A Non-local non-linear convection-diffusion problem on the Hyperbolic space

Dragoș Manea

"Simion Stoilow" Mathematical Institute of the Romanian Academy

Joint work with María del Mar González (Universidad Autónoma de Madrid), Liviu Igant and Sergiu Moroianu (IMAR)

Partially supported by national Grant 0794/2020
"Spectral Methods in Hyperbolic Geometry" PN-III-P4-ID-PCE-2020-0794

The convection-diffusion problem

$$
\begin{cases}
\partial_t u(t,x) = \displaystyle\int_{\mathbb{H}^N} J(d(x,y))(u(t,y)-u(t,x))dy \\
\qquad\qquad + \displaystyle\int_{\mathbb{H}^N} G(x,y)\left[f(u(t,y)) - f(u(t,x))\right] dy, \quad x \in \mathbb{H}^N, t \geq 0; \\
u(0,x) = u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{H}^N.
\end{cases}
\tag{1}
$$

Both $J = J_\varepsilon$ and $G = G_\varepsilon$ are shrunk as $\varepsilon \to 0$ to obtain:

$$
\begin{cases}
u_t(t,x) = A\Delta u(t,x) - \mathrm{div}(f(u(t)) \cdot X)(x), \quad x \in \mathbb{H}^N, t \geq 0; \\
u(0,x) = u_0(x), \qquad\qquad\qquad\qquad\qquad x \in \mathbb{H}^N.
\end{cases}
\tag{2}
$$

where $A > 0$ depends on J, X is a bounded C^1 vector field, depending on G.

The transport problem

$$\begin{cases} \partial_t u(t,x) = \int_{\mathbb{H}^N} G(x,y)(u(t,y) - u(t,x))dy, & x \in \mathbb{H}^N, t \geq 0 \\ u(0,x) = u_0(x), & x \in \mathbb{H}^N \end{cases} \tag{3}$$

Shrinking $G = G_\varepsilon$, $\varepsilon \to 0$ to obtain:

$$\begin{cases} \partial_t u(t,x) = -\mathrm{div}(u(t) \cdot X)(x), & x \in \mathbb{H}^N, t \geq 0 \\ u(0,x) = u_0(x), & x \in \mathbb{H}^N \end{cases} \tag{4}$$

X bounded C^1 vector field

The diffusion case

[Bandle et al., 2018] – diffusion equation

$$\begin{cases} \partial_t u^\varepsilon(t,x) := \int_{\mathbb{H}^N} \varepsilon^{-N-2} J\left(\frac{d(x,y)}{\varepsilon}\right)(u(t,y)-u(t,x))dy & ,x\in\mathbb{H}^N, t\geq 0 \\ u^\varepsilon(0,x) = u_0(x) & ,x\in\mathbb{H}^N \end{cases} \tag{5}$$

$$\begin{cases} \partial_t u(t,x) := -q\cdot\Delta u(t,x) & ,x\in\mathbb{H}^N, t\geq 0 \\ u(0,x) = u_0(x) & ,x\in\mathbb{H}^N \end{cases} \tag{6}$$

$$q = \frac{1}{2N}\int_{\mathbb{H}^N} J(d(0,x))\cdot(d(0,x))^2 dx$$

The euclidean case I

[Ignat and Rossi, 2007] – convection-diffusion equation on $\mathbb{R}^N$

$$\begin{cases} \partial_t u(t,x) = \int_{\mathbb{H}^N} J(x-y)(u(t,y)-u(t,x))dy \\ \qquad\qquad + \int_{\mathbb{H}^N} G(x-y)\left[f(u(t,y)) - f(u(t,x))\right] dy, \quad x \in \mathbb{R}^N, t \geq 0; \\ u(0,x) = u_0(x), \qquad\qquad x \in \mathbb{R}^N. \end{cases} \tag{7}$$

$f(r) = |r|^{q-1} r, q \geq 1$

J is a radial function, G is not radial.

