# An eigenvalue problem of Steklov type as $p \to +\infty$

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Unterstützt von / Supported by

Alexander von Humboldt
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AND NUMERICS
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# First non-zero Steklov eigenvalue

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded, connected, open set with Lipschitz boundary.

The first non-zero Steklov eigenvalue of $\Omega$ is defined by

$$\sigma(\Omega) := \min \left\{ \frac{\displaystyle\int_\Omega |\nabla v|^2 dx}{\displaystyle\int_{\partial\Omega} v^2 d\mathcal{H}^{n-1}} : v \in H^1(\Omega) \setminus \{0\}, \int_{\partial\Omega} v \, d\mathcal{H}^{n-1} = 0 \right\}.$$

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Any minimizer satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \sigma(\Omega) u & \text{on } \partial\Omega. \end{cases}$$

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The sequence of Steklov eigenvalues

$$0 = \sigma_1(\Omega) < \sigma_2(\Omega)(= \sigma(\Omega)) \leq \sigma_3(\Omega) \leq \sigma_3(\Omega) \cdots \nearrow +\infty$$

starts with zero.

# First non-zero Steklov eigenvalue

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The first non-trivial Steklov eigenvalue $\sigma(\Omega)$ coincides with the value of the best constant in the Poincaré-Wirtinger trace inequality:

$$c_\Omega \int_{\partial\Omega} |u - \bar{u}_\Omega|^2 \, d\mathcal{H}^{n-1} \leq \int_\Omega |\nabla u|^2 \, dx,$$

for $u \in W^{1,2}(\Omega)$, where $\bar{u}_\Omega$ is the average of the trace of $u$ on $\partial\Omega$.

# Motivation

- The Steklov eigenvalues can be interpreted as the eigenvalues of the Dirichlet-to-Neumann operator

$$D : H^{1/2}(\partial\Omega) \to: H^{-1/2}(\partial\Omega)$$

that maps a function $f \in H^{1/2}(\partial\Omega)$ to $Df = \frac{\partial Hf}{\partial \nu}$, where $Hf$ is the harmonic extension of $f$ to $\Omega$.
Application to: electrical impendance tomography (used in medical and geophysical imaging) and in the analysis of photonic crystals.

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- The Steklov boundary condition is often considered in the more general form

$$\frac{\partial u}{\partial \nu} = \sigma \rho u,$$

where $\rho \in L^\infty(\partial\Omega)$. If $\Omega \subset \mathbb{R}^2$, the Steklov eigenvalues can be thought of as the squares of the natural frequencies of a vibrating free membrane with its mass concentrated along its boundary with density $\rho$.

# Shape Optimization problem in dimension 2

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**Theorem [Weinstock, *J. Rational Mech. Anal.*, 1954]**

If $\Omega \subseteq \mathbb{R}^2$ is a bounded, Lipschitz simply connected open set, then

$$\sigma(\Omega)P(\Omega) \leq \sigma(B)P(B), \tag{1}$$

where $P(\Omega)$ stands for the perimeter of $\Omega$ and $B \subseteq \mathbb{R}^2$ is a ball. Equality holds if and only if $\Omega$ is a ball.

In other words: "among all simply connected sets of $\mathbb{R}^2$ with prescribed perimeter, the disc maximises the first non-zero Steklov eigenvalue".

**Remark (Girouard-Polterovich, *J. Spectral Theory*, 2017)**

Weinstock inequality fails for planar domains which are not simply connected. Namely, one can find an annulus $\Omega_\varepsilon = B_1 \setminus \overline{B}_\varepsilon$, $\varepsilon \approx 0$, such that

$$\sigma(\Omega_\varepsilon)P(\Omega_\varepsilon) > \sigma(B)P(B).$$

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What about the *n*-dimensional case, $n \geq 3$?

# Shape Optimization problem in $\mathbb{R}^n$ with volume constraint

**Theorem [Brock, *ZAMM*, 2001]**

For every Lipschitz bounded open set $\Omega \subseteq \mathbb{R}^n$, it holds true

$$\sigma(\Omega)V(\Omega)^{\frac{1}{n}} \leq \sigma(B)V(B)^{\frac{1}{n}}.$$

The equality holds iff $\Omega$ is a ball.

In other words: "Among all Lipschitz sets of $\mathbb{R}^n$ with prescribed volume, balls maximise the first non-zero Steklov eigenvalue".

