On the analysis of the Transport-Stokes system

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IX Partial differential equations, optimal design and numerics
Special session on sedimentation

Outline

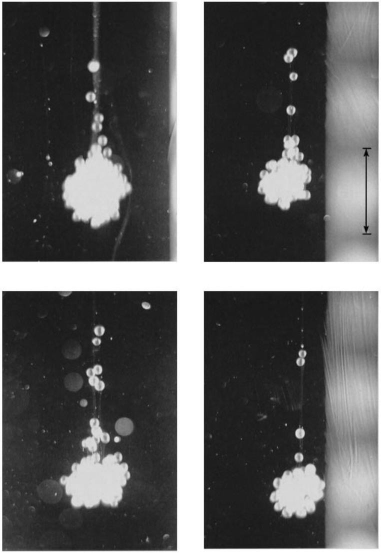

Figure: Representative photographs of blobs containing glass particles sedimenting through glycerine. Batchelor and Nitsche (1997)

Figure: Dispersed phase (thin spray) with small volume fraction

Transport-Stokes system

$(t,x) \mapsto \rho(t,x)$ probability of particles at time $t \in \mathbb{R}^+$ and position $x \in \mathbb{R}^3$
$(t,x) \mapsto u(t,x)$ velocity at time $t \in \mathbb{R}^+$ and position $x \in \mathbb{R}^3$
$(t,x) \mapsto p(t,x)$ pressure at time $t \in \mathbb{R}^+$ and position $x \in \mathbb{R}^3$

$$\left\{ \begin{array}{rcll} \partial_t \rho + \operatorname{div}(u\rho) & = & 0\,, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\ -\Delta u + \nabla p & = & -\rho e_3\,, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u & = & 0\,, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\ \lim\limits_{|x|\to\infty} |u| & = & 0, & \text{on } \mathbb{R}^+, \\ \rho(0,\cdot) & = & \rho_0\,, & \text{on } \mathbb{R}^3. \end{array} \right. \tag{TS}$$

Transport-Stokes system

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$$\begin{cases} \partial_t \rho + \mathrm{div}(u\rho) &= 0\,, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\ u &:= -\mathcal{U} \star (\rho e_3), & \\ \rho(0,\cdot) &= \rho_0\,, & \text{on } \mathbb{R}^3. \end{cases}$$

$\mathcal{U}$ the Stokes fundamental solution

$$\mathcal{U}(x) = \frac{1}{8\pi}\left(\frac{I}{|x|} + \frac{x \otimes x}{|x|^3}\right)$$

Well posedness results

→ R. Höfer. 2018.

- Well posedness for regular initial data $\rho_0, \nabla\rho_0 \in X_\beta$, $\beta > 2$.

$$X_\beta := \left\{ f \in L^\infty(\mathbb{R}^3), \sup_{x\in\mathbb{R}^3} (1+|x|^\beta)|f(x)| < +\infty \right\}$$

Well posedness results

→ R. Höfer. 2018.

→ A. M. 2020.

- Well posedness for $\mathcal{P}_1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ initial data
- Persistence of patch solutions with propagation of $C^{1,\mu}$ regularity of the boundary
- Invariance of the spherical patch (Hadamard-Rybczinski)
- Derivation of a 1d free surface model for the evolution of the contour of axisymmetric drops and numerical simulations

Well posedness results

→ R. Höfer. 2018.

→ A. M. 2020.

→ R. Höfer and R. Schubert. 2020.

- Well posedness for $\mathcal{P}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ initial data

Well posedness results

→ R. Höfer. 2018.

→ A. M. 2020.

→ R. Höfer and R. Schubert. 2020.

→ A. Leblond. 2021.