The euclidean case II

$$
\begin{cases}
\partial_t u^\varepsilon(t,x) = \displaystyle\int_{\mathbb{H}^N} \varepsilon^{-N-2} J\left(\frac{x-y}{\varepsilon}\right)(u(t,y)-u(t,x))dy \\
\qquad\qquad + \displaystyle\int_{\mathbb{H}^N} \varepsilon^{-N-1} G\left(\frac{x-y}{\varepsilon}\right)\left[f(u(t,y)) - f(u(t,x))\right] dy, \quad x \in \mathbb{R}^N, t \geq 0; \\
u^\varepsilon(0,x) = u_0(x), \qquad x \in \mathbb{R}^N.
\end{cases}
\tag{8}
$$

The euclidean case III

$$\begin{cases} \partial_t u(t,x) = A\Delta u(t,x) + \nabla_x f(u)(t,x)\cdot X & , x\in\mathbb{R}^N, t\geq 0 \\ u(0,x) = u_0(x) & , x\in\mathbb{R}^N \end{cases} \tag{9}$$

$$A = \frac{1}{2N}\int_{\mathbb{R}^N} J(x)|x|^2 dx$$

$$X = -\int_{\mathbb{R}^N} G(x)\cdot x\, dx \in \mathbb{R}^N \text{ constant vector.}$$

Essential property for the convergence:

$$\int_{\mathbb{R}^N} [G(y-x) - G(x-y)]dx = 0, \forall y \in \mathbb{R}^N$$

The Hyperbolic space I

The Poincaré ball model
$B^N \subset \mathbb{R}^N$ the unit ball.

$$g_{i,j} = (\lambda(x))^2 \delta_{i,j},$$

$$\lambda(x) = \frac{2}{1 - |x|_e^2},$$

The half-space model
$\mathbb{R}^N_+ = \left\{ x = (x', x_N) \in R^N : x_N > 0 \right\}$

$$g_{i,j} = \frac{1}{x_N^2} \delta_{i,j}.$$

The Hyperbolic space II

Isometry between models

Cayley transform $\mathcal{C} : R^N_+ \to B^N$:

$$\mathcal{C}(x', x_N) = \left(\frac{2x'}{1 + |x|_e^2 + 2x_N}, \frac{|x|_e^2 - 1}{1 + |x|_e^2 + 2x_N} \right), \forall x = (x', x_N) \in R^N_+.$$

Defining G on the Hyperbolic space I

Definition (Geodesic flow)

$(x, V) \in T\mathbb{H}^N$
$\gamma_{x,V}$ be the unique geodesic s.t. $\gamma(0) = x$, $\gamma'(0) = V$

$exp_x : T_x\mathbb{H}^N \to \mathbb{H}^N$

$$exp_x(V) = \gamma_{x,V}(1)$$

$\Phi_t(x, V) : T\mathbb{H}^N \to T\mathbb{H}^N$

$$\Phi_t(x, V) = (\gamma_{x,V}(t), \gamma'_{x,V}(t))$$

The vector $V_{x,y} \in T_x\mathbb{H}^N$ such that $exp_x(V_{x,y}) = y$ accounts for $y - x$.
Moreover, $|V_{x,y}| = d(x, y)$.

Defining G on the Hyperbolic space II

Remark

$$\Phi_1(x, V_{x,y}) = (y, -V_{y,x})$$

This relation accounts for

$$(y - x) = -(x - y)$$

We define

$$G(x, y) = \widetilde{G}(x, V_{x,y})$$

$\widetilde{G} : T\mathbb{H}^N \to [0, \infty)$

$$\widetilde{G}(\Phi_t(x, V)) = (x, V)$$

An important property of G

$$\widetilde{G}(x, V_{x,y}) = \widetilde{G}(y, -V_{y,x})$$

Proposition

$$\int_{\mathbb{H}^N} [G(x,y) - G(y,x)]dx = 0, \forall y \in \mathbb{H}^N$$

Proof idea.

$$\int_{\mathbb{H}^N} [\widetilde{G}(y, -V_{y,x}) - G(y, V_{y,x})]dx = 0$$

equivalent to

$$\int_{T_y\mathbb{H}^N} [\widetilde{G}(y, -V) - G(y, V)] \left| J_{\exp_y}(V) \right| dV = 0$$

□

Concentrating the kernel G

$$G_\varepsilon(x,y) = \varepsilon^{-N-1}\, \widetilde{G}\left(x, \frac{1}{\varepsilon} V_{x,y}\right)$$

Still invariant to the geodesic flow.

