i_\ell(\lambda a) & -k'_\ell(\lambda a) - \gamma\lambda k_\ell(\lambda a) \\ j'_\ell(\lambda) + \gamma\lambda j_\ell(\lambda) & y'_\ell(\lambda) + \gamma\lambda y_\ell(\lambda) & -i'_\ell(\lambda) + \gamma\lambda i_\ell(\lambda) & -k'_\ell(\lambda a) + \gamma\lambda k_\ell(\lambda) \end{pmatrix}$$

$n = 2$



Proposition 5.3. *Assume that $n \geq 2$ and $\gamma \in (0, \frac{1}{n})$. There exists $a_* \in (0,1)$ such that, or $a_* < a < 1$, the eigenfunction corresponding to the lowest non-zero eigenvalue is radially symmetric, while if $0 < a < a_*$, the eigenfunction is not radially symmetric but corresponds to the mode $\ell = 1$. For $n = 2$, $a_* = 1 - 2\gamma$.*

EIGENVALUE PROBLEMS IN CONFORMAL GEOMETRY

Flat case: $\Omega \subset \mathbb{R}^n$

Ω $\quad Lu = \Delta^2 u$

$\partial\Omega$ $\quad Nu = \partial_\nu u$

$\partial\Omega$ $\quad Bu = -\partial_\nu \Delta u$

Conformal version: (M^n, g)

$Lu = \Delta_g^2 u + \mathrm{div}_g(4A_g - 2J_g)d$

$Nu = \partial_\nu u$

$Bu = -\partial_\nu \Delta u + \ldots$

$$B_g^3 u = -\eta\Delta u - 2\tilde{\Delta}\eta u + 2\langle II_0, \tilde{D}^2 u\rangle - \frac{2}{3}H\tilde{\Delta}u + \frac{2}{3}\langle\tilde{\nabla}H, \tilde{\nabla}u\rangle + \Big(-\frac{1}{3}H^2 - 2A(\eta,\eta) + 2\tilde{J} + \frac{1}{2}|II_0|^2\Big)\eta u.$$

(Chang-Qing case, ...).

GEOMETRIC INTEREST:

- Conformal invariance: $\tilde{g} = e^{2f} g$
  $\tilde{L} = e^{-4f} L$
  $\tilde{B} = e^{-3f} B$

- Chern-Gauss-Bonnet formula in dim = 4:

$$8\pi^2\chi(M) = \int_M \underbrace{\Big(\frac{|W|_g^2}{4}}_{\text{conformal invariant}} + \underbrace{Q_g}_{L}\Big)\, dv_g + \int_\Sigma \Big(\underbrace{T_g}_{B} - \frac{2}{3}\underbrace{\operatorname{tr} II_0^3}_{\text{umbilic boundary}}\Big)\, d\sigma_h$$

- Generalize 2-dim geometry to 4-dim

CLASSIFICATION RESULTS:

M^4 lcf, umbilic boundary

- M simply connected, ∂M one component $\Longrightarrow$ $M \approx$ half-sphere

- $\chi(M) = 1$, $\mathcal{Y}(M) > 0 \Rightarrow M \approx$ half-sphere [Raulot]

- Theorem: M simply connected, ∂M two boundary components, $R > 0$, $Q > 0 \Rightarrow M \cong$ annulus $\{\underbrace{\rho}_{\text{conformal modulus}} \leq r \leq 1\}$

BACK TO THE EIGENVALUE PROBLEM:

$$\begin{cases} Lu = 0 & \text{in } M \\ Nu = 0 & \text{on } \partial M \\ Bu = \lambda u & \text{on } \partial M \end{cases}$$

Theorem 1.4. *We get the following bounds for $\lambda_1^g > 0$:*

a. *If M is conformally equivalent to a half-sphere $\mathbb{S}^4_+$,*

$$\lambda_1^g \operatorname{Vol}(\Sigma) \leq 24\pi^2.$$

and it is attained at a flat disk.

b. *If M is conformally equivalent to an annulus $\mathcal{A}_\rho$ (with boundaries Σ_1, Σ_ρ),*

$$\lambda_1^g \operatorname{Vol}(\Sigma) \leq c\Big(\rho, \frac{\operatorname{Vol}(\Sigma_\rho)}{\operatorname{Vol}(\Sigma_1)}\Big), \tag{1.24}$$

where this is a constant can be explicitly computed.

In addition, there is $\rho^* > 0$ such that for $\rho \leq \rho^*$ the bound is sharp.

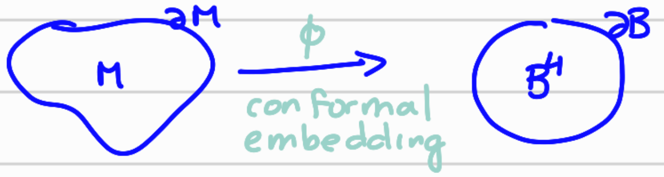
Proof:

- Variational characterization of eigenvalues

$$\lambda_1 = \min_{\substack{Lu=0 \\ Nu=0}} \frac{\int_M (\Delta u)^2 + \dots}{\int_{\partial M} u^2}$$

conformally invariant quantity

- Use conformal properties to transform to a model geometry



- Use as test functions: $U_i = (u_i^{(1)}\ Y_1) \circ \Phi$, $i = 0, 1, \dots, 4$

eigenfunctions for λ_1 on S^n

- Hersch trick (calibration).

- Lengthy calculations in model cases. $\big\langle$ ball / annulus

- Bifurcation in annulus!