On the control of the Srödinger equation with moving Dirac potentials

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Outline

1. Controllability of the Schrödinger equation with moving Diracs
2. Existence of solutions
3. Numerical approximation of the control
4. Experiments

Controllability of the Sródinger equation

We consider the system with a potential $a(t,x)$

$$\begin{cases} i\partial_t \psi = -\partial_{xx}^2 \psi + a(t,x)\psi, & x \in \Omega = (0,1),\ t \geq 0, \\ \psi(t,0) = \psi(t,1) = 0, & t \geq 0, \\ \psi(0) = \psi^0 \in L^2(\Omega, \mathbb{C}). \end{cases}$$

The L^2-norm is conserved, i.e.

$$\|\psi(t)\|_{L^2} = \|\psi^0\|_{L^2}, \quad t > 0.$$

We are interested in the following controllability result: Given ψ^f with $\|\psi^f\|_{L^2} = \|\psi^0\|_{L^2}$, find $T > 0$ and $a(t,x)$ such that

$$\psi(T,x) = \psi^f(x).$$

Our result (dimension $d = 1$ and $\Omega = (0,1)$)

Theorem (CC-A. Duca, preprint)

Assume that

$$\psi^0 = \sum_{j=1}^{N_0} c_j \varphi_j(x), \quad \psi^f = \sum_{j=1}^{N_f} d_j \varphi_j(x).$$

For any $\varepsilon > 0$ there exist $T > 0$ (large) and $a(t,x)$ such that the solution of the above system can be written as

$$\psi(T) = \sum_{k=1}^{\infty} c_j(T) \varphi_j,$$

where

$$\sum_{k=1}^{K} \left||c_k(T)| - |d_k|\right|^2 + \sum_{k=K+1}^{\infty} |c_k(T)|^2 < \varepsilon$$

Example

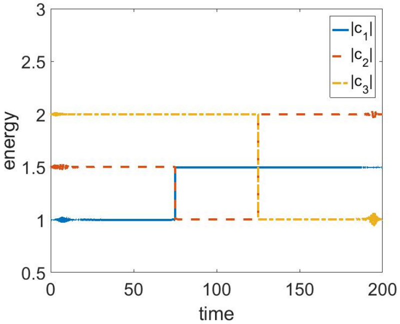
Assume that we want to permute the 'energy' of the first three modes:

$$\psi^0 = \varphi_1 + \tfrac{3}{2}\varphi_2 + 2\varphi_3$$
$$\psi^f = \tfrac{3}{2}\varphi_1 + 2\varphi_2 + \varphi_3$$

The control will produce a solution for which

$$\psi(x,T) \sim c_1(T)\varphi_1 + c_2(T)\varphi_2 + c_3(T)\varphi_3,$$

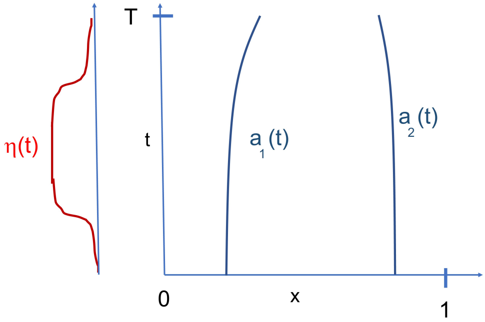
where $|c_1(T)| \sim 3/2$, $|c_2(T)| \sim 2$ and $|c_3(T)| \sim 1$.



The control $a(t, x)$

The control is explicit and given by

$$a(t, x) = \eta(t) \sum_{k=1}^{K} \delta_{a_k(t)}(x)$$



The controls are $\{a_k(t)\}_{k=1}^{K}$

Most of the works consider electric fields $a(t,x) = v(t)\mu(x)$ where $v(t)$ is the intensity of the field (control) and $\mu(x)$ the dipolar moment (smooth)

- **Global approximate controllability:** Mirrahimi and Beauchard' 09, Boscain and Adami' 05, Boscain, Chittaro, Gauthier, Mason, Rossi and Sigalotti' 12, Boussaid, Caponigro and Chambrion' 22, ...
- **Local exact controllability:** Ball, Marsden and Slemrod' 85 (Negative result), Beauchard and Laurent 11', Puel' 16...
- Nonlinear models, systems, networks, etc...

The peculiarity of our result is in the explicit form of the control. It produces an adibatic regime almost any time.

Existence of solutions in presence of a moving Dirac potential

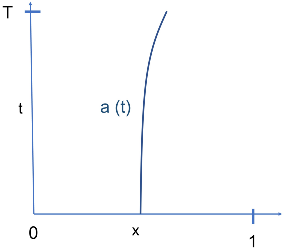
$$\begin{cases} i\partial_t \psi = -\partial^2_{xx}\psi + \eta(t)\delta_{x=a(t)}\psi, & x \in (0,1),\ t \in (0,T), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T), \\ \psi(0) = \psi^0 \in L^2((0,1),\mathbb{C}). & \end{cases}$$

Theorem

Let $a \in C^3([0,T],(0,1))$ and $\eta \in C^2([0,T],\mathbb{R}^+)$. For any $\psi_0 \in H^1_0$, the above system admits a unique solution in $C^0([0,T];H^1_0) \cap C^1([0,T];H^{-1})$.

