Averaged turnpike property for differential equations with random constant coefficients

IX Partial differential equations, optimal design and numerics

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September, 2022

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Turnpike Property

Consider the Evolutive problem

$$\min_{u\in L^2(0,T;U)} \left\{ J^T(u) = \frac{1}{2}\left(\int_0^T \|u(t)\|^2 + \|Cx(t) - z\|^2 \, dt \right) \right\},$$

where $x(t)$ is the solution of

$$\begin{cases} x_t(t) + Ax(t) = Bu(t) & t \in (0,T), \\ x(0) = x_0. \end{cases}$$

Also, consider the stationary problem

$$\min_{u\in U} \left\{ J^s(u) = \frac{1}{2}\left(\|u\|^2 + \|Cx - z\|^2 \right) \right\},$$

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Assume that there exists

- (x^T, u^T): Optimal evolutive pair.
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Then we will say that the turnpike property holds if:
There exist two positive constants C and δ,
independent of the time horizon T, such that

$$\left\| x^T(t) - \overline{x} \right\|_X + \left\| u^T(t) - \overline{u} \right\|_U \leq C(e^{-\delta t} + e^{-\delta (T-t)}),$$

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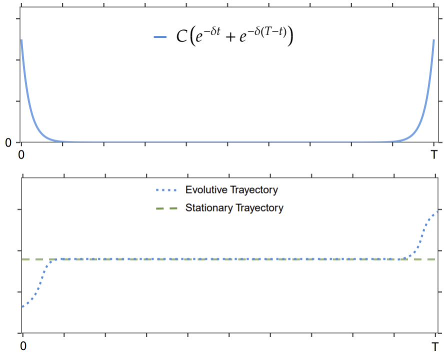

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2. Stationary control is not very different from evolutionary control.

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Contenidos

Random differential equations (RDE)

Following the book

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Random differential equations in science and engineering.
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Let us consider the probability space $(\Omega, \mathcal{F}, \mu)$ with $\omega \in \Omega$

1. RDE with random initial condition

$$\begin{cases} x_t(t,\omega) = f(x(t,\omega),t), \\ x(t_0) = x_0(\omega). \end{cases}$$

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3. Nonhomogeneous RDE with random initial condition

$$\begin{cases} x_t(t,\omega) = f(x(t,\omega),t) + g(t,\omega), \\ x(t_0) = x_0(\omega). \end{cases}$$

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Control system of random differential equations

Some possibilities can be

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Evolutive random problem

Consider the following evolutive problem with random coefficients

$$\begin{cases} x_t(t,\omega) + A(\omega)x(t,\omega) = B(\omega)u(t) \quad t \in (0,T), \\ x(0) = x_0, \end{cases} \tag{1}$$

and the minimization problem

$$\min_{u \in L^2(0,T;\mathbb{R}^m)} \left\{ J^T(u) = \frac{1}{2}\left(\int_0^T \|u(t)\|^2_{\mathbb{R}^m}\, dt + \int_0^T \int_\Omega \|C(\omega)x(t,\omega) - z\|^2_{\mathbb{R}^n}\, d\mu dt \right) \right\}, \tag{2}$$

with $x(t,\omega)$ the solution of (1) and $z \in \mathbb{R}^n$.

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with $x(t,\omega)$ the solution of (1) and $z \in \mathbb{R}^n$.

Stationary random problem

Also, consider the following minimization stationary problem

$$\min_{u\in\mathbb{R}^m}\left\{ J^s(u) = \frac{1}{2}\left(\|u\|^2_{\mathbb{R}^m} + \int_\Omega \|C(\omega)x(\omega) - z\|^2_{\mathbb{R}^n}\, d\mu \right)\right\}, \tag{3}$$

with $x(\omega)$ the solution of $A(\omega)x(\omega) = B(\omega)u$.

Turnpike inequality

Let us assume that

- (x^T, u^T) : The optimal pair of the **evolutive** problem.
- $(\overline{x}, \overline{u})$: The optimal pair of the **stationary** problem.

We will say that the average turnpike property hold, if the optimal pairs satisfy in average the exponential turnpike property, that is,

$$\left\| x^T(t) - \overline{x} \right\|_{L^2(\Omega;\mathbb{R}^n)} + \left\| u^T(t) - \overline{u} \right\|_{\mathbb{R}^m} \leq \mathscr{C}(e^{-\delta(T-t)} + e^{-\delta t}),$$

for all $t \in (0, T)$. In particular, the previous inequality implies

$$\left\| \mathbb{E}(x^T(t)) - \mathbb{E}(\overline{x}) \right\|_{\mathbb{R}^n} + \left\| u^T(t) - \overline{u} \right\|_{\mathbb{R}^m} \leq \mathscr{C}(e^{-\delta(T-t)} + e^{-\delta t}),$$

where $\mathbb{E}$ denote the mathematical expectation.