Particular case: G compactly supported around the diagonal of $\mathbb{H}^N \times \mathbb{H}^N$:

$$d(x,y) > M \Rightarrow G(x,y) = 0$$

$$\widetilde{G}(x,V) = 0, \text{ if } |V| > M$$

$$d(x,y) > \varepsilon M \Rightarrow G_\varepsilon(x,y) = 0$$

The non-local and local transport problems

$$\begin{cases} u \in C^1([0,\infty), L^2(\mathbb{H}^N)) \\ \partial_t u^\varepsilon(t,x) = L_{G_\varepsilon}(u^\varepsilon(t))(x) = \varepsilon^{-N-1} \int_{\mathbb{H}^N} \widetilde{G}\left(x, \frac{1}{\varepsilon} V_{x,y}\right) (u^\varepsilon(t,y) - u^\varepsilon(t,x)) dy, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{H}^N, t \geq 0 \\ u(0,x) = u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{H}^N \end{cases} \tag{10}$$

$$\begin{cases} \partial_t u(t,x) = -\mathrm{div}(u(t) \cdot X)(x) & , x \in \mathbb{H}^N, t \geq 0 \\ u(0,x) = u_0(x) & , x \in \mathbb{H}^N \end{cases} \tag{11}$$

$$X(x) = -\int_{T_x \mathbb{H}^N} \widetilde{G}(x, W) \cdot W \, dW$$

Weak formulations for the transport problems

For every $\varphi \in C_c^1([0,\infty), H^1(\mathbb{H}^N))$,

$$\int_0^\infty \int_{\mathbb{H}^N} u^\varepsilon \partial_t \varphi dxdt = -\int_{\mathbb{H}^N} u_0 \varphi(0) dx - \int_0^\infty \int_{\mathbb{H}^N} u^\varepsilon L^*_{G_\varepsilon}(\varphi) dxdt \qquad (12)$$

$$L^*_{G_\varepsilon}(\varphi)(x) = \int_{\mathbb{H}^N} G_\varepsilon(y,x)(\varphi(y) - \varphi(x)) dy$$

$$\int_0^\infty \int_{\mathbb{H}^N} u \partial_t \varphi dxdt = -\int_{\mathbb{H}^N} u_0 \varphi(0) dx - \int_0^\infty \int_{\mathbb{H}^N} u \nabla\varphi \cdot X dxdt \qquad (13)$$

Remark

$$\nabla\varphi \cdot X = X(\varphi)$$

Transport: convergence non-local → local I

$$\|u^{\varepsilon}(t)\|_{L^2(\mathbb{H}^N)} \leq \|u_0\|_{L^2(\mathbb{H}^N)}$$

$$u^{\varepsilon} \xrightarrow{\varepsilon \to 0} U \text{ in } L^2((0,T) \times \mathbb{H}^N)$$

It is enough to prove

$$\|L^*_{G_\varepsilon}(\varphi) - X(\varphi)\|_{L^2(\mathbb{H}^N)} \xrightarrow{\varepsilon \to 0} 0, \forall \varphi \in H^1(\mathbb{H}^N).$$

Transport: convergence non-local → local II

Proof of convergence

$$L^*_{G_\varepsilon}(\varphi)(x) = \varepsilon^{-N-1}\int_{\mathbb{H}^N} \widetilde{G}\left(y, \frac{1}{\varepsilon}V_{y,x}\right)(\varphi(y)-\varphi(x))dy$$
$$= \varepsilon^{-N-1}\int_{\mathbb{H}^N} \widetilde{G}\left(x, -\frac{1}{\varepsilon}V_{x,y}\right)(\varphi(y)-\varphi(x))dy$$

C.V. $W = V_{x,y}$, that is $y = exp_x(W)$

$$L^*_{G_\varepsilon}(\varphi)(x) = \varepsilon^{-N-1}\int_{T_x\mathbb{H}^N} \widetilde{G}\left(x, -\frac{1}{\varepsilon}W\right)(\varphi(exp_x(W)) - \varphi(x))\left|J_{exp_x}(W)\right| dW.$$

