Transport-Stokes equations as mean-field limit of inertialess suspensions

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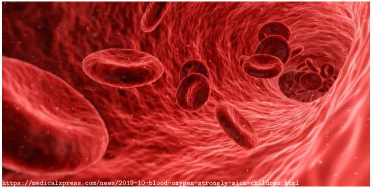
Suspensions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Microscopic dynamics for spherical particles in a Navier-Stokes flow

- N particles $B_i \subset \mathbb{R}^3$ pairwise disjoint, center of mass X_i, translational/rotational velocities V_i, Ω_i, mass density ρ_p.
- Incompressible fluid with constant mass density ρ_f, viscosity μ, velocity u, pressure p.
- Gravitational acceleration g.

$$\rho_f(\partial_t u + u \cdot \nabla u) - \mu \Delta u + \nabla p = 0, \quad \operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i},$$

$$u(x) \to 0 \quad \text{as } x \to \infty, \qquad u = V_i + \Omega_i \times (x - X_i) \quad \text{in } B_i.$$

- Newton's laws for the particles

$$\rho_p |B_i| \frac{d}{dt} V_i = (\rho_p - \rho_f)|B_i| g + \int_{\partial B_i} \sigma[u,p] n \, \mathrm{d}\mathcal{H}^2,$$

$$\frac{d}{dt}(I_i \Omega_i) = \int_{\partial B_i} (x - X_i) \times \sigma[u,p] n \, \mathrm{d}\mathcal{H}^2$$

- $I_i \in \mathbb{R}^{3\times 3}$ moment of inertia of B_i,
- $\sigma[u,p] = \mu(\nabla u + (\nabla u)^T) - p\mathrm{Id}$ fluid stress.

A single spherical particle

- Single particle $B = B_R(0)$ with velocity V

$$
\begin{aligned}
-\Delta u + \nabla p = 0, \quad \operatorname{div} u = 0 \qquad & \text{in } \mathbb{R}^3 \setminus B, \\
u = V \qquad & \text{in } B, \\
u(x) \to u_\infty \qquad & \text{as } |x| \to \infty.
\end{aligned}
$$

- Explicit solution (Stokes 1845): In $\mathbb{R}^3 \setminus B$

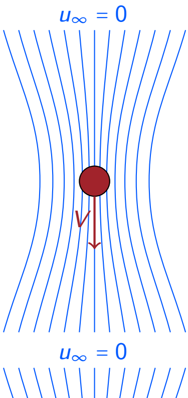
$$
u = u_\infty + 6\pi R(V - u_\infty)\left(\Phi + \frac{R^2}{6}\Delta\Phi\right)
$$

where Φ, is the fundamental solution

$$
-\Delta\Phi + \nabla P = \mathrm{Id}\delta
$$

- Equivalent problem

$$
\begin{aligned}
-\Delta u + \nabla p = 6\pi R(V - u_\infty)\delta_{\partial B} \quad & \text{in } \mathbb{R}^3, \\
\operatorname{div} u = 0, \qquad u(x) \to u_\infty \quad & \text{as } |x| \to \infty,
\end{aligned}
$$



The Vlasov-Navier-Stokes equations

Formally, the limit for many small particles with identical radii R is given by

$$\partial_t f + v \cdot \nabla_x f + \lambda \operatorname{div}_v \left(\hat{g} f + \frac{9}{2} \gamma (u - v) f \right) = 0,$$

$$\mathrm{Re}(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla p = 6\pi\gamma \int_{\mathbb{R}^3} (v - u) f \, \mathrm{d}v, \quad \operatorname{div} u = 0, \quad \text{"Brinkman eq."} \qquad \text{(VS)}$$

- $\gamma = \frac{NR}{L}$ "interaction strength", where L is the diameter of the particle cloud,
- Stokes number $\mathrm{St} = \frac{1}{\gamma\lambda} = \frac{\rho_p(\rho_p - \rho_f)|g| N R^5}{L^2 \mu^2}$,
- Reynolds number $\mathrm{Re} = \frac{NR^3 \rho_f (\rho_f - \rho_p)|g|}{|\mu|^2}$.

Derivation of the Brinkman equations:
Allaire '90, Desvillettes-Golse-Ricci '09, Feireisl-Namlyeyeva-Nečasová '16, Hillairet-Moussa-Sueur '19, Giunti-H. '19, Carrapatoso-Hillairet '20, H.-Jansen '20, ...

Formal derivation from Boltzman equations: Bernard-Desvillettes-Golse-Ricci '16

Derivation from mollified approximated dynamics: Flandoli-Leocata-Ricci '18

Derivation for almost inertialess particles: H.-Schubert '22

Theorem (H. '18)

Let $\gamma > 0$ and $\mathrm{Re} = 0$. Assume $f_0 \in W^{1,\infty}(\mathbb{R}^3 \times \mathbb{R}^3)$ is compactly supported. Let

Microscopic dynamics of inertialess particles in a Stokes flow

$$-\Delta u_N + \nabla p_N = 0, \quad \operatorname{div} u_N = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N B_i,$$

$$u_N = V_i + (x - X_i) \times \Omega_i \quad \text{in } B_i, \quad 1 \le i \le N, \qquad \frac{\mathrm{d}}{\mathrm{d}t} X_i(t) = V_i,$$

$$\int_{\partial B_i} \sigma[u_N, p_N] n \, \mathrm{d}\mathcal{H}^2 = -\frac{g}{N}, \qquad \int_{\partial B_i} \sigma[u_N, p_N] n \times (x - X_i) \, \mathrm{d}\mathcal{H}^2 = 0.$$

The particle velocities V_i and Ω_i are not given but determined though solving the fluid equations.

