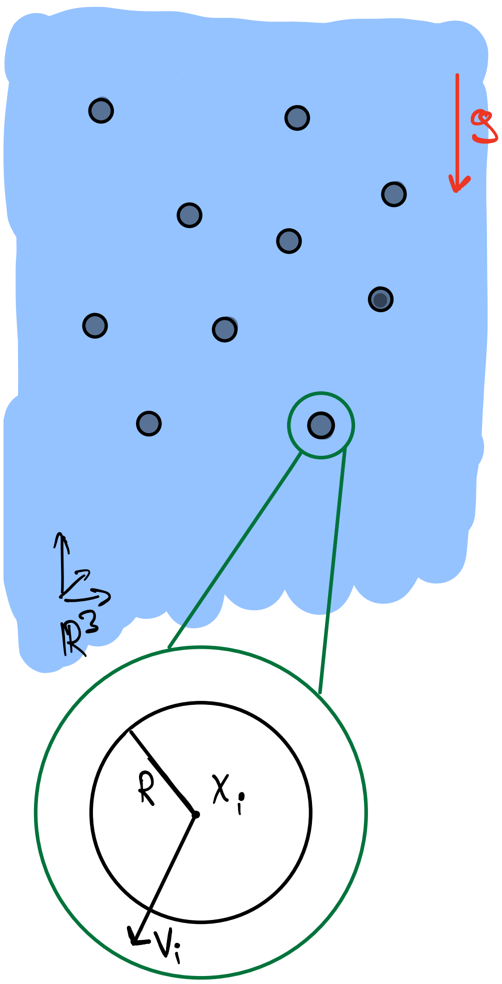
Mean-field limits for sedimentation of particles with inertia

joint work w/ Richard Höfer

Setting



- N identical particles in Stokes fluid
- positions $(X_i)_i$, velocities $(V_i)_i$
- $B_i = B(X_i, R)$
- Fluid velocity $u : \mathbb{R}^3 \setminus \bigcup_i B_i \to \mathbb{R}^3$

$$\boxed{\text{Fluid} \quad \begin{cases} -\Delta u + \nabla p = 0 & \text{in } \mathbb{R}^3 \setminus \bigcup_i B_i \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \setminus \bigcup_i B_i \\ u = V_i & \text{on } \overline{B_i} \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}}$$

Stress : $\sigma[u] = -p\,\mathrm{Id} + (\nabla u + (\nabla u)^T)$

$$-\Delta u + \nabla p = 0 \iff \operatorname{div} \sigma = 0$$

Forces

Force of particle B_i on fluid: $F_i = \int_{\partial B_i} \sigma[u]n \, dS$

$$\boxed{\text{particle dynamics} \begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = \lambda \left(g - \frac{\gamma}{R} F_i \right) \end{cases}}$$

$\gamma = RN$ – interaction strength

$\lambda = \dfrac{\mu^2}{\rho_p(\rho_p - \rho_f)\ \phi^2\ |g| L^3}$, λ^{-1} – strength of inertial effects

$$\rho_N = \frac{1}{N} \sum_i \delta_{X_i}$$

Consider the limit: $N \to \infty$, $R \to 0$, $\gamma \to \gamma_* \in (0, \infty)$, $\lambda^{-1} \to 0$.

Inertialess Dynamics

For fixed N, formally take $\lambda^{-1} \to 0$: $\quad F_i = \frac{1}{N} g$

Sedimentation BC

V_i unknown!

$$\begin{cases} -\Delta u + \nabla p = 0 & \text{in } \mathbb{R}^3 \setminus \bigcup_i B_i \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \setminus \bigcup_i B_i \\ u = V_i & \text{on } \overline{B_i} \\ \int_{\partial B_i} \sigma[u] n \, dS = \frac{1}{N} g & i = 1, \dots, N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

Resistance matrix $\quad R(X) : (V_i)_i \mapsto (F_i)_i$

$$R(X)[V] \cdot V = F \cdot V = \sum_i F_i \cdot V_i = \frac{1}{2} \int_{\mathbb{R}^3} |(\nabla u + (\nabla u)^T)|^2 \, dx$$

Stokes-Transport equation

If $\rho_N(0) \to \rho_{*,0}$, $\gamma \to \gamma_* \in (0, \infty]$ we expect $\rho_N \to \rho_*$ with

$$\boxed{(\text{Stokes-Transport}) \begin{cases} \partial_t \rho_* + (\underbrace{u_*}_{\text{mean-field}} + \underbrace{(6\pi \gamma_*)^{-1} g}_{\text{self-interaction}}) \cdot \nabla \rho_* = 0 \\ \rho_*(0) = \rho_{*,0} \\ u_* = St^{-1} * (\rho g) \end{cases}}$$

$$u_* = St^{-1} * (\rho g) \iff \begin{cases} -\Delta u_* + \nabla p = \rho_* g & \text{in } \mathbb{R}^3 \\ \operatorname{div} u_* = 0 & \text{in } \mathbb{R}^3 \end{cases}$$

What can we expect for $\lambda^{-1} \searrow 0$?

