Euler equations on networks: Parabolic limit and asymptotic preserving numerics

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Overview

Overview

Introduction

Port-Hamiltonian structure and stability

Extension to networks

Structure preserving discretization

Gas transport in pipeline networks

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We study barotropic Euler equations on networks, as a model for gas transport in pipeline networks.

Specific features, different from standard scenario for (barotropic) Euler equations

- slow flow velocities (1-2 m/s), i.e. low Mach number flows
- interest in large space and time scales $\rightarrow$ friction dominated flow
- specific non-linear friction law is important to get correct steady states
- solutions are expected to be continuous (no shock waves)

Our analysis works for networks but most parts of this talk focus on single pipes simlify the presentation.

Numerische Mathematik

Modeling

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Conservation of mass

$$\partial_t \rho + \partial_x m = 0$$

and balance of momentum

$$\varepsilon^2 \partial_t m + \partial_x \left(\varepsilon^2 \frac{m^2}{\rho} + p(\rho) \right) = -\frac{|m|m}{\rho}$$

where $p = p(\rho)$ is a constitutive law for the pressure, we only assume $p' > 0$; $\varepsilon \geq 0$ small parameter related to Mach number (i.e. reference fluid velocity divided by speed of sound).

Equations can be rewritten in different variables (equivalent for strong solutions but not for weak entropy solutions): Fluid velocity v; $m = \rho v$.

$$\partial_t \rho + \partial_x (\rho v) = 0$$

$$\varepsilon^2 \partial_t v + \partial_x \left(\varepsilon^2 \frac{v^2}{2} + P'(\rho) \right) = -|v|v$$

with P such that $\rho P''(\rho) = p'(\rho)$.

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Low Mach – large friction limit

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By formally setting $\varepsilon = 0$ in

$$\varepsilon^2 \partial_t v + \partial_x \left(\varepsilon^2 \frac{v^2}{2} + P'(\rho) \right) = -|v|v$$

we obtain friction dominated flow model ($\rightarrow$ talk by Günther Leugering)

$$\partial_t \rho + \partial_x (\rho v) = 0 \qquad \partial_x P'(\rho) = -|v|v$$

- Can we make this convergence rigorous (on networks)?
- Speed of convergence?
- Asymptotic preserving numerical schemes?

For linear friction $\rightarrow$ Marcati & Milani 1990, Lattanzio & Tzavaras 2016.

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port-Hamiltonian structure

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The formulation in (ρ, v) variables reveales a port-Hamiltonian structure:
Energy of the system

$$\mathscr{H}(\rho, v) := \int_\Omega \frac{\varepsilon^2}{2} \rho v^2 + P(\rho)\, dx$$

where Ω is the computational domain.
Taking variational derivatives we see

$$\frac{\delta \mathscr{H}}{\delta \rho} = \varepsilon^2 \frac{v^2}{2} + P'(\rho), \qquad \frac{\delta \mathscr{H}}{\delta v} = \varepsilon^2 \rho v$$

Thus, the equations can be expressed as

$$\begin{aligned} \partial_t \rho + \frac{1}{\varepsilon^2} \partial_x \frac{\delta \mathscr{H}}{\delta v} &= 0 \\ \varepsilon^2 \partial_t v + \partial_x \frac{\delta \mathscr{H}}{\delta \rho} &= -\frac{1}{\varepsilon^2} \frac{|v|}{\rho} \frac{\delta \mathscr{H}}{\delta v} \end{aligned}$$

port-Hamiltonian structure

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Based on

$$\partial_t \rho + \frac{1}{\varepsilon^2}\partial_x \frac{\delta \mathscr{H}}{\delta v} = 0$$
$$\varepsilon^2 \partial_t v + \partial_x \frac{\delta \mathscr{H}}{\delta \rho} = -\frac{1}{\varepsilon^2}\frac{|v|}{\rho}\frac{\delta \mathscr{H}}{\delta v}$$