Weak formulation: Find $u_N \in V$,

$$V := \left\{ v \in \dot{H}^1(\mathbb{R}^3) : \operatorname{div} v = 0, \; Dv = 0 \text{ in } \cup_i B_i \right\},$$

such that

$$\int_{\mathbb{R}^3} \nabla u_N : \nabla v = 2 \int_{\mathbb{R}^3} D u_N : Dv = \sum_i \int_{\partial B_i} v = \sum_i v(X_i) \quad \text{for all } v \in V.$$

Rigorous derivation of the transport-Stokes system

Jabin-Otto '04: If $\Lambda(0) := \frac{RN^{2/3}}{d_{\min}(0)} \leq \varepsilon_0$, then

$$|V_i - V_{\mathrm{St}}| \leq C\Lambda(0)|V_{\mathrm{St}}| \text{ for all } 1 \leq i \leq N.$$

Note that $d_{\min} \leq CLN^{-1/3}$, thus $\Lambda(0) \geq \gamma = \frac{NR}{L}$.

H. '18: Assume $\rho_N(0) = \frac{1}{N}\sum_i \delta_{X_i(0)} \to \rho_0$ and

$$d_{\min} \geq cN^{-1/3}, \tag{A1}$$

$$\phi \log N := NR^3 \log N \to 0, \tag{A2}$$

$$NR \to \gamma \in (0, \infty]. \tag{A3}$$

Then $\rho_N(t) \to \rho(t)$ which solves the transport-Stokes system.

$$\begin{aligned} \partial_t \rho + (u + (6\pi\gamma)^{-1}\hat{g}) \cdot \nabla_x \rho &= 0, \\ -\Delta u + \nabla p = \rho g, \qquad \operatorname{div} u &= 0. \end{aligned} \tag{TS}$$

Mecherbet '19: Relaxation of (A1) in the case $\gamma \leq \varepsilon_0$, quantitative convergence in Wasserstein distance, particle rotations included.

Additional $O(\phi_N)$ correction due to an increased effective viscosity

Theorem (H. & Schubert '20)

Assume

$$d_{\min} \geq cN^{-\frac{1}{3}},$$
$$\phi_N \log N \to 0,$$
$$\mathcal{W}_\infty(\rho_N(0), \rho_0) = o(\phi_N) \quad \textit{for some } \rho_0 \in W^{1,\infty}(\mathbb{R}^3).$$

where $\mathcal{W}_p$, $1 \leq p \leq \infty$ denotes the Wasserstein distance.
Let ρ be the unique solution to

$$\partial_t \rho + (u_{eff} + (6\pi NR)^{-1} g) \cdot \nabla \rho = 0, \quad \rho(0, \cdot) = \rho_0,$$
$$\operatorname{div}\left(2\left(1 + \frac{5}{2}\phi_N \rho\right) Du_{eff}\right) + \nabla p = \rho g, \quad \operatorname{div} u_{eff} = 0.$$

Then, for all $1 \leq p < \infty$ and all $T > 0$, for all N sufficiently large and all $t \leq T$

$$\mathcal{W}_p(\rho_N(t), \rho(t)) \leq C\left(\phi_N^2 |\log \phi_N| + \mathcal{W}_p(\rho_N(0), \rho(0))\right) e^{Ct}.$$

Moreover, for $q < 3$ and $p > \max\{1, \frac{3q}{3+q}\}$

$$\|u_N(t) - u_{eff}\|_{L^q_{\mathrm{loc}}} \leq C\left(\phi_N^2 |\log \phi_N| + \mathcal{W}_p(\rho_N(0), \rho(0))\right) e^{Ct}$$

First step: explicit approximation of the particle velocities

Using the method of reflections, one can show

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) &= u_N(X_i) \\
&\approx \frac{g}{6\pi R N} + \frac{1}{N} \sum_{i \neq j} \Phi(X_i - X_j) g - 5\phi_N \frac{1}{N^2} \sum_{j \neq i} \sum_{k \neq j} D\Phi(X_i - X_j) D\Phi(X_j - X_k) g,
\end{aligned}
$$

where

$$
\Phi(x) = \frac{1}{8\pi} \left(\frac{Id}{|x|} + \frac{x \otimes x}{|x|^3} \right)
$$

is the fundamental solution of the Stokes equations.

The convergence of the method of reflections for u_N in $L^\infty(\mathbb{R}^3)$ relies on the assumption $\phi_N \log N \ll 1$.
The estimates for the method of reflections are based on a previous analysis in the case of Dirichlet boundary conditions in [H.-Velázquez '18].

