Damped wave equation with unbounded damping

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Based on

[1] P. Freitas, P. Siegl, and C. Tretter [2018]. "Damped wave equation with unbounded damping". In: *J. Differential Equations* 264, pp. 7023–7054

[2] A. Arifoski and P. Siegl [2020]. "Pseudospectra of damped wave equation with unbounded damping". In: *SIAM J. Math. Anal.* 52, 1343–1362

[3] P. Freitas, N. Hefti, and P. Siegl [2020]. "Damped wave equation with singular damping". In: *Proc. Amer. Math. Soc.* 148, pp. 4273–4284

[4] A. Arnal [2022]. *Resolvent estimates for the one-dimensional damped wave equation with unbounded damping.* arXiv:2206.08820v1 [math.SP]

Damped wave equation in $\Omega \subset \mathbb{R}^d$

$$u_{tt}(t,x) + 2a(x)\, u_t(t,x) = (\Delta - q(x))u(t,x)$$

- $t \in \mathbb{R}_+$ time, $x \in \Omega$ spatial coordinate
- $a : \Omega \to \mathbb{R}_+$ damping, $q : \Omega \to \mathbb{R}_+$ potential
- Dirichlet boundary conditions at $\partial\Omega$ (if $\partial\Omega \neq \emptyset$)

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- $a : \Omega \to \mathbb{R}_+$ damping, $q : \Omega \to \mathbb{R}_+$ potential
- Dirichlet boundary conditions at $\partial\Omega$ (if $\partial\Omega \neq \emptyset$)
- very extensive literature on $a \in L^\infty(\Omega)$ or “small” w.r.t to Δ

  e.g. references in [Gesztesy and Holden, 2011]
- our goal: a can be unbounded at infinity

  [Freitas, Siegl, and Tretter, 2018; Ikehata and Takeda, 2020; Sobajima and Wakasugi, 2018]
- for example: $\Omega = \mathbb{R}$ and $a(x) = x^2$, $q(x) = 0$

Focus of this talk

- spectral and pseudospectral effects caused by unbounded damping
- more on operator theory & functional analysis: talk of B. Gerhat

Operator matrix

$$u_{tt}(t,x) + 2a(x)u_t(t,x) = (\Delta - q(x))u(t,x)$$

- can be rewritten as a system

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I \\ \Delta - q & -2a \end{pmatrix}}_{G} \begin{pmatrix} u \\ v \end{pmatrix}$$

- G acts in a Hilbert space $\mathcal{H} = \mathcal{W}(\Omega) \oplus L^2(\Omega)$ where

$$\mathcal{W}(\Omega) = \overline{C_0^\infty(\Omega)}^{(\|\nabla \cdot\|^2 + \|q^{\frac{1}{2}} \cdot\|^2)^{\frac{1}{2}}}$$

- $a = 0 \implies \mathrm{i}G$ is symmetric w.r.t. $\langle \nabla \cdot, \nabla \cdot \rangle_1 + \langle q^{\frac{1}{2}} \cdot, q^{\frac{1}{2}} \rangle_1 + \langle \cdot, \cdot \rangle_2$

Associated quadratic function

$$T(\lambda) = -\Delta + q + 2\lambda a + \lambda^2$$
$$\mathrm{Dom}(T(\lambda)) = \{f \in \mathcal{D}(\Omega) \,:\, T(\lambda)f \in L^2(\Omega)\}$$
$$\mathcal{D}(\Omega) = W_0^{1,2}(\Omega) \cap \mathrm{Dom}(a^{\frac{1}{2}}) \cap \mathrm{Dom}(q^{\frac{1}{2}})$$

- arises for special solutions $e^{\lambda t}u(x)$ (or Schur complement)

Theorem (semigroup, relation of spectra) [Freitas, Siegl, and Tretter, 2018; Gerhat, 2022]

Let $0 \leq a, q \in L^1_{\text{loc}}(\Omega)$, let T be as above and let

$$G = \begin{pmatrix} 0 & I \\ \Delta - q & -2a \end{pmatrix}$$

$$\text{Dom}(G) = \{(f,g) \in \mathcal{W}(\Omega) \times \mathcal{D}(\Omega) \,:\, (\Delta - q)f - 2ag \in L^2(\Omega)\}$$

Then

- $-G$ is m-accretive $\rightsquigarrow$ G generates a contraction semigroup in $\mathcal{W}(\Omega) \times L^2(\Omega)$

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- $-G$ is m-accretive $\rightsquigarrow$ G generates a contraction semigroup in $\mathcal{W}(\Omega) \times L^2(\Omega)$
- $\forall \lambda \in \mathbb{C} \setminus (-\infty, 0]$: $\lambda \in \sigma_\iota(G) \Longleftrightarrow 0 \in \sigma_\iota(T(\lambda))$ $\iota \in \{\cdot, \mathrm{p}, \mathrm{ess}\}$