it is 'straightforward' to derive an energy balance

$$d_t \mathscr{H}(\rho, v) = \int_\Omega \frac{\delta \mathscr{H}}{\delta \rho}\partial_t \rho + \frac{\delta \mathscr{H}}{\delta v}\partial_t v = -\int_\Omega \frac{\delta \mathscr{H}}{\delta \rho}\frac{1}{\varepsilon^2}\partial_x \frac{\delta \mathscr{H}}{\delta v} + \frac{\delta \mathscr{H}}{\delta v}\frac{1}{\varepsilon^2}\left(\partial_x \frac{\delta \mathscr{H}}{\delta \rho} + \frac{1}{\varepsilon^2}\frac{|v|}{\rho}\frac{\delta \mathscr{H}}{\delta v}\right)$$
$$= -\frac{1}{\varepsilon^2}\int_{\partial\Omega} \underbrace{\frac{\delta \mathscr{H}}{\delta v}\frac{\delta \mathscr{H}}{\delta \rho}\nu}_{\text{energy flux}} - \frac{1}{\varepsilon^4}\int_\Omega \underbrace{\frac{|v|}{\rho}\left|\frac{\delta \mathscr{H}}{\delta v}\right|^2}_{\text{energy dissipation}}$$

Relative Energy

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We abbreviate further $\mathbf{u} = (\rho, v)$ and write

$$\mathscr{C}\partial_t \mathbf{u} + \mathcal{J}\mathbf{z}(\mathbf{u}) = \mathcal{R}(\mathbf{u})\mathbf{z}(\mathbf{u})$$

with

$$\mathbf{z}(\mathbf{u}) = \mathscr{C}^{-1}\frac{\delta \mathscr{H}}{\delta \mathbf{u}}, \quad \mathscr{C} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, \quad \mathcal{R}(\mathbf{u}) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{|v|}{\rho} \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}$$

For two (approximate) solutions $\mathbf{u} = (\rho, v)$ and $\hat{\mathbf{u}} = (\hat{\rho}, \hat{v})$ we define the relative energy

$$\mathscr{H}(\mathbf{u}|\hat{\mathbf{u}}) := \mathscr{H}(\mathbf{u}) - \mathscr{H}(\hat{\mathbf{u}}) - \int_\Omega \frac{\delta \mathscr{H}}{\delta \mathbf{u}}(\mathbf{u} - \hat{\mathbf{u}})$$

and have the relative energy balance

$$\begin{aligned} d_t \mathscr{H}(\mathbf{u}|\hat{\mathbf{u}}) = &-\langle \mathcal{R}(\mathbf{u})\mathbf{z}(\mathbf{u}) - \mathcal{R}(\hat{\mathbf{u}})\mathbf{z}(\hat{\mathbf{u}}), \mathbf{z}(\mathbf{u}) - \mathbf{z}(\hat{\mathbf{u}})\rangle_\Omega + \langle \mathbf{z}_1(\mathbf{u}) - \mathbf{z}_1(\hat{\mathbf{u}}), \mathbf{z}_2(\mathbf{u}) - \mathbf{z}_2(\hat{\mathbf{u}})\rangle_{\partial\Omega} \\ &+ \langle \mathscr{C}\partial_t \hat{\mathbf{u}}, \mathbf{z}(\mathbf{u}) - \mathbf{z}(\hat{\mathbf{u}}) - \mathcal{G}(\hat{\mathbf{u}})(\mathbf{u} - \hat{\mathbf{u}})\rangle_\Omega \end{aligned}$$

where $\mathcal{G}(\hat{\mathbf{u}}) = \mathscr{C}^{-1}\frac{\delta^2 \mathscr{H}}{\delta \mathbf{u}^2}(\hat{\mathbf{u}})$.

Convexity of Energy

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$\mathscr{H}$ is strictly convex on the set of states such that $\rho > 0$ and $|v|^2 < p'(\rho)$, i.e. subsonic flows without vacuum.