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Problem formulation

Let us consider the previews optimal control problems and suppose that $A, C \in C(\Omega; \mathcal{L}(\mathbb{R}^n))$ and $B \in C(\Omega; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$, and are uniformly bounded. Also $u \in L^2(0, T; \mathbb{R}^m)$ and $x_0 \in \mathbb{R}^n$.

Theorem (Existence, Uniqueness and Characterization of the optimal control)

There exist a unique optima control $u^T \in L^2(0, T; \mathbb{R}^m)$ for the problem of minimize the functional J^T, and unique optimal state x^T associated to u^T. Furthermore, the optimal control u^T can be characterized by

$$u^T(t) = -\int_\Omega B^*(\omega)\varphi^T(t,\omega)d\mu, \qquad \begin{cases} -\varphi_t^T + A^*(\omega)\varphi^T = C^*(Cx^T - z), \\ \varphi^T(T) = 0. \end{cases} \tag{4}$$

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Necessary hypotheses

The following two hypotheses allow us to conclude the turnpike property

- **Hypotheses 1:** There exist an feedback operator $K_C \in C(\Omega; \mathcal{L}(\mathbb{R}^n))$ and a positive constant ζ such that

$$\langle (A + K_C C)v, v\rangle_{L^2(\Omega;\mathbb{R}^n)} \geq \zeta \|v\|^2_{L^2(\Omega;\mathbb{R}^n)} \quad \forall v \in L^2(\Omega; \mathbb{R}^n).$$

- **Hypotheses 2:** There exist two constants $\lambda > 0$ and $\kappa \in \mathbb{R}$ such that

$$\langle A^* v, v\rangle_{L^2(\Omega;\mathbb{R}^n)} + \kappa \left\| \int_\Omega B^* v d\mu \right\|^2_{\mathbb{R}^m} \geq \lambda \|v\|^2_{L^2(\Omega;\mathbb{R}^n)} \quad \forall v \in L^2(\Omega; \mathbb{R}^n).$$

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Evolutive estimations

Assuming the previous hypotheses, there exist a constant $K > 0$ such that

- Evolutive primal inequality

$$\left\| x^T(T) \right\|^2_{L^2(\Omega;\mathbb{R}^n)} + \int_0^T \left\| x^T \right\|^2_{L^2(\Omega;\mathbb{R}^n)} dt$$
$$\leq K \left(\int_0^T \|u\|^2_{\mathbb{R}^m} + \left\| C(\omega) x^T \right\|^2_{L^2(\Omega;\mathbb{R}^n)} dt + \|x_0\|^2_{\mathbb{R}^n} \right).$$

- Evolutive adjoint inequality

$$\left\| \varphi^T(0) \right\|^2_{L^2(\Omega;\mathbb{R}^n)} + \int_0^T \left\| \varphi^T \right\|^2_{L^2(\Omega;\mathbb{R}^n)} dt$$
$$\leq K \left(\int_0^T \|u\|^2_{\mathbb{R}^m} + \left\| C(\omega) x^T - z \right\|^2_{L^2(\Omega;\mathbb{R}^n)} dt \right).$$

Stacionary estimations

- Stationary primal inequality

$$\|x\|^2_{L^2(\Omega;\mathbb{R}^n)} \leq K\left(\|Ax\|^2_{L^2(\Omega;\mathbb{R}^n)} + \|Cx\|^2_{L^2(\Omega;\mathbb{R}^n)} \right) \quad \forall x \in L^2(\Omega;\mathbb{R}^n).$$

- Stationary adjoint inequality

$$\|p\|^2_{L^2(\Omega;\mathbb{R}^n)} \leq K\left(\|A^*p\|^2_{L^2(\Omega;\mathbb{R}^n)} + \left\| \int_\Omega B^* p d\mu \right\|^2_{\mathbb{R}^m} \right) \quad \forall p \in L^2(\Omega;\mathbb{R}^n).$$

Stationary problem

As a consequence of the energy estimations shown above, we have

Theorem (Existence and uniqueness of the stationary problem)

Under the **Hypothesis 1** the functional J^s has a unique minimum $\overline{u} \in \mathbb{R}^m$. Moreover, there exist a unique optimal trajectory $\overline{x} \in L^2(\Omega; \mathbb{R}^n)$ associated to $\overline{u}$.