# Shape Optimization problem in $\mathbb{R}^n$ with perimeter constraint

**Theorem [Bucur-Ferone-Nitsch-Trombetti, *J. Differential Geom.*, 2018]**

Let $\Omega$ be a bounded, open and convex set of $\mathbb{R}^n$. Then

$$\sigma(\Omega)P(\Omega)^{\frac{1}{n-1}} \leq \sigma(B)P(B)^{\frac{1}{n-1}}.$$

Equality holds only if $\Omega$ is a ball.

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The above inequality cannot hold for simply connected sets in $\mathbb{R}^n$ . Namely, one can find a spherical shell $\Omega_\varepsilon = B_1 \setminus \overline{B}_\varepsilon$, $\varepsilon \approx 0$, ($B_r$ denotes the ball of radius $r$ centered at the origin) such that

$$\sigma(\Omega_\varepsilon)P(\Omega_\varepsilon)^{\frac{1}{n-1}} > \sigma(B)P(B)^{\frac{1}{n-1}}.$$

# The orthotropic $p$-Laplace operator

Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded and convex set and let $p > 1$. We consider the orthotropic $p-$Laplace operator:

$$\widetilde{\Delta}_p u = \sum_{i=1}^{n} \left( |u_{x_i}|^{p-2} u_{x_i} \right)_{x_i} .$$

- For $p = 2$ it coincides with the Laplacian
- For $p \neq 2$ it differs from the usual $p-$Laplacian, that is defined as $\Delta_p u := \operatorname{div}\left( |\nabla u|^{p-2} \nabla u \right)$.

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This operator has been considered by several authors:

- Visik, 1963
- Lions, 1969
- Belloni-Kawohl, 2004
- Rossi-Saez, 2007
- Brasco-Franzina, 2013

# Orthotropic $p$-Laplacian

Case of the $p$-Laplace operator: Suppose we are considering a non linear elastic membrane, fixed on a boundary $\partial\Omega$ of a plane domain. If $u(x)$ denotes its vertical displacements, its deformation energy is given by $\int_\Omega |\nabla u|^p$ and a minimizer of the Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}$$

on $W_0^{1.p}(\Omega)$ satisfies the Euler-Lagrange equation in $\Omega$:

$$-\Delta_p u = \lambda_p |u|^{p-2} u$$

# Orthotropic $p$-Laplacian

Case of the orthotropic $p$-Laplace operator: If the membrane is woven out of elastic strings in a rectangular fashion, then its deformation energy is given by

$$\int_\Omega \sum_i |u_{x_i}|^p dx.$$

A minimizer of the Rayleigh quotient

$$\frac{\int_\Omega \sum_i |u_{x_i}|^p dx}{\int_\Omega |u|^p dx}$$

on $W_0^{1,p}$, if it exists, will satisfy

$$\widetilde{\Delta}_p u = \tilde{\lambda}_p |u|^{p-2} u.$$

# Steklov problem for the orthotropic $p-$Laplacian

Let us consider the Steklov-eigenvalue problem for the pseudo $p-$Laplace operator:

$$\begin{cases} -\widetilde{\Delta}_p u = 0 & \text{on } \Omega \\ \sum_{j=i}^{n} |u_{x_j}|^{p-2} u_{x_j} \nu^j_{\partial\Omega} = \sigma |u|^{p-2} u \rho_p & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where

- $\nu_{\partial\Omega} = (\nu^1_{\partial\Omega}, \dots, \nu^n_{\partial\Omega})$ is the outer normal of $\partial\Omega$
- $\rho_p(x) = \|\nu_{\partial\Omega}(x)\|_{\ell^{p'}}$;
- $p'$ is the coniugate exponent of $p$;
- $\|x\|^p_{\ell^p} = \sum_{j=1}^{n} |x^j|^p$.

The real number $\sigma$ is called Steklov eigenvalue whenever problem (2) admits a non-trivial weak $W^{1,p}$ solution.

# Steklov problem for the orthotropic $p-$Laplacian

We can write

$$-\widetilde{\Delta}_p u = \operatorname{div}\left(\mathcal{A}_p(\nabla u)\right), \qquad \mathcal{A}_p(\nabla u) = \left(|u_{x_1}|^{p-2}u_{x_1}, \dots, |u_{x_n}|^{p-2}u_{x_n}\right).$$

Let $u \in W^{1,p}(\Omega)$. We say that $u$ is a weak solution of (2) if

$$\int_\Omega \langle \mathcal{A}_p(\nabla u), \nabla\varphi\rangle dx = \sigma \int_{\partial\Omega} |u|^{p-2}u\varphi\rho_p d\mathcal{H}^{n-1} \qquad \forall\varphi \in W^{1,p}(\Omega).$$

# Some results

The following results are proved in [L. Brasco - G. Franzina, *NoDEA*, 2013].