To infer (finite time) stability from relative energy balance we need:

$$-\langle \mathcal{R}(\mathbf{u})\mathbf{z}(\mathbf{u}) - \mathcal{R}(\hat{\mathbf{u}})\mathbf{z}(\hat{\mathbf{u}}), \mathbf{z}(\mathbf{u}) - \mathbf{z}(\hat{\mathbf{u}})\rangle_\Omega \leq \hat{C}_1 \mathscr{H}(\mathbf{u}|\hat{\mathbf{u}}) - 2\mathscr{D}(\mathbf{u}|\hat{\mathbf{u}}), \tag{C1}$$

$$\langle \mathscr{C}\partial_t \hat{\mathbf{u}}, \mathbf{z}(\mathbf{u}) - \mathbf{z}(\hat{\mathbf{u}}) - \mathcal{G}(\hat{\mathbf{u}})(\mathbf{u} - \hat{\mathbf{u}})\rangle_\Omega \leq \hat{C}_2 \mathscr{H}(\mathbf{u}|\hat{\mathbf{u}}), \tag{C2}$$

$$\langle \mathbf{z}_1(\mathbf{u}) - \mathbf{z}_1(\hat{\mathbf{u}}), \mathbf{z}_2(\mathbf{u}) - \mathbf{z}_2(\hat{\mathbf{u}})\rangle_{\partial\Omega} \leq \mathcal{P}_\partial(\mathbf{z}(\mathbf{u}) - \mathbf{z}(\hat{\mathbf{u}})), \tag{C4}$$

If we allow $\hat{\mathbf{u}}$ to solve the equation only up to some residual e, i.e.

$$\mathscr{C}\partial_t \hat{\mathbf{u}} + \mathcal{J}\mathbf{z}(\hat{\mathbf{u}}) = \mathcal{R}(\hat{\mathbf{u}})\mathbf{z}(\hat{\mathbf{u}}) + e$$

we aditinally require

$$\langle e, \mathbf{z}(\mathbf{u}) - \mathbf{z}(\hat{\mathbf{u}})\rangle_\Omega \leq \hat{C}_3 \mathscr{H}(\mathbf{u}|\hat{\mathbf{u}}) + \mathscr{D}(\mathbf{u}|\hat{\mathbf{u}}) + \mathscr{P}(e), \tag{C3}$$

with some suitable $\mathscr{P}(e)$.

Stability and parabolic limit

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Lemma (Egger, JG 2020)

Let (ρ, v) and $(\hat{\rho}, \hat{v})$ be two Lipschitz continuous solutions of the barotropic Euler equations that are subsonic and bounded away from vacuum and share identical Dirichlet boundary conditions for the enthalpy, then

$$\|\rho(t) - \hat{\rho}(t)\|^2_{L^2(\Omega)} + \varepsilon^2 \|v(t) - \hat{v}(t)\|^2_{L^2(\Omega)} + \int_0^t \|v(s) - \hat{v}(s)\|^3_{L^3(\Omega)} ds$$
$$\leq Ce^{Ct} \left(\|\rho(0) - \hat{\rho}(0)\|^2_{L^2(\Omega)} + \varepsilon^2 \|v(0) - \hat{v}(0)\|^2_{L^2(\Omega)} + |\varepsilon^2 - \hat{\varepsilon}^2| \right)$$

where C depends on $\|\partial_t \hat{\mathbf{u}}\|_{L^\infty((0,t)\times\Omega)}$

The estimate is uniform in $\hat{\varepsilon}$ and allows us to show that for $\varepsilon \searrow 0$ solutions to the barotropic Euler equations converge to solutions of

$$\partial_t \hat{\rho} + \partial_x(\hat{\rho}\hat{v}) = 0$$
$$\partial_x P'(\hat{\rho}) = -|\hat{v}|\hat{v}$$

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Network topology

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$(\mathscr{V}, \mathscr{E})$ directed and connected finite graph with vertices $v \in \mathscr{V}$ and edges $e \in \mathscr{E}$, which are identified with intervals $(0, \ell^e)$.