Stationary problem

Theorem (Adjoint stationay system)

There exist $\overline{\varphi} \in L^2(\Omega;\mathbb{R}^n)$ such that for every $\omega \in \Omega$ satisfy

$$A^*(\omega)\overline{\varphi} = C^*(\omega)(C(\omega)\overline{x} - z).$$

Moreover, $\overline{u}$ the unique solution of the stationary problem is characterized by

$$\langle \overline{u}, v\rangle_{\mathbb{R}^m} + \langle \overline{\varphi}, Bv\rangle_{L^2(\Omega;\mathbb{R}^n)} = 0 \quad \forall v \in D.$$

where

$$D = \{v \in \mathbb{R}^m \ : \ B(\omega)v \in Ran(A(\omega)), \text{ for each } \omega \in \Omega\}$$

Main theorems

In the following, we present our first main result

Theorem (Integral turnpike property)

Under the **Hypotheses 1** and **2** the optimal pair (x^T, u^T) and $(\overline{x}, \overline{u})$ satisfy

$$\frac{1}{T}\int_0^T x^T(t,\cdot)dt \longrightarrow \overline{x}(\cdot) \quad en \;\; L^2(\Omega;\mathbb{R}^n),$$

$$\frac{1}{T}\int_0^T u^T(t)dt \longrightarrow \overline{u} \quad en \;\; \mathbb{R}^m.$$

when $T \to \infty$.

Main theorems

Theorem (Exponential turnpike property)[M.H, S.Zamorano, R.Lecaros]

Let us assume that **Hypothesis 1** and **2** hold. Let (u^T, x^T, φ^T) be the solution of the evolutive system and $(\overline{u}, \overline{x}, \overline{\varphi})$ the corresponding stationary solution of the stationary system. Then, there exists two positive constants K and δ (independent of T) such that

$$\left\|x^T(t) - \overline{x}\right\|^2_{L^2(\Omega;\mathbb{R}^n)} + \left\|\varphi^T(t) - \overline{\varphi}\right\|^2_{L^2(\Omega;\mathbb{R}^n)} + \left\|u^T(t) - \overline{u}\right\|_{\mathbb{R}^m} \leq K(e^{-\delta(T-t)} + e^{-\delta t}),$$

for every $t \in (0, T)$. In particular, we obtain an averaged exponential turnpike as follows

$$\left\|\mathbb{E}(x^T) - \mathbb{E}(\overline{x})\right\|_{\mathbb{R}^n} + \left\|\mathbb{E}(\varphi^T) - \mathbb{E}(\overline{\varphi})\right\|_{\mathbb{R}^n} + \left\|u^T(t) - \overline{u}\right\|_{\mathbb{R}^m} \leq K(e^{-\delta(T-t)} + e^{-\delta t}),$$

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Proof (Sketch)

L. Grüne and M. Schaller and A. Schiela (2019).
Sensitivity Analysis of Optimal Control for a Class of Parabolic PDEs Motivated by Model Predictive Control.
SIAM Journal on Control and Optimization.

- **Step 1:** Let $m = x^T - \overline{x}$ and $n = \varphi^T - \overline{\varphi}$. We write the system that satisfy m, n in a matrix structure

$$\underbrace{\begin{pmatrix} -C^*C & -\frac{d}{dt} + A^* \\ 0 & E_T \\ \frac{d}{dt} + A & B\left(\int_\Omega B^* \cdot d\mu\right) \\ E_0 & 0 \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} m \\ n \end{pmatrix}}_{z} = \underbrace{\begin{pmatrix} 0 \\ n_T \\ 0 \\ m_0 \end{pmatrix}}_{\mathcal{Y}},$$

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$$\Lambda \mathcal{Z} = \mathcal{Y}.$$

We prove that Λ^{-1} is well defined and that there exists $K > 0$ independent of the time horizon such that

$$\left\|\Lambda^{-1}\right\|_{\mathcal{L}((L^2(\Omega;\mathbb{R}^n))^2,(\mathcal{X})^2)} < K,$$

where $\mathcal{X} = C([0,T]; L^2(\Omega;\mathbb{R}^n))$.

Proof (Sketch)

- **Step 2:** We consider a new variable change

$$\hat{m} = \frac{m}{e^{-\delta(T-t)} + e^{-\delta t}}, \quad \hat{n} = \frac{n}{e^{-\delta(T-t)} + e^{-\delta t}},$$

and we prove that there exist $K > 0$ independent of the time horizon such that

$$\|\hat{m}\|_{\mathcal{X}} + \|\hat{n}\|_{\mathcal{X}} \leq K,$$

$\mathcal{X} = C([0, T]; L^2(\Omega; \mathbb{R}^n))$.

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$$\|\hat{m}(t)\|_{L^2(\Omega,\mathbb{R}^n)} + \|\hat{n}(t)\|_{L^2(\Omega,\mathbb{R}^n)} \leq \|\hat{m}\|_{\mathcal{X}} + \|\hat{n}\|_{\mathcal{X}} \leq K,$$

$\mathcal{X} = C([0, T]; L^2(\Omega; \mathbb{R}^n))$. We conclude the proof by returning to the original variables.