- These Steklov eigenvalues form at least a countably infinite sequence of positive numbers diverging at infinity.
- The first eigenvalue is 0 and corresponds to constant eigenfunctions.
- Denoting by $\Sigma_p^p(\Omega)$ the first non-trivial, it has the following variational characterization

$$\Sigma_p^p(\Omega) = \min \left\{ \frac{\int_\Omega \|\nabla u\|_{\ell^p}^p \, dx}{\int_{\partial\Omega} |u|^p \rho_p d\mathcal{H}^{n-1}}, \ u \in W^{1,p}(\Omega), \ \int_{\partial\Omega} |u|^{p-2} u \rho_p d\mathcal{H}^{n-1} = 0 \right\}.$$

# Some results

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- These Steklov eigenvalues form at least a countably infinite sequence of positive numbers diverging at infinity.
- The first eigenvalue is 0 and corresponds to constant eigenfunctions.
- $\Sigma_p^p(\Omega)$ represents the optimal constant in the weighted trace-type inequality

$$\int_\Omega \|\nabla u\|_{\ell^p}^p \, dx \geq \Sigma_p^p(\Omega) \int_{\partial\Omega} |u|^p \rho_p d\mathcal{H}^{n-1}$$

in the class of Sobolev functions $u \in W^{1,p}(\Omega)$, such that

$$\int_{\partial\Omega} |u|^{p-2} u \rho_p d\mathcal{H}^{n-1} = 0.$$

# The orthotropic $\infty-$Laplace operator

## Aim

We want to study the first non-trivial Steklov eigenvalue for the orthotropic $p-$Laplacian, as $p \to +\infty$:

$$\lim_{p \to +\infty} \Sigma_p(\Omega).$$

We define the following operator

$$\widetilde{\Delta}_\infty u(x) := \sum_{j \in I(\nabla u(x))} u^2_{x_j}(x) u_{x_j,x_j}(x),$$

where

$$I(x) := \{j \leq n : \ |x_j| = \|x\|_{\ell^\infty}\}, \quad \|x\|_{\ell^\infty} = \max_{j=1,\dots,n} |x^j|.$$

# Some heuristics: the limit problem as $p \to +\infty$

If we assume that $u \in C^2$, we can write

$$\widetilde{\Delta}_p u = (p-1) \sum_{j=1}^{n} |u_{x_j}|^{p-4} u_{x_j}^2 u_{x_j,x_j}.$$

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Dividing by $(p-1)\,\|\nabla u\|_{\ell^\infty}^{p-4}$, we achieve

$$\frac{\widetilde{\Delta}_p u}{(p-1)\,\|\nabla u\|_{\ell^\infty}^{p-4}} = \sum_{j=1}^{n} \left| \frac{u_{x_j}}{\|\nabla u\|_{\ell^\infty}} \right|^{p-4} u_{x_j}^2 u_{x_j,x_j}. \tag{3}$$

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Let us define

$$I(x) := \{j \le n : \ |x_j| = \|x\|_{\ell^\infty}\}, \quad \|x\|_{\ell^\infty} = \max_{j=1,\dots,n} |x^j|.$$

we can rewrite (3) as

$$\frac{\widetilde{\Delta}_p u}{(p-1)\, \|\nabla u\|_{\ell^\infty}^{p-4}} = \sum_{j \in I(\nabla u(x))} u_{x_j}^2 u_{x_j,x_j} + \sum_{j \notin I(\nabla u(x))} \left| \frac{u_{x_j}}{\|\nabla u\|_{\ell^\infty}} \right|^{p-4} u_{x_j}^2 u_{x_j,x_j}.$$

# Some heuristics

Starting from

$$\frac{\widetilde{\Delta}_p u}{(p-1)\,\|\nabla u\|_{\ell^\infty}^{p-4}} = \sum_{j\in I(\nabla u(x))} u_{x_j}^2 u_{x_j,x_j} + \sum_{j\notin I(\nabla u(x))} \left|\frac{u_{x_j}}{\|\nabla u\|_{\ell^\infty}}\right|^{p-4} u_{x_j}^2 u_{x_j,x_j},$$

we take the limit as $p \to +\infty$ and, since for any $j \notin I(\nabla u(x))$ we have $\left|\frac{u_{x_j}}{\|\nabla u\|_{\ell^\infty}}\right| < 1$, we obtain

$$\widetilde{\Delta}_\infty u = \lim_{p\to+\infty} \frac{\widetilde{\Delta}_p u}{(p-1)\,\|\nabla u\|_{\ell^\infty}^{p-4}} = \sum_{j\in I(\nabla u(x))} u_{x_j}^2 u_{x_j,x_j}.$$