$\mathscr{E}(v)$ denotes the set of edges incident to the vertex v, andv$|\mathscr{E}(v)|$ its cardinality.
We decompose $\mathscr{V} = \mathscr{V}_0 \cup \mathscr{V}_\partial$ into sets of interior and boundary vertices, characterized by $\mathscr{V}_0 = \{v \in \mathscr{V} : |\mathscr{E}(v)| > 1\}$ and $\mathscr{V}_\partial = \{v \in \mathscr{V} : |\mathscr{E}(v)| = 1\}$.

We associate to any vertex $v \in \mathscr{V}$ and edge $e \in \mathscr{E}(v)$ a number

$$n^e(v) = \begin{cases} 1 & \text{if } e = (\cdot, v), \\ -1 & \text{if } e = (v, \cdot). \end{cases}$$

The vertex v thus corresponds to the end point ℓ^e or the start point 0 of the interval $(0, \ell^e)$ representing the edge e.

Equations and Coupling conditions

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Superscript ^{e} denotes functions restricted to the edge e. As before, on every edge of the network we consider

$$\partial_t \rho^e + \partial_x m^e = 0, \qquad e \in \mathscr{E}$$

$$\varepsilon^2 \partial_t w^e + \partial_x h^e + |w^e| w^e = 0, \qquad e \in \mathscr{E}$$

with co-state variables defined by

$$h^e = \varepsilon^2 \frac{|w^e|^2}{2} + P'(\rho^e),\ e \in \mathscr{E}, \qquad m^e = \rho^e w^e,\ e \in \mathscr{E}.$$

We consider coupling conditions across pipe junctions that ensure conservation of mass and energy (Reigstad 2014)

$$\sum_{e \in \mathscr{E}(v)} m^e(v) n^e(v) = 0, \qquad v \in \mathscr{V}_0,$$

$$h^e(v) = h^f(v), \qquad e, f \in \mathscr{E}(v),\ v \in \mathscr{V}_0,$$

the second condition means continuity of the total specific enthalpy h.

Weak form

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The coupling conditions are natural in the sense that they do not show up explicitly when we multiply the equations by suitable test functions, integrate over the edges, use integration-by-parts, and then sum over all edges:

$$\sum_{e \in \mathscr{E}} (\partial_t \rho^e, q^e)_e + (\partial_x m^e, q^e)_e = 0 \quad \forall (q^e)_{e \in \mathscr{E}}$$

$$\sum_{e \in \mathscr{E}} (\varepsilon^2 \partial_t w^e, r^e)_e - (h^e, \partial_x r^e)_e + (\frac{|w^e|}{\rho^e} m^e, r^e)_e = -\sum_{v \in \mathscr{V}_\partial} \sum_{e \in \mathscr{E}(v)} h^e(v) r^e(v) n^e(v) \quad \forall (r^e)_{e \in \mathscr{E}},$$

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Mixed Finite Element Scheme

$\mathscr{T}_h$ 'triangulation' of our interval Ω.
At each time we seek $\rho_h^n \in Q_h := \mathbb{P}_0(\mathscr{T}_h)$ and $m_h^n \in R_h := \mathbb{P}_1(\mathscr{T}_h) \cap H^1(\Omega)$ such that

$$(\bar{d}_t \rho_h^n, q_h)_\Omega + (\partial_x m_h^n, q_h)_\Omega = 0 \quad \forall q_h \in Q_h$$
$$\varepsilon^2 (\bar{d}_t v^n, r_h)_\Omega - (h^n, \partial_x r_h)_\Omega + h_\partial^n r_h|_{\partial\Omega} + (|v^n| v^n, r_h)_\Omega = 0 \quad \forall r_h \in R_h$$

where $\bar{d}_t$ denotes the backward difference quotient, e.g.