- Well posedness for L^∞ initial data in bounded domains in $\mathbb{R}^2$ and $\mathbb{R}^3$ and in infinite strip $\mathbb{R} \times (0, 1)$

Well posedness results

- → R. Höfer. 2018.
- → A. M. 2020.
- → R. Höfer and R. Schubert. 2020.
- → A. Leblond. 2021.
- → H. Grayer II. 2022
  - Well posedness for L^∞compactly supported initial data in $\mathbb{R}^2$
  - Persistence of spherical patch with propagation of $C^{k+\mu}$, $k = 0, 1, 2$ regularity of the boundary
  - Numerical investigation of the evolution of patchs

Outline

Well posedness for $L^p(\mathbb{R}^3) \cap \mathcal{P}(\mathbb{R}^3)$ initial densities with $p \geq 3$

Theorem

Let $p \geq 3$ and $\rho_0 \in E_p := \mathcal{P}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$. There is a unique solution $\rho \in C([0, +\infty), E_p)$ of the (TS) *equation and the associated velocity and pressure* $(u, q) \in$

$$\begin{cases} C([0,+\infty), W^{2,p}(\mathbb{R}^3) \times W^{1,p}(\mathbb{R}^3)) & \text{if } p > 3 \\ C([0,+\infty), \dot{W}^{2,3}(\mathbb{R}^3) \cap \dot{W}^{1,3}(\mathbb{R}^3) \cap \bigcap_{q \in (3,+\infty]} L^q(\mathbb{R}^3) \times W^{1,3}(\mathbb{R}^3)) & \text{if } p = 3 \end{cases}$$

Moreover,

$$u \in \begin{cases} C([0,+\infty), C^{1,\mu}(\mathbb{R}^3)), & \text{if } p > 3, \\ C([0,+\infty), \mathrm{LLip}(\mathbb{R}^3)), & \text{if } p = 3. \end{cases}$$

Finally, for all $s \in [0,+\infty)$, there exists a unique

$$X(\cdot, s, \cdot) \in \begin{cases} C([0,+\infty), C^{1,\mu}(\mathbb{R}^3)), & \text{if } p > 3, \\ C([0,+\infty), C^{0,r_t}(\mathbb{R}^3)), & \text{if } p = 3, \end{cases}$$

$$\mu = 1 - 3/p, \quad r_t = e^{-Ct}, \quad C = C\|\rho_0\|_{L^1(\mathbb{R}^3)\cap L^3(\mathbb{R}^3)}$$

$$\rho(t,\cdot) = X(t,0,\cdot)\#\rho_0$$

Theorem (Stability estimate)

Let $p \geq 3$. For any $R > 0$, there is $C > 0$ such that for any couple of initial densities (ρ_1^0, ρ_2^0) in E_p with $\max_{i=1,2}(\|\rho_i^0\|_{L^1(\mathbb{R}^3)}, \|\rho_i^0\|_{L^p(\mathbb{R}^3)}) \leq R$, if (u_1, ρ_1) and (u_2, ρ_2) satisfy the transport-Stokes equation (TS) *for any $t \geq 0$, then*

$$W_1(\rho_1(t), \rho_2(t)) \leq W_1(\rho_1^0, \rho_2^0)e^{Ct}, \quad \text{if } p > 3,$$
$$W_1(\rho_1(t), \rho_2(t)) \leq W_1(\rho_1^0, \rho_2^0)^{e^{-Ct}}, \quad \text{if } p = 3.$$

Stability estimates for Liouville equations

→ R. Dobrushin (1979). Case of particles with a Lipschitz interacting kernel

→ G. Loeper (2006). Case of the Vlasov-Poisson system

→ M. Hauray (2009). Case of first order interacting system with kernel $K : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $|\partial^\alpha K(x)| \leq \frac{C}{|x|^{|\alpha|+s}}$, $s < d - 1$

Outline

Known results on analyticity of trajectories for incompressible fluid models

Case of incompressible Euler in 2D/3D in bounded/unbounded domains

→ J-Y. Chemin 1991- 1992.
→ P. Gamblin 1994.
→ P. Serfaty 1995.
→ T. Kato 2000.
→ F. Sueur 2011.
→ O. Glass, F. Sueur and Takéo Takahashi. 2012.
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Idea: $\partial_t^{n+1} X(t,x) = D_t^n u(t, X(t,x))$, with $D_t := \partial_t + (u \cdot \nabla)$ the material derivative