# Some new results

## Theorem [Ascione-P., *JMAA*, 2021]

It holds

$$\lim_{p\to+\infty} \Sigma_p(\Omega) = \frac{2}{\mathrm{diam}_1(\Omega)} =: \Sigma_\infty(\Omega),$$

where $\mathrm{diam}_1(E) := \sup_{x,y\in E} ||x-y||_{\ell^1}$. Moreover, the following variational characterization holds:

$$\Sigma_\infty(\Omega) = \min\left\{ \frac{\| \| \nabla u \|_{\ell^\infty} \|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\partial\Omega)}},\ u \in W^{1,\infty}(\Omega),\ \max_{x\in\partial\Omega} u(x) = -\min_{x\in\partial\Omega} u(x) \neq 0 \right\}.$$

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$$\Sigma_\infty(\Omega)=\min\left\{\frac{\|\|\nabla u\|_{\ell^\infty}\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\partial\Omega)}},\ u\in W^{1,\infty}(\Omega),\ \max_{x\in\partial\Omega}u(x)=-\min_{x\in\partial\Omega}u(x)\neq 0\right\}.$$

Notation:

- We denote by $u_p$ an eigenfunction with related eigenvalue $\Sigma_p^p(\Omega)$, with the following normality condition

$$\frac{1}{V(\Omega)}\int_{\partial\Omega}|u_p|^p\rho_p d\mathcal{H}^{n-1}=1,$$

  where $V(\cdot)$ is the volume.

# Some new results

## Theorem [Ascione-P., *JMAA*, 2021]

Let $\Omega$ be a bounded open convex set such that $\partial\Omega$ is $C^1$. There exists a sequence $p_i \to +\infty$ such that $u_{p_i}$ converges uniformly in $\overline{\Omega}$ to $u_\infty$, that is solution of

$$\begin{cases} -\widetilde{\Delta}_\infty u = 0 & \text{on } \Omega \\ \Lambda(x, u, \nabla u) = 0 & \text{on } \partial\Omega \end{cases}$$

in the viscosity sense, where, for $(x, u, \eta) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^N$, we set

$$\Lambda(x, u, \eta) = \begin{cases} \min\left\{ \|\eta\|_{\ell^\infty} - \Sigma_\infty(\Omega)|u| \,,\, \sum_{j\in I(\eta)} \eta_j \nu^j_{\partial\Omega}(x) \right\} & \text{if } u > 0 \\ \max\left\{ \Sigma_\infty(\Omega)|u| - \|\eta\|_{\ell^\infty} \,, \sum_{j\in I(\eta)} \eta_j \nu^j_{\partial\Omega}(x) \right\} & \text{if } u < 0 \\ \sum_{j\in I(\eta)} \eta_j \nu^j_{\partial\Omega}(x) & \text{if } u = 0. \end{cases}$$

# Shape Optimization results

Let $p \in (1, \infty]$. Let us recall that

$$\|x\|_{\ell^p}^p = \sum_{j=1}^{n} |x^j|^p, \quad \|x\|_{\ell^\infty} = \max_{j=1,\dots,n} |x^j|$$

and

$$\rho_p(x) = \|\nu_{\partial\Omega}(x)\|_{\ell^{p'}} \, .$$

We introduce the following notation:

- the ball relative to the $\ell^p$ norm:

$$\mathcal{W}_p = \{x \in \mathbb{R}^n \mid \|x\|_{\ell^p} \leq 1\};$$

- the perimeter relative to the $\ell^p$ norm:

$$\mathcal{P}_p(\Omega) := \int_{\partial\Omega} \rho_p(x) d\mathcal{H}^{n-1}(x).$$

# Shape Optimization Results

**Theorem [Ascione-P., *JMAA*, 2021]**

Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex set and $p > 1$. Consider $q \geq 0$ and $r \in [0, n]$ such that $\frac{p}{n} = q + \frac{r}{n}$. Then, we have

$$\Sigma_p^p(\Omega)\,\mathcal{P}_p(\Omega)^{\frac{r-1}{n-1}}\,V(\Omega)^q \leq \mathcal{P}_p(\mathcal{W}_p)^{\frac{r-1}{n-1}}\,V(\mathcal{W}_p)^q.$$