$$\bar{d}_t \rho_h^n := \frac{\rho_h^n - \rho_h^{n-1}}{t^{n+1} - t^n},$$

and we abbreviate

$$v^n := \frac{m_h^n}{\rho_h^n}, \quad h^n := \frac{\varepsilon^2}{2} \frac{(m_h^n)^2}{(\rho_h^n)} + P'(\rho_h^n)$$

We have carried out integration by parts in the second equation which allows us to include Dirichlet boundary conditions for enthalpy h_∂ in a natural way.

Energy stability

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The scheme reads

$$(\bar{d}_t \rho_h^n, q_h)_\Omega + (\partial_x m_h^n, q_h)_\Omega = 0 \quad \forall q_h \in Q_h$$
$$\varepsilon^2(\bar{d}_t v^n, r_h)_\Omega - (h^n, \partial_x r_h)_\Omega + h_\partial r_h|_{\partial\Omega} + (|v^n|v^n, r_h)_\Omega = 0 \quad \forall r_h \in R_h$$

Use $r_h = m_h^n$ and $q_h = \pi_h h^n$ as test functions and use convexity of $\mathscr{H}$, i.e.

$$\bar{d}_t \mathscr{H}(\rho_h^n, v^n) \leq \left(\frac{\delta \mathscr{H}}{\delta \rho}(\rho_h^n, v^n), \bar{d}_t \rho_h^n \right)_\Omega + \left(\frac{\delta \mathscr{H}}{\delta v}(\rho_h^n, v^n), \bar{d}_t v^n \right)_\Omega$$

to obtain

$$\bar{d}_t \mathscr{H}(\rho_h^n, v^n) \leq -\frac{1}{\varepsilon^2} \int_{\partial\Omega} h_\partial^n m_h^n \nu - \frac{1}{\varepsilon^2} \int_\Omega \rho_h^n |v^n|^3.$$

where ν is outward unit normal vector.

Robust a priori error estimate

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Lemma (Egger, JG, Philippi, Kunkel 2021)

As long as the exact solution (ρ, v) *is sufficiently regular, subsonic and away from vacuum, there exists a constant* $C > 0$ *independent of* ε *such that*

$$\|\rho(t^n) - \rho_h^n\|_{L^2(\Omega)}^2 + \varepsilon^2 \|v(t^n) - v^n\|_{L^2(\Omega)}^2 + \sum_{k=0}^{n} \Delta t \|v(t^k) - v^k\|_{L^3(\Omega)}^3 \leq C((\Delta t)^2 + (\Delta x)^2)$$

Idea of the proof:
Split the error into discrete error and appropriation error

$$\rho(t^n) - \rho_h^n = \rho(t^n) - \pi_h \rho(t^n) + \pi_h \rho(t^n) - \rho_h^n, \quad m(t^n) - m_h^n = m(t^n) - \pi_h m(t^n) + \pi_h m(t^n) - m_h^n$$

The approximation errors $\rho(t^n) - \pi_h \rho(t^n)$ and $m(t^n) - \pi_h m(t^n)$ can be bounded using Bramble-Hilbert lemma.
The discrete errors $\pi_h \rho(t^n) - \rho_h^n$ and $\pi_h m(t^n) - m_h^n$ can be bounded using a discrete version of relative energy.

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Summary, Outlook, Open Questions

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Summary:

- Stability framework for certain 2×2 systems of PDEs in port-Hamiltonian form
- application to barotropic Euler equations on networks
- Conditional convergence to friction dominated flow limit, with convergence rate
- Structure preserving discretisation of problems in this form and robust error estimate.

Outlook/Questions:

- Other problems with similar structure?
- Can this analysis be extended to cover more general problems in port-Hamiltonian form?
- Well-posedness theory for barotropic Euler with non-linear friction (on networks)?
- What happens if only weak entropy solutions are available?