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- the spectral correspondence holds also in $(-1/M, 0)$ if

$$\forall f \in \mathcal{D}(\Omega) \qquad 2\|a^{\frac{1}{2}} f\|^2 \leq M(\|\nabla f\|^2 + \|q^{\frac{1}{2}} f\|^2) + C\|f\|^2$$

[mostly NOT in this talk]

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- Ω bounded or $\lim_{|x| \to \infty} a(x) = \infty$ $\rightsquigarrow$

  $\sigma(G) \setminus (-\infty, 0]$ = isolated eigenvalues of finite multiplicity

  (possible accumulations only to $(-\infty, 0]$)

$$T(\lambda) = -\Delta + q + 2\lambda a_0 + \lambda^2, \quad a_0 \in \mathbb{R}_+$$

Finite interval $\Omega = (-1, 1)$ and $q = 0$

- bounded Ω and bounded $a \rightsquigarrow$ only EV's in $\mathbb{C}$

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$$-u'' = (-\lambda^2 - 2\lambda a_0)u \quad \Longrightarrow \quad -\lambda_k^2 - 2\lambda_k a_0 = \mu_k, \quad \mu_k = \left(\frac{k\pi}{2}\right)^2$$

$$\lambda_k = -a_0 \pm \mathrm{i}\sqrt{\mu_k - a_0^2}, \quad k \in \mathbb{N}$$

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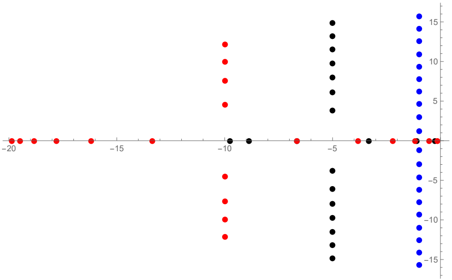
- solutions of the time-dependent equation: $e^{\lambda_k t} u_k(x)$



Finite interval $\Omega = (-1, 1)$, $a \in L^\infty(\Omega)$

- asymptotic behavior of eigenvalues ($a \in BV(-1, 1)$)

[Cox and Zuazua, 1994; Freitas and Zuazua, 1996]

$$\lambda_k = \pm \mathrm{i}\frac{k\pi}{2} - \frac{\int a(x)\mathrm{d}x}{2} + \mathcal{O}(k^{-1}), \quad k \to \infty$$

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- eigenfunctions form a Riesz basis

[Cox and Zuazua, 1994; Djakov and Mityagin, 2010; Lunyov and Malamud, 2016]

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Reminder: Gearhart-Prüss theorem

Let A be a densely defined closed operator in a Hilbert space $\mathcal{H}$ such that A generates a C_0 semigroup and let $\omega \in \mathbb{R}$. If

$$\sup_{\operatorname{Re} z \geq \omega} \|(A - z)^{-1}\| < \infty,$$

then $\exists M > 0$ such that

$$\|e^{tA}\| \leq M e^{\omega t}, \quad t \geq 0.$$

Theorem [Castro and Cox, 2001]

Let $\Omega = (0,1)$. Every solution of

$$u_{tt} + \frac{2}{x} u_t = u_{xx}$$

(Dirichlet BC, initial condition $(u_0, u_1) \in W_0^{1,2}(0,1) \times L^2(0,1)$) vanishes in a finite time

$$u(t,\cdot) = 0, \quad t > 2.$$

Proposition [Freitas, Hefti, and Siegl, 2020]

Let $\alpha > 0$ and $a(x) = \alpha/x$. Then

- if $\alpha = n+1$, $n \in \mathbb{N}_0$, then

$$\sigma(G) = \{\mu_k^{(n)}\}_{k=1}^{n} \subset (-\infty, 0), \quad \text{determined by} \quad L_n^{(1)}(-2\mu) = 0$$

- $\alpha = 0 + 1 \rightsquigarrow \sigma(G) = \emptyset$
- if $\alpha \notin \mathbb{N}$, then $\sigma(G)$ contains exactly $\lceil \alpha - 1 \rceil$ negative eigenvalues and infinitely many complex conjugated eigenvalues $\{\lambda_k^{(\alpha)}\}$ satisfying

$$\lambda_k^{(\alpha)} = \mp \frac{2k+1-\alpha}{2} \pi \mathrm{i} - \frac{1}{2} \log \left(- \frac{\Gamma(1-\alpha)}{\Gamma(2+\alpha)} (\pm 2k\pi \mathrm{i})^{2\alpha} \right) + \mathcal{O}(k^{-1} \log k), \quad k \to \infty$$