A single particle with inertia

$u_\infty = 0$

$$\dot V = \frac{1}{\frac{4\pi}{3}\rho_p R^3}\left[\frac{4\pi}{3}(\rho_p-\rho_f)R^3 g - 6\pi R\mu V\right]$$

$$= \frac{(\rho_p-\rho_f)}{\rho_p} g - \underbrace{\frac{9}{2}\frac{\mu}{\rho_p R^2}}_{\lambda} V$$

$$= \lambda\left(\frac{2}{9}\frac{(\rho_p-\rho_f)}{\mu} g R^2 - V\right)$$

$$\Rightarrow \quad V(t) = e^{-\lambda t}\left(V_0 - \frac{2}{9}\frac{(\rho_p-\rho_f)}{\mu} g R^2\right) + \underbrace{\frac{2}{9}\frac{(\rho_p-\rho_f)}{\mu} g R^2}_{V_\infty}$$

$$X(T) = X_0 + \int_0^T V(t) = X_0 + V_\infty t + (V_0 - V_\infty)\int_0^T e^{-\lambda t}$$

$$= \underbrace{X_0 + V_\infty t}_{\text{inertialess particle}} + \frac{1}{\lambda}(V_0 - V_\infty)\left(1 - e^{\lambda T}\right)$$

$$\Rightarrow \quad \left|X(T) - (X_1 + V_\infty t)\right| \le |X_0 - X_1| + \frac{1}{2}|V_0 - V_\infty|$$

Result

Assumptions:

(A1) $\min_{i,j} |X_i(0)-X_j(0)| \geq c N^{-\frac{1}{3}}$ → minimal distance

(A2) $|V_i(0)-V_j(0)| \leq 3\pi \gamma_N \lambda_N |X_i(0)-X_j(0)|$ → Lipschitz-velocity

(A3) $\max_i |V_i(0)| \leq C$ → bounded velocity

(A4) $\lambda_N^{-1} \to 0$ → vanishing inertia

(A5) $\gamma_N = NR \to \gamma_* \in (0,\infty)$ → finite interaction

Theorem (Höfer–S. '22): $\forall T \geq 0\ \exists N_0 \in \mathbb{N} : \forall N \geq N_0, t \leq T$

$$W_2(\rho_N, \rho_*) \leq \Big[\underbrace{W_2(\rho_N(0), \rho_0)}_{\text{initial data}} + \frac{1}{\lambda} C \Big(\underbrace{N^{-\frac{1}{2}} |V(0) - \overline{V}(0)|_2}_{\text{relaxation}} + \underbrace{t}_{\text{error propagation}} \Big) + \underbrace{CNR^3}_{\substack{\text{changed viscosity effect} \\ =\phi}} \Big] e^{Ct}$$

where W_2 – Wasserstein-2 distance

$$\overline{V}(0) = R^{-1}(X) \frac{\vec{g}}{N} \qquad \vec{g}_i = g$$

Why should relaxation still hold?

$$\dot{V}_i = \lambda \left(g - \frac{\gamma}{R} F_i \right)$$

Lemma: $F_i \approx 6\pi R V_i - 6\pi R v_*$

$\Rightarrow$ V_i wants to relax to $\frac{g}{6\pi\gamma}$

$\rightarrow$ Transport-Stokes!

Problem:

$$|X_i - X_j| \leq |X_i^0 - X_j^0| + \int_0^t |V_i - V_j|$$

$$|V_i - V_j| \leq |V_i^0 - V_j^0| + \lambda \int_0^t |F_i - F_j|$$

$$\int$$

$$|V_i - V_j|$$

$$\Rightarrow |V_i - V_j| \leq e^{\lambda t}$$

$$\Rightarrow |X_i - X_j| \leq e^{\lambda t}$$

How to get rid of error terms?

<u>Keeping track of the velocities</u>

Consider $\lambda^{-1} > 0$ (inertia are present).

Let $f_N = \frac{1}{N}\sum \delta_{x_i} \otimes \delta_{v_i}$ and $f_N(0) \longrightarrow f_{*,0}$. We expect $f_N \to f_*$

$$\boxed{\begin{pmatrix}\text{Vlasov-}\\ \text{Brinkman}\end{pmatrix} \begin{cases} \partial_t f_* + v\cdot\nabla_x f_* + \lambda \operatorname{div}_v((\underbrace{g}_{\text{gravitational acceleration}} + \underbrace{u_* - v}_{\text{difference to mean-field}}) f_*) = 0 \\ -\Delta u_* + \nabla p = \underbrace{\int_{\mathbb{R}^3} (v-u_*) f_* \, dv}_{\text{Brinkman-term}}, \quad \operatorname{div} u_* = 0 \\ f_*(0) = f_{*,0}. \end{cases}}$$

$\varrho = \int f \, dv$ $\qquad$ f_0 monokin.!

<u>Thm</u>: $W_2(\varrho_N, \varrho) \le \left(W_2(\varrho_N^0, \varrho_0) + \frac{C}{\lambda} E_0 + o(N^{-?})\right) e^{Ct}$

?

$$W_2^2(\varrho_N^0, \varrho_0) = \int_{\mathbb{R}^3} |x - T_0(x)|^2 \, d\varrho_0$$

$$E_0 = \frac{1}{2}\int_{\mathbb{R}^3} |V - V(T_0(x))|^2 \, dS_0$$