Numerical Simulations

Consider the minimization problem

$$\min_{u\in L^2(0,T;\mathbb{R})}\left\{J^T(u)=\frac{1}{2}\left(\int_0^T\|u(t)\|^2_{\mathbb{R}}\,dt+\int_0^T\int_\Omega\|C(\omega)x(t,\omega)-z\|^2_{\mathbb{R}^2}\,d\mu dt\right)\right\},$$

subject to

$$\begin{cases}x_t(t,\omega)+A(\omega)x(t,\omega)=B(\omega)u(t) & t\in(0,T),\\ x(0)=x_0\in\mathbb{R}^2,\end{cases}$$

Here we take

$$A(\omega)=\alpha(\omega)\cdot\begin{pmatrix}2 & -5\\ 5 & 0{,}5\end{pmatrix},\quad B(\omega)=\beta(\omega)\cdot\begin{pmatrix}5\\ 7\end{pmatrix},\quad C(\omega)=\lambda(\omega)\cdot\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad z=\begin{pmatrix}1\\ 2\end{pmatrix}$$

$$\alpha,\ \beta,\ \lambda\sim Bin(7,1/2),\qquad T=300.$$

Numerical Simulations

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$$\min_{u\in L^2(0,T;\mathbb{R})}\left\{ J^T(u)=\frac{1}{2}\left(\int_0^T \|u(t)\|_{\mathbb{R}}^2\,dt+\int_0^T\int_\Omega \|C(\omega)x(t,\omega)-z\|_{\mathbb{R}^2}^2\,d\mu dt\right)\right\},$$

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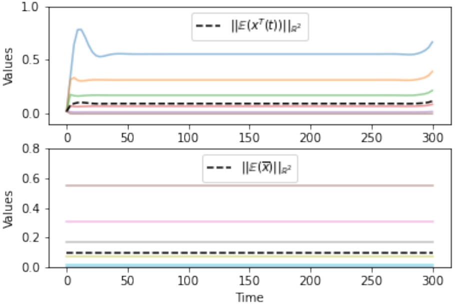
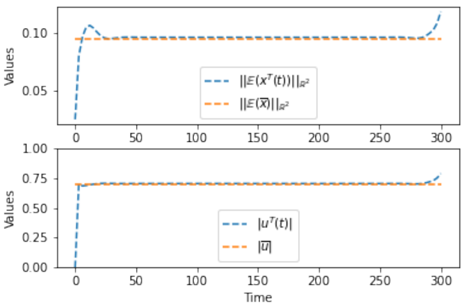
Numerical Simulations

Using Gekko library in Python, we obtain the following simulations



(a) In color, the norm in space of the different realizations of the random variables.



(b) 1) Average of the evolutive and stationary states. 2) Optimal controls

Numerical Simulations

$$\left\|\mathbb{E}(x^T(t)) - \mathbb{E}(\overline{x})\right\|_{\mathbb{R}^n} \leq \left\|x^T(t) - \overline{x}\right\|_{L^2(\Omega;\mathbb{R}^n)} + \left\|u^T(t) - \overline{u}\right\|_{\mathbb{R}^m} \leq K(e^{-\delta(T-t)} + e^{-\delta t}).$$

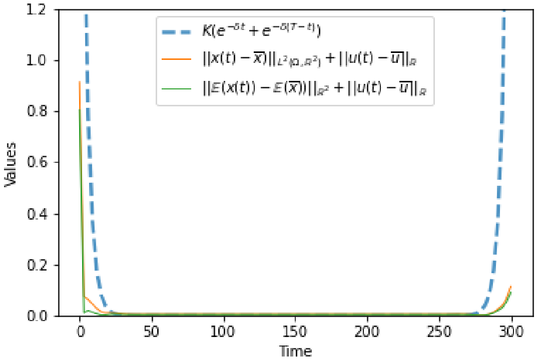


Figura: Averaged turnpike property, with constants $K = 3{,}5$ and $\delta = 0{,}25$.

Article

M. Hernández and R. Lecaros and S. Zamorano (2022).
Averaged turnpike property for differential equations with random constant coefficients.
Mathematical Control and Related Fields.

Index

Open problems

1. **Turnpike and average control:** We say that the system

$$\begin{cases} x_t + A(\omega)x = B(\omega)u \quad t \in (0, T), \\ x(0) = x_0, \end{cases}$$

is average controllable, if for every x_0 and x_1 exist u such that $\mathbb{E}(x(T)) = x_1$. Under average controllability assumptions, is possible to prove the average turnpike?

2. It is possible to prove this property when $A(\omega)$ is an unbounded operator, and $B(\omega)$ a bounded for each $\omega \in \Omega$?

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Thanks