Schur complement dominance

and damped wave equations

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- B. Gerhat [arXiv:2205.11653]
  Schur complement dominant operator matrices
- B. Gerhat and P. Siegl
  Schrödinger operators with accretive potentials in weighted spaces

Outline

Introduction

Operator matrices

Damped wave equations

$$\partial_t^2 u(x,t) + 2a(x)\partial_t u(x,t) = (\Delta_x - q(x))u(x,t), \quad x \in \Omega \subseteq \mathbb{R}^d, \quad t \geq 0$$

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transformation to first order (in time) problem

$$\partial_t \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ \Delta_x - q(x) & -2a(x) \end{pmatrix}}_{=\mathcal{A}} \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix}$$

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- dense domain, non-empty resolvent set
- structure and location of spectrum
- norm of resolvent

Lax-Milgram theorem

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- equivalence between spectra of Schur complement and matrix

Schur complement dominance

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- very "non-linear" approach, dominance of Schur complement encoded in spaces of test functions and distributions
- previous works on (abstract) Dirac operators
  [Esteban-Loss'07, Esteban-Loss'08, Schimmer-Solovej-Tokus'20]

Setting

- dense, continuously embedded triples of Hilbert spaces

$$\mathcal{D}_S \subseteq \mathcal{H}_1 \subseteq \mathcal{D}_{-S}, \quad \mathcal{D}_2 \subseteq \mathcal{H}_2 \subseteq \mathcal{D}_{-2}$$

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$$\operatorname{dom} \mathcal{A} := \widehat{\mathcal{A}}^{-1}(\mathcal{H}), \quad \operatorname{dom} S_\lambda := \widehat{S}_\lambda^{-1}(\mathcal{H}_1)$$

Spectral equivalence

Theorem [G'22]

If for all $\lambda \in \Theta \subseteq \rho(\widehat{D})$ there exists $z_\lambda \in \mathbb{C}$ such that

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- condition $(\star)$ established e.g. by form representation theorem
- generalises standard patterns like e.g. diagonal dominance

[Nagel'89, Tretter'08]

Damped wave equations

Non-negative distributional dampings

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta - q & -2\mathbf{a} \end{pmatrix}, \quad \mathcal{H} = \mathcal{W}(\Omega) \oplus L^2(\Omega), \quad q \in L^1_{\mathrm{loc}}(\Omega), \quad q \geq 0$$

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  → Dirac delta type [Krejčiřík-Kurimaiová'20, Krejčiřík-Royer'22] [Ammari-Nicaise'15]

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- (second) Schur complement

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- $\mathcal{D}_S$ closure of $C_0^\infty(\Omega)$ w.r.t.

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$$\|f\|_S^2 = \|\nabla f\|_{L^2}^2 + \|q^{\frac{1}{2}} f\|^2 + \mathbf{a}(f,f) + \|f\|_{L^2}^2$$

$$S_\lambda = -\frac{1}{\lambda}\left(-\Delta + q + 2\lambda \mathbf{a} + \lambda^2\right)$$

- remaining spaces are

$$\mathcal{D}_1 = \mathcal{D}_{-1} = \mathcal{H}_1 = \mathcal{W}(\Omega), \quad \mathcal{D}_{-S} = \mathcal{D}_S^*$$

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- domains of $\mathcal{A}$ and S_λ read

$$\operatorname{dom}\mathcal{A} = \{(f,g) \in \mathcal{W}(\Omega) \times \mathcal{D}_S \,:\, (\Delta - q)f - 2\mathbf{a}(g,\cdot) \in L^2(\Omega)\}$$

$$\operatorname{dom} S_\lambda = \{f \in \mathcal{D}_S \,:\, (-\Delta + q)f + 2\lambda\mathbf{a}(f,\cdot) \in L^2(\Omega)\}$$

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- previously implemented under more restrictive assumptions

$$\mathbf{a} = a \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega}), \quad |\nabla a| \leq \varepsilon a^{\frac{3}{2}} + C_\varepsilon (q^{\frac{1}{2}} + 1)$$

[Freitas-Siegl-Tretter'18]

Accretive differential dampings in weighted spaces

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta & -2(a - \nabla \cdot M\nabla) \end{pmatrix}, \quad \mathcal{H}_w = \mathcal{W}_w(\Omega) \oplus L^2_w(\Omega)$$

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- there exist $\varepsilon_0 \in (0, 2)$ and $C_0 \geq 0$ with

$$|M^{\frac{1}{2}} \nabla(w^2)| \leq \sqrt{2}\varepsilon_0 w^2 (\operatorname{Re} a + C_0)^{\frac{1}{2}}$$

Further applications

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- second order matrix differential operators

$$\mathcal{A} = \begin{pmatrix} -\Delta + q & \nabla \cdot \mathbf{b} \\ \mathbf{c} \cdot \nabla & d \end{pmatrix}$$

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  $\rightarrow$ generalise / recover previous results from symmetric setting

[Esteban-Loss'07, Esteban-Loss'08, Schimmer-Solovej-Tokus'20]

Thank you for your attention!

Y. Almog and B. Helffer.
On the spectrum of non-selfadjoint Schrödinger operators with compact resolvent.
Comm. Partial Differential Equations, 40:1441–1466, 2015.

K. Ammari and S. Nicaise.
Stabilization of elastic systems by collocated feedback, volume 2124 of *Lecture Notes in Mathematics*.
Springer, Cham, 2015.

J. Dolbeault, M. J. Esteban, M. Loss, and L. Vega.
An analytical proof of Hardy-like inequalities related to the Dirac operator.
Journal of Functional Analysis, 216:1–21, 2004.

J. Dolbeault, M. J. Esteban, and E. Séré.
On the eigenvalues of operators with gaps. Application to Dirac operators.
Journal of Functional Analysis, 174:208–226, 2000.

M. J. Esteban and M. Loss.
Self-adjointness for Dirac operators via Hardy-Dirac inequalities.
Journal of Mathematical Physics, 48:112107, 8, 2007.

M. J. Esteban and M. Loss.
Self-adjointness via partial Hardy-like inequalities.
In *Mathematical results in quantum mechanics*, pages 41–47. World Sci. Publ., Hackensack, NJ, 2008.

P. Freitas, P. Siegl, and C. Tretter.
The damped wave equation with unbounded damping.
Journal of Differential Equations, 264:7023–7054, 2018.

O. O. Ibrogimov.
Essential spectrum of non-self-adjoint singular matrix differential operators.
Journal of Mathematical Analysis and Applications, 451:473–496, 2017.

O. O. Ibrogimov, P. Siegl, and C. Tretter.
Analysis of the essential spectrum of singular matrix differential operators.
Journal of Differential Equations, 260:3881–3926, 2016.

O. O. Ibrogimov and C. Tretter.
Essential spectrum of elliptic systems of pseudo-differential operators on $L^2(\mathbb{R}^N) \oplus L^2(\mathbb{R}^N)$.
Journal of Pseudo-Differential Operators and Applications, 8:147–166, 2017.

D. Krejčiřík and T. Kurimaiová.
From lieb–thirring inequalities to spectral enclosures for the damped wave equation.
Integral Equations and Operator Theory, 92(6), oct 2020.

R. Nagel.
Towards a "matrix theory" for unbounded operator matrices.
Mathematische Zeitschrift, 201:57–68, 1989.

L. Schimmer, J. P. Solovej, and S. Tokus.
Friedrichs extension and min-max principle for operators with a gap.
Ann. Henri Poincaré, 21(2):327–357, 2020.

Christiane Tretter.
Spectral theory of block operator matrices and applications.
Imperial College Press, London, 2008.