Known results on analyticity of trajectories for incompressible fluid models

Lagrangian formulation

→ P. Constantin, V. Vicol and J. Wu. 2015

the 2D surface quasi-geostrophic equation
the 2D incompressible porous medium equation
the 2D and the 3D incompressible Euler equations
the 2D Boussinesq equation

Self-contained Lagrangian formulation for the TS equation

Since $\rho(t,\cdot) = X(t,\cdot)\#\rho_0$ we have for all $x \in \mathbb{R}^3$, $t \geq 0$

$$\dot{X}(t,x) = \int_{\mathbb{R}^3} \mathcal{U}_3(X(t,x) - X(t,y))\rho_0(y)dy,$$

where

$$\mathcal{U}_3(x) := \mathcal{U}(x)e_3$$

satisfies for all multi-index $\alpha \in \mathbb{N}_0^3$ and $x \neq 0$

$$|\partial^\alpha \mathcal{U}_3(x)| \leq \frac{C^{|\alpha|}|\alpha|!}{|x|^{1+|\alpha|}}$$

for some universal constant $C > 0$.

Analyticity of the trajectories in the case $p > 3$

Theorem

Let $p > 3$, ρ_0 in E_p and (u, ρ) in $C([0, +\infty), W^{2,p} \times E_p)$ the unique corresponding solution of the transport-Stokes equation (TS). Let $\mu := 1 - 3/p$. Let X the flow associated with u, then X is analytic from $[0, +\infty)$ to $\mathcal{C}^{1,\mu}(\mathbb{R}^3)$.

Proposition

There exists $T > 0$ depending only on $\|\rho_0\|_{L^1 \cap L^p}$ and C_0 depending only on $(\|\rho_0\|_{L^1 \cap L^p}, T, \mu)$ such that for $t \in [0, T]$ and $n \geq 1$,

$$\|\partial_t^n X(t, \cdot)\|_{1,\mu} \leq n! C_0^n.$$

Outline

Exact controllability

Theorem

Let $T > 0$*. Let* ρ_0 *and* ρ_f *in* L_c^p *with* $p > 3$ *and* ω *an open non empty set of* $\mathbb{R}^3$*. Then there exists* $(u, \rho) : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}$ *such that*

$$(u, p) \in C([0, +\infty), W^{2,p}(\mathbb{R}^3 \setminus \omega) \times L_p)$$

and

$$\begin{cases} -\Delta u + \nabla p = -\rho e_3, & \\ \operatorname{div} u = 0, & \text{on } [0, T] \times \mathbb{R}^3 \setminus \omega \\ \partial_t \rho + \operatorname{div}(\rho u) = 0, & \\ \rho(0, \cdot) = \rho_0, \ \rho(T, \cdot) = \rho_f, & \text{on } \mathbb{R}^3 \end{cases}$$

Idea of proof

Coron's return method

Proposition

Let $T > 0$, ω and ρ_0 as above. There exists $u_{aux} : (0, T) \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ satisfying $u_{aux} \in \mathcal{C}_c^\infty((0, T) \times \mathbb{R}^3 \setminus \omega; \mathbb{R}^3)$,

$$-\Delta u_{aux} + \nabla p_{aux} = \operatorname{div} u_{aux} = 0 \text{ on } \mathbb{R}^3 \setminus \omega,$$

and there exists $\bigcup_{i=1}^{L} B(x_i, r_i)$ a covering of $\operatorname{supp} \rho_0$ such that the flow map associated to u_{aux} sends the parts $B(x_i, r_i)$ to the control zone and back to $B(x_i, r_i)$ on several time intervals $]t_i, t_{i+1/2}[$, $1 \leq i \leq L$.

Ongoing works/ Open questions

- Lagrangian controllability
- Gevrey regularity in the critical case $p = 3$?
- characterization of the spherical patch as the unique invariant patch?