1D examples on $\Omega = \mathbb{R}$

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- eigenvalues $\lambda_k = 2^{\frac{1}{2n+1}} e^{\pm \mathrm{i}\pi \frac{n+1}{2n+1}} \mu_{k;n}^{\frac{n+1}{2n+1}} - \frac{2(n+1)}{2n+1} a_0 + o_k(1), \quad k \to \infty$
- $\{\mu_{k;n}\}_k$ are eigenvalues of the self-adjoint $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^{2n}$ in $L^2(\mathbb{R})$
- for $n = 1$: $\mu_{k;1} = 2k + 1$, $k = 0, 1, 2, \ldots$

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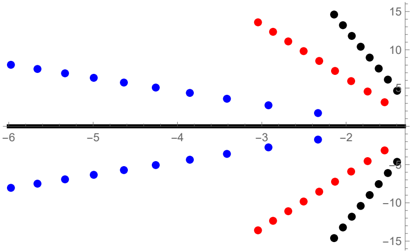

Infinite 2D strip

- $\Omega = \mathbb{R} \times (-1, 1)$ and $a(x, y) = x^2 + a_0$
- separation of variables: algebraic equation for eigenvalues λ

$$2\lambda\mu_k = (\lambda^2 + \sigma_j^2 + 2\lambda a_0)^2$$

$$\sigma_j^2 = \left(\frac{j\pi}{2l}\right)^2, \qquad \mu_k = 2k + 1,\ j, k \in \mathbb{N}$$

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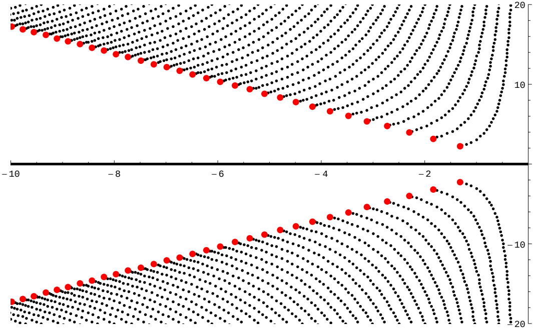

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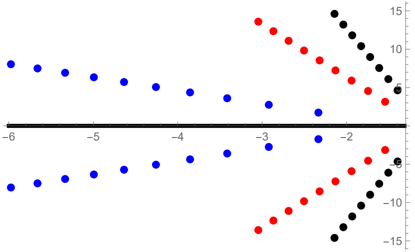

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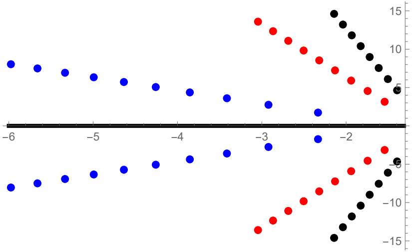

- it seems that EV's converge to those of $a_\infty(x) = a_0$ on $(-1, 1)$
- similar effect as for Schrödinger operators (convergence to the “square-well”)

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^{2n} \quad \text{in} \quad L^2(\mathbb{R}) \xrightarrow{n\to\infty} -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \quad \text{with Dirichlet BC in} \quad L^2(-1,1)$$

Theorem [Freitas, Siegl, and Tretter, 2018]

Assume we have $\Omega_\infty \subset \Omega$ open and sufficiently regular, $a_\infty \in L^1_{\text{loc}}(\Omega_\infty)$ and $\{a_n\} \subset L^1_{\text{loc}}(\Omega)$ such that

- $\lim_{|x|\to\infty} a_0(x) = \infty$ and $\forall n \in \mathbb{N}^*$, $a_n \geq a_0$
- $a_n \to a_\infty$ in Ω_∞ and $a_n \to \infty$ in $\Omega \setminus \Omega_\infty$ (convergence of $a_n^{\frac{1}{2}}$ in L^2_{loc})

Then

- every $\lambda \in \sigma_{\text{p}}(G_\infty) \setminus (-\infty, 0]$ is approximated:

$$\forall \lambda \in \sigma_{\text{p}}(G_\infty) \setminus (-\infty, 0], \quad \exists \{\lambda_n\}, \quad \lambda_n \in \sigma_{\text{p}}(G_n), \quad \lambda_n \to \lambda$$

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- no pollution: if

$$\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C} \setminus (-\infty, 0], \quad \lambda_n \in \sigma_{\mathrm{p}}(G_n)$$

  has an accumulation point $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, then $\lambda \in \sigma_{\mathrm{p}}(G_\infty)$

What about $\mathbb{R}_-$?