Denote $F_x = \varphi \circ exp_x$, C.V. $W \leftarrow -\varepsilon W$

$$L^*_{G_\varepsilon}(\varphi)(x) = \varepsilon^{-1}\int_{T_x\mathbb{H}^N} \widetilde{G}(x, W)\left(F_x(-\varepsilon W)\right) - F_x(0))\left|J_{exp_x}(-\varepsilon W)\right| dW.$$

Transport: convergence non-local $\to$ local III

Leibniz-Newton formula:

$$L^*_{G_\varepsilon}(\varphi)(x) = \int_{T_x\mathbb{H}^N} \widetilde{G}(x, W) \cdot W \int_0^1 \nabla F_x(-\tau\varepsilon W) d\tau \, |J_{\exp_x}(-\varepsilon W)| \, dW.$$

$$\frac{d}{ds} F_x(sW) = dF_x(sW)(W) = W \cdot \nabla F_x(sW)$$

Transport: convergence non-local → local IV

$$W(\varphi)(x) = W(x) \cdot \nabla F_x(0)$$

Remember

$$X(\varphi)(x) = -\int_{T_x\mathbb{H}^N} \widetilde{G}(x, W) \cdot W(\varphi)(x)\, dW$$

The studied difference $\|L^*_{G_\varepsilon}(\varphi) - X(\varphi)\|_{L^2(\mathbb{H}^N)}$ is equal to:

$$\int_{\mathbb{H}^N}\left(\int_{T_x\mathbb{H}^N} \widetilde{G}(x, W) \cdot W\left[\int_0^1 \nabla F_x(-\tau\varepsilon W) d\tau\, |J_{exp_x}(-\varepsilon W)| - \nabla F_x(0)\right] dW\right)^2 dy$$

$$|J_{exp_x}(0)| = 1$$

Transport: convergence non-local → local V

Ingredients for the general case:

We assume

$$G(x,y) \leq k(d(x,y))$$

$$M(G) = \int_0^{\infty} k(r) \cdot r \cdot e^{2(N-1)r} dr < \infty.$$

We obtain:

$$\|L^*_{G_\varepsilon}(\varphi)\|_{L^2(\mathbb{H}^N)} \leq M(G) \|\nabla \varphi\|_{L^2(\mathbb{H}^N)}, \forall \varphi \in H^1(\mathbb{H}^N) \tag{14}$$

Half-space model: $\forall V \in \mathbb{R}^N, \quad \exists C_V \in \left[e^{-|V|_e}, e^{|V|_e}\right]$

$$\|\nabla \varphi(exp_x(x_N V))\|_{L^2_x(\mathbb{H}^N)} \leq C_V^{\frac{N-1}{2}} \|\nabla \varphi\|_{L^2(\mathbb{H}^N)} \tag{15}$$

Accounts for: $\int_{\mathbb{R}^N} |f(x+a)|^2 dx = \int_{\mathbb{R}^N} |f(x)|^2 dx, \forall a \in \mathbb{R}^N$

The non-local and local non-linear convection-diffusion problems

$u_0 \in Y = L^1(\mathbb{H}^N) \cap L^\infty(\mathbb{H}^N), \quad f(r) = |r|^{q-1} \cdot r, \ q \geq 1.$

$$\begin{cases} \partial_t u^\varepsilon(t,x) = \varepsilon^{-N-2} \displaystyle\int_{\mathbb{H}^N} J\left(\frac{d(x,y)}{\varepsilon}\right)(u^\varepsilon(t,y) - u^\varepsilon(t,x))dy \\ \qquad + \varepsilon^{-N-1} \displaystyle\int_{\mathbb{H}^N} G_\varepsilon(x,y)\left[f(u^\varepsilon(t,y)) - f(u^\varepsilon(t,x))\right] dy, \quad t \geq 0, x \in \mathbb{H}^N; \\ u^\varepsilon(0,x) = u_0(x), \qquad x \in \mathbb{H}^N. \end{cases} \tag{16}$$

$$\begin{cases} u_t(t,x) = A\Delta u(t,x) - \operatorname{div}(f(u(t)) \cdot X)(x), & x \in \mathbb{H}^N, t \geq 0; \\ u(0,x) = u_0(x), & x \in \mathbb{H}^N. \end{cases} \tag{17}$$

$$A = \frac{1}{2N}\int_0^\infty J(r) \cdot r^{N+1} dr \qquad X(x) = -\int_{T_x\mathbb{H}^N} \widetilde{G}(x,W) \cdot W \, dW$$