Second step: adaptation of a result by Hauray

Let K satisfies $\mathrm{div} K = 0$, $|K| + |x||\nabla K| \leq C|x|^{-\alpha}$ for some $\alpha < d - 1$.

Theorem (Hauray '09)

Let

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j).$$

Let ρ be the solution to $\partial_t \rho + (K * \rho) \cdot \nabla \rho = 0$ with initial datum $\rho_0 \in L^\infty \cap \mathcal{P}$. Denote $W_\infty(t) = W_\infty(\rho_N(t), \rho(t))$. Then,

$$\frac{(W_\infty(0))^d}{(d_{\min}(0))^{1+\alpha}} \to 0 \quad \Rightarrow \quad \forall T > 0 \ \exists N_0 \in \mathbb{N} \ \forall N > N_0 \ \forall t \leq T \quad W_\infty(t) \leq W_\infty(0) e^{Ct}.$$

Theorem (H.-Schubert '20 (rough statement))

Let

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j) + \phi_N(X_i) + E_i(t).$$

Assume $\mathrm{div}\, \phi_N = 0$ and let ρ be the solution to $\partial_t \rho + (K * \rho + \phi_N) \cdot \nabla \rho = 0$ with initial datum $\rho_0 \in L^\infty \cap \mathcal{P}$. Assume there exists a monotone function $e_N(t)$ such that

$$\forall \lambda > 0 \ \exists N_0 \in \mathbb{N} \ \forall N > N_0 \quad \frac{d_{\min}(0)}{d_{\min}(t)} + \frac{W_\infty(t)}{W_\infty(0) + e_N(t)} \leq \lambda \quad \Rightarrow \quad E_i(t) \leq e_N(t).$$

Then,

$$\forall t > 0 \ \frac{(W_\infty(0) + e_N(t))^{d-(1+\alpha)}}{(d_{\min}(0))^{1+\alpha} N^{(1+\alpha)/d}} \to 0 \ \Rightarrow \ \forall T > 0 \ldots \quad W_\infty(t) \leq C(W_\infty(0) + e_N(t)) e^{Ct}.$$

First application of the abstract theorem

Recall

$$\frac{\mathrm{d}}{\mathrm{d}t}X_i(t) \approx \frac{g}{6\pi RN} + \frac{1}{N}\sum_{i\neq j}\Phi(X_i - X_j)g - 5\phi_N\frac{1}{N^2}\sum_{j\neq i}\sum_{k\neq j}D\Phi(X_i - X_j)D\Phi(X_j - X_k)g$$

Treat last term as error to deduce

$$\mathcal{W}_\infty(\rho_N(t), \tau(t)) \le C(\phi_N + W_\infty(\rho_N(0), \rho_0))e^{Ct}.$$

where τ is the solution to

$$\begin{aligned} \partial_t\tau + (v + (6\pi\gamma)^{-1}g)\cdot\nabla\tau = 0, \quad \tau(0,\cdot) = \rho_0, \\ -\Delta v + \nabla p = \tau g, \quad \operatorname{div} v = 0 \end{aligned}$$

Using this,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}X_i(t) &\approx \frac{g}{6\pi RN} + \frac{1}{N}\sum_{i\neq j}\Phi(X_i - X_j)g - 5\phi_N D\Phi * (\tau(D\Phi g * \tau))(X_i) \\ &= \frac{g}{6\pi RN} + \frac{1}{N}\sum_{i\neq j}\Phi(X_i - X_j)g - 5\phi_N D\Phi * (\tau Dv)(X_i) \end{aligned}$$

Conclusion of the argument

We apply again the abstract theorem, this time to

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) \approx \frac{g}{6\pi RN} + \frac{1}{N} \sum_{i \neq j} \Phi(X_i - X_j) g - 5\phi_N D\Phi * (\tau D v)(X_i)$$

We observe that $5\phi_N D\Phi * (\tau D v)(X_i)$ is the solution to the Stokes equations with source term $5\phi_N \operatorname{div}(\tau D v)$.

Thus we obtain as the mean field limit

$$\partial_t \tilde{\rho} + (\tilde{u} + (6\pi NR)^{-1} g) \cdot \nabla \tilde{\rho} = 0, \quad \tilde{\rho}(0, \cdot) = \rho_0,$$
$$\operatorname{div}(2D\tilde{u} + 5\phi_N \tau D v) + \nabla p = \tilde{\rho} g, \quad \operatorname{div} \tilde{u} = 0.$$

Finally, we compare this to the desired limit system

$$\partial_t \rho + (u_{eff} + (6\pi NR)^{-1} g) \cdot \nabla \rho = 0, \quad \rho(0, \cdot) = \rho_0,$$
$$\operatorname{div}(2D u_{eff} + 5\phi_N \rho D u_{eff}) + \nabla p = \rho g, \quad \operatorname{div} u_{eff} = 0.$$

and show that $\|\tilde{u} - u_{eff}\| + W_p(\tilde{\rho}, \rho) \leq C\phi^2$.

Thank you!