The issue

- for e.g. $\lambda = -1$ and $a(x) = x^{2n}$: $T(-1) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x^{2n} + 1$

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- "tricky" spectra of $T(\lambda)$ with $\lambda < 0$:

$$\sigma(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^2) = \mathbb{R} \quad \text{vs.} \quad \sigma(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^4) = \sigma_{\mathrm{disc}}(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^4)$$

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Theorem (negative essential spectrum)

[Freitas, Siegl, and Tretter, 2018]

For $d = 1$, if $\mathbb{R}_+ \subset \Omega$ and $\lambda < 0$ is such that

- $A(x) := 2|\lambda| a(x) + \lambda^2 - q(x)$ is (eventually) increasing on $\mathbb{R}_+$,
- $\lim_{x\to+\infty} A(x) = +\infty$,
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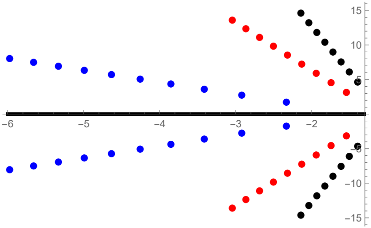
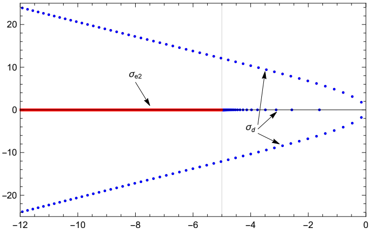
- $\sigma_{\mathrm{ess}}(G) = (-\infty, 0]$

For $d > 1$, the same holds if Ω contains a (possibly shrinking) neighborhood of a semi-infinite segment where A grows to $+\infty$.

Overdamping due to $\sigma_{\mathrm{ess}}(G)$





Remarks

- no exponential decay due to $0 \in \sigma(G)$
- shift of σ_{ess} by adding "large" q: e.g. $q(x) = 10x^2$, $a(x) = x^2$ [Arnal, 2022]
- jump in spectra:
  $a_n(x) = x^{2n} + a_0$ on $\mathbb{R}$ vs. $a_\infty(x) = a_0$ on $(-1, 1)$
- eigenvalues (and eigenfunctions) do converge vs. "jump" in σ_{ess}

Theorem [Ikehata and Takeda, 2020]

Let $d \geq 3$, $a \in C(\mathbb{R}^d)$ such that $a(x) \geq a_0 > 0$, $x \in \mathbb{R}^d$. Let initial data satisfy

- $(u_0, u_1) \in (H^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)) \times (H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))$
- $au_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

Then

$$\|u(t,\cdot)\|^2 \lesssim (1+t)^{-1}, \qquad \|u_t(t,\cdot)\|^2 + \|\nabla u(t,\cdot)\|^2 \lesssim (1+t)^{-2}.$$

Theorem [Sobajima and Wakasugi, 2018]

Let $\Omega \subset \mathbb{R}^d$ be an exterior domain with a smooth $\partial\Omega$ and $0 \notin \Omega$. Let (with $\alpha > 0$)

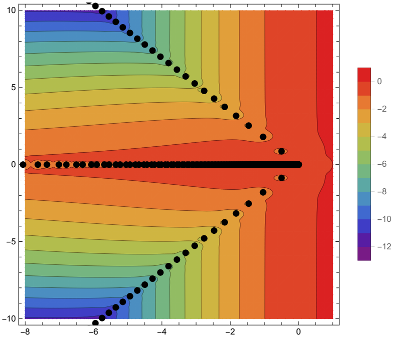
$$\lim_{|x| \to \infty} |x|^{-\alpha} a(x) = a_0 > 0.$$

Let $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ with compact support. Then (with $\delta > 0$ arb. small)

$$\|a^{\frac{1}{2}} u(t,\cdot)\|^2 \lesssim (1+t)^{-\frac{d+\alpha}{2+\alpha}+\delta} \|(u_0,u_1)\|^2_{H^2 \times H^1}.$$

(and a comparison to the solution of a related heat equation).

- $a(x) = x^2$, $d = 1$
- resolvent norm (log scale)



Theorem (simplified)

Let $\Omega = \mathbb{R}$, $q(x) = 0$, $a(x) = x^{2n}$, $n \in \mathbb{N}$. Then

i) for every $\omega \in (\frac{\pi}{2}, \pi) \cup (\pi, \frac{3}{2}\pi)$ [Arifoski and Siegl, 2020]

$$\lim_{r \to +\infty} \|(G - re^{i\omega})^{-1}\| = \infty;$$

ii) for every $c \geq 0$ [Arnal, 2022]

$$\|(G - (-c + ib))^{-1}\| \approx 1, \quad |b| \to +\infty.$$

Based on (as $|b| \to \infty$)

$$\|T(-c + ib)^{-1}\| = \|(A - c)^{-1}\|(2|b|)^{-1} \times (1 + \mathcal{O}_c(|b|^{-1})),$$

where $A = -\partial_x + a(x)$ in $L^2(\mathbb{R})$.