Remarks and Corollaries:

- This result implies the Brock-Weinstock inequality proved in [Brasco-Franzina, 2013]:
$$\Sigma_p^p(\Omega)V(\Omega)^{\frac{p-1}{n}} \leq V(\mathcal{W}_p)^{\frac{p-1}{n}}.$$
- If $p \in (1, n]$, we can choose $r = p$ and $q = 0$, obtaining
$$\Sigma_p^p(\Omega)\,\mathcal{P}_p(\Omega)^{\frac{p-1}{n-1}} \leq \mathcal{P}_p(\mathcal{W}_p)^{\frac{p-1}{n-1}}.$$

# Shape Optimization Results

## Theorem [Ascione-P., *JMAA*, 2021]

For any bounded open convex set $\Omega \subseteq \mathbb{R}^n$, it holds

$$\Sigma_\infty(\Omega)V(\Omega)^{\frac{1}{n}} \leq \Sigma_\infty(\mathcal{W}_1)V(\mathcal{W}_1)^{\frac{1}{n}}.$$

Equality holds if and only if $\Omega$ is equivalent to $\mathcal{W}_1$ up to translations and scalings Moreover, if $n = 2$, it also holds

$$\Sigma_\infty(\Omega)P_\infty(\Omega) \leq \Sigma_\infty(\mathcal{W}_1)P(\mathcal{W}_1).$$

Equality holds if and only if $\Omega$ is of constant width, i.e. if and only if $\omega(v) \equiv \mathrm{diam}_1(\Omega)$ for every directions $v$.

Fix $n = 2$ and consider any bounded open convex set $\Omega$. For each direction $v$ there exists two supporting lines $r_1, r_2$ for $\Omega$ that are orthogonal to $v$ in the Euclidean sense. We call width of $\Omega$ in the direction $v$ the distance $\omega(v) = d_1(r_1, r_2)$.

Thank you for your attention!

# Orthotropic $p$-Laplacian

Case of the $p$-Laplace operator: Suppose we are considering a non linear elastic membrane, fixed on a boundary $\partial\Omega$ of a plane domain. If $u(x)$ denotes its vertical displacements, its deformation energy is given by $\int_\Omega |\nabla u|^p$ and a minimizer of the Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}$$

on $W_0^{1.p}(\Omega)$ satisfies the Euler-Lagrange equation in $\Omega$:

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# Viscosity Solution

We denote

$$F_p : (\xi, X) \in \mathbb{R}^N \times \mathbb{R}^{N\times N} \mapsto -\sum_{j=1}^{N} (p-1)|\xi_j|^{p-2} X_{j,j}$$

and

$$B_p : (\sigma, x, u, \xi) \in \mathbb{R} \times \partial\Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \sum_{j=1}^{N} |\xi_j|^{p-2}\xi_j \nu^j_{\partial\Omega}(x) - \sigma|u|^{p-2}u\rho_p(x).$$

The Steklov problem can be *formally* rewritten as

$$\begin{cases} F_p(\nabla u, \nabla^2 u) = 0, & \text{on } \Omega \\ B_p(\sigma, x, u, \nabla u) = 0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Let $u$ be a lower (upper) semi-continuous function on the closure $\overline{\Omega}$ of $\Omega$ and $\Phi \in C^2(\overline{\Omega})$. We say that $\Phi$ is **touching from below** (**above**) $u$ in $x_0 \in \overline{\Omega}$ if and only if $u(x_0) - \Phi(x_0) = 0$ and $u(x) > \Phi(x)$ ($u(x) < \Phi(x)$) for any $x \neq x_0$ in $\overline{\Omega}$.
A lower (upper) semi-continuous function $u$ on $\overline{\Omega}$ is said to be a **viscosity supersolution** (**subsolution**) of (4) if for any function $\Phi \in C^2(\overline{\Omega})$ touching from below (above) $u$ in $x_0 \in \overline{\Omega}$ one has

$$F_p(\nabla\Phi(x_0), \nabla^2\Phi(x_0)) \geq (\leq) 0 \qquad x_0 \in \Omega;$$
$$\max\{F_p(\nabla\Phi(x_0), \nabla^2\Phi(x_0)), B_p(\sigma, x_0, \Phi(x_0), \nabla\Phi(x_0))\} \geq (\leq) 0 \qquad x_0 \in \partial\Omega.$$

Finally, we say that a continuous function $u$ on $\overline{\Omega}$ is a **viscosity solution** if it is both viscosity subsolution and supersolution.