Weak formulations for non-local convection-diffusion problem

$u^\varepsilon \in C([0,\infty), L^2(\mathbb{H}^N)) \cap L^\infty([0,\infty)\times \mathbb{H}^N)$

$$\int_0^\infty \int_{\mathbb{H}^N} u^\varepsilon \partial_t \varphi dxdt + \int_{\mathbb{H}^N} u_0 \varphi(0) dx$$

$$= - \int_0^\infty \int_{\mathbb{H}^N} u^\varepsilon(t,x) \varepsilon^{-N-2} \int_{\mathbb{H}^N} J\left(\frac{d(x,y)}{\varepsilon}\right) (\varphi(t,y)-\varphi(t,x)) dydx$$
$$- \int_0^\infty \int_{\mathbb{H}^N} f(u^\varepsilon(t,x)) \int_{\mathbb{H}^N} G_\varepsilon(y,x) (\varphi(t,y)-\varphi(t,x)) dydx$$

$$= - \int_0^\infty \int_{\mathbb{H}^N} u^\varepsilon \widetilde{L}_J^\varepsilon(\varphi) dxdt - \int_0^\infty \int_{\mathbb{H}^N} f(u^\varepsilon) L^*_{G_\varepsilon}(\varphi) dxdt \quad (18)$$

for every $\varphi \in C_c^1([0,\infty), H^1(\mathbb{H}^N))$.

Weak formulations for local convection-diffusion problem

$u \in C([0,\infty), L^2(\mathbb{H}^N)) \cap L^2_{loc}([0,\infty), H^1(\mathbb{H}^N)) \cap L^\infty([0,\infty) \times \mathbb{H}^N)$

$$\begin{aligned} \int_0^\infty \int_{\mathbb{H}^N} u \cdot \partial_t \varphi(t,x) dx dt + \int_{\mathbb{H}^N} u_0 \varphi(0) dx \\ = \int_0^\infty \int_{\mathbb{H}^N} \left[A \nabla u \cdot \nabla \varphi - f(u) X \cdot \nabla \varphi \right] dx dt \end{aligned} \tag{19}$$

for every $\varphi \in C^1_c([0,\infty), H^1(\mathbb{H}^N))$.

One can prove uniqueness of the weak solution for the local problem.

Compactness result for the sequence $(u^\varepsilon)_\varepsilon$ I

Inspired from [Ignat et al., 2015] – Euclidean case.

M is a N-dimensional complete connected Riemannian Manifold.
$J : [0, \infty) \to [0, \infty)$ continuous function, $J(0) \neq 0$.

$$J_\varepsilon(r) = \frac{1}{\varepsilon^N} J\left(\frac{r}{\varepsilon}\right)$$

$(u^\varepsilon)_\varepsilon$ a bounded in $L^2([0, T] \times M)$

$$A = \liminf_{\varepsilon \to 0} \varepsilon^{-2} \int_0^T \int_M \int_M J_\varepsilon(d(x, y)) |u^\varepsilon(y) - u^\varepsilon(x)|^2 dV_g(x) dV_g(y) dt < \infty \tag{20}$$

Compactness result for the sequence $(u^\varepsilon)_\varepsilon$ II

Then

1. If $u^\varepsilon \rightharpoonup u$ in $L^2([0,T], L^2(M))$, then:

$$u \in L^2([0,T], H^1(M))$$

$$\int_0^T \|\nabla u(t)\|^2_{L^2(M)} dt \leq C \cdot A.$$

2. If $D \subseteq M$ open, bounded and

$$\|\partial_t u^\varepsilon\|_{L^2([0,T],H^{-1}(D))} \text{ uniformly bounded in } \varepsilon > 0$$

then $(u^\varepsilon)_\varepsilon$ converges strongly in $L^2([0,T] \times D)$ on a subsequence.

Convection-diffusion: convergence non-local $\to$ local I

Aim:

$$-\int_0^\infty \int_{\mathbb{H}^N} u^\varepsilon \widetilde{L}_J^\varepsilon(\varphi) dy dt - \int_0^\infty \int_{\mathbb{H}^N} f(u^\varepsilon) L^*_{G_\varepsilon}(\varphi) dy dt$$

$$\downarrow$$

$$A \int_0^\infty \int_{\mathbb{H}^N} \nabla u \cdot \nabla \varphi dx dt - \int_0^\infty \int_{\mathbb{H}^N} f(u) X \cdot \nabla \varphi dx dt$$

Convection-diffusion: convergence non-local $\rightarrow$ local II

From the compactness result:

$$f(u^{\varepsilon}) \rightharpoonup f(u) \text{ in } L^2([0,T] \times \mathbb{H}^N).$$

Since

$$\|L^*_{G_\varepsilon}(\varphi) - X(\varphi)\|_{L^2(\mathbb{H}^N)} \xrightarrow{\varepsilon \to 0} 0, \forall \varphi \in H^1(\mathbb{H}^N).$$

it follows that:

$$\int_0^\infty \int_{\mathbb{H}^N} f(u^\varepsilon) L^*_{G_\varepsilon}(\varphi) dydt \to \int_0^\infty \int_{\mathbb{H}^N} f(u) X \cdot \nabla \varphi dxdt$$

Convection-diffusion: convergence non-local $\to$ local III

We are left to prove:

$$\int_0^\infty \int_{\mathbb{H}^N} u^\varepsilon \widetilde{L}_J^\varepsilon(\varphi) dy dt \to -A \int_0^\infty \int_{\mathbb{H}^N} \nabla u \cdot \nabla \varphi dx dt$$

$$\|\widetilde{L}_J^\varepsilon(\varphi) - A\Delta\varphi\|_{L^2(\mathbb{H}^N)} \xrightarrow{\varepsilon \to 0} 0, \text{ for } \varphi \in C_c^\infty(\mathbb{H}^N)$$

It can be done similarly to the proof of:

$$\|L^*_{G_\varepsilon}(\varphi) - X(\varphi)\|_{L^2(\mathbb{H}^N)} \xrightarrow{\varepsilon \to 0} 0$$

Convection-diffusion: convergence non-local $\to$ local IV

In the general case, $\varphi \in C_c^1([0,\infty), H^1(\mathbb{H}^N))$, we need to impose on J the following:

Hypothesis

$J : [0,\infty) \to [0,\infty)$ *continuous and* $J(0) \neq 0$*.*

$$\int_0^\infty J(r) \cdot r^2 \cdot e^{2(N-1)r} dr < \infty, \qquad \textit{if } N \geq 3;$$

$$\int_0^\infty J(r) \cdot (r^2 \cdot e^{2r} + \sinh(r)) dr < \infty, \qquad \textit{if } N = 2.$$

Further directions of research

1. Study the convergence of $u^{\varepsilon} \to u$ in other L^p or Sobolev norms.
2. Study the long-time asymptotic behaviour of the difference

$$\|U(t) - u(t)\|_{L^p(\mathbb{H}^N)}$$

U is the solution of the non-local convection-diffusion equation with initial data u_0 and some fixed J and G.

u is the solution of the local convection-diffusion equation with the same initial data and the corresponding A and X.

References I

[Bandle et al., 2018] Bandle, C., del Mar González, M., Fontelos, M. A., and Wolanski, N. (2018).
A nonlocal diffusion problem on manifolds.
Communications in Partial Differential Equations, 43(4):652–676.

[Ignat et al., 2015] Ignat, L. I., Ignat, T. I., and Stancu-Dumitru, D. (2015).
A compactness tool for the analysis of nonlocal evolution equations.
SIAM Journal on Mathematical Analysis, 47(2):1330–1354.

[Ignat and Rossi, 2007] Ignat, L. I. and Rossi, J. D. (2007).
A nonlocal convection–diffusion equation.
Journal of Functional Analysis, 251(2):399–437.