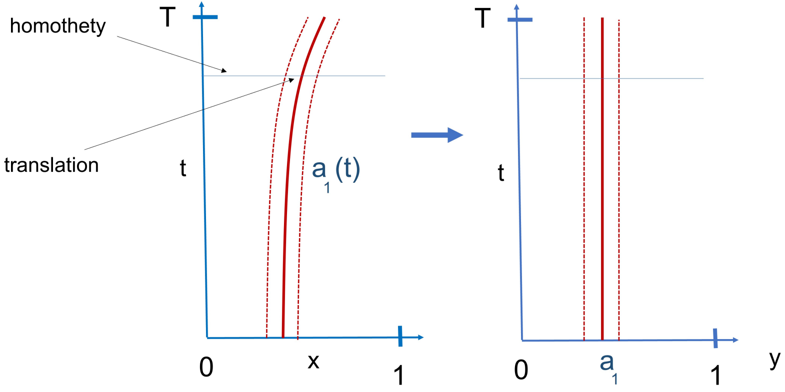
Step 1. Write the system as,

$$\begin{cases} i\partial_t\psi = -\partial_{xx}^2\psi, \quad x \in (0, a(t)) \cup (a(t), 1), \quad t > 0, \\ \psi(t, 0) = \psi(t, 1) = 0, \\ \psi(t, a(t)^-) = \psi(t, a(t)^+), \\ \partial_x\psi(t, a(t)^+) - \partial_x\psi(t, a(t)^-) = \eta(t)\psi(t, a(t)), \\ \psi(0) = \psi^0 \in L^2((0, 1), \mathbb{C}). \end{cases}$$



Step 2. Introduce a change of variables that fix the Dirac

$$(t, x) \longrightarrow (t, y) = (t, h(t, x))$$



Step 3. After the change of variables the system can be written for $\phi = h^\sharp \psi$ as

$$i\partial_t \phi(t) = h^\sharp H(t) h_\sharp \phi(t), \quad \text{in} \quad (0,1) \quad \text{with}$$

$$H(t) := -\Big[\big(\partial_x + iM_h\big) \circ \big(\partial_x + iM_h\big) + M_h^2 \Big],$$

$$M_h(t,x) = -\frac{1}{2}(h_* \partial_t h)(t,x).$$

where,

$$h^\sharp(t) \,:\, \psi \in L^2((0,1),\mathbb{C}) \longmapsto \sqrt{|\partial_x h(t,\cdot)|}\,(\psi \circ h)(t) \in L^2((0,1),\mathbb{C}).$$

$$h_\sharp(t) = (h^\sharp(t))^{-1} \,:\, \phi \mapsto (\phi / \sqrt{|\partial_x h(t,\cdot)|}) \circ h^{-1} \in L^2((0,1),\mathbb{C}).$$

This is basically a Schrödinger equation with smooth coefficients and an interior fixed Dirac. Existence is known for such systems [Kisynsky, 64].

Control strategy (proof of the main result)

The proof is based on an idea from Turaev' 19, and further developed by Duca, Joly and Turaev' 20 with L^1-potentials. Main ingredients:

1. Adiabatic regime. For slow variation of $\eta(t)$ and $a_k(t)$ the energy associated to each mode is almost preserved. (Nenciu, 80)
2. The dynamics for large η is similar to the split domain $(0, a(t)) \cup (a(t), 1)$ with homogeneous Dirichlet conditions. This requires a convergence result for the eigenvalues and eigenfuntions.
3. Continuity result. Assuming smoothness of $\eta(t)$ and $a_k(t)$ the solution maintains its spatial energy distribution in a sufficiently small time interval.

Lemma Take any function $f \in C^1([t_1, t_2], H^2((0,1), \mathbb{C}) \cap H_0^1((0,1), \mathbb{C}))$ such that $f(t,x)$ vanishes in $a(t)$ for every $t \in [t_1, t_2]$. Then, for all $t \in [t_1, t_2]$,

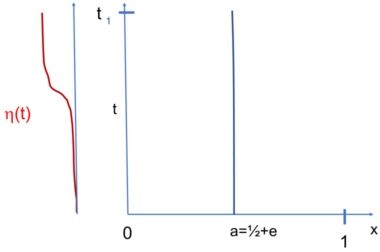
$$\|\psi(t) - f(t)\|_{L^2}^2 \leq \|\psi(t_1) - f(t_1)\|_{L^2}^2 + C(t_2 - t_1),$$

with C independent of the choice of functions η and a

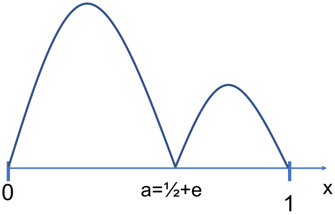
Assume that we want to permute the energy of the first two modes. Take

$$\psi^0 = 2\varphi_1 + \varphi_2, \quad \psi^f = 1\varphi_1 + 2\varphi_2$$

First subinterval $(0, t_1)$ Take $a(t,x) = \eta(t)\delta_{a_1}(x)$ with $a_1 = 1/2 + \varepsilon$ and $\eta(t)$ growing slowly from zero to a large value η_M (adiabatic regime)

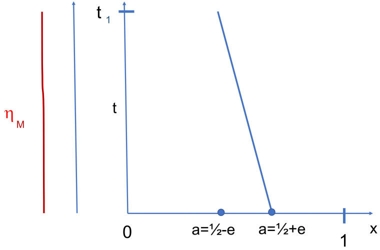


$\eta(t)$ and $a(t)$

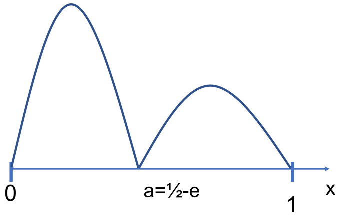


$|\psi(t_1,x)|^2$

Second subinterval (t_1, t_2) Take $a(t,x) = \eta_M \delta_{a_1(t)}(x)$ with $a_1(t)$ that moves fast from $a_1 + \varepsilon$ to $a_1 - \varepsilon$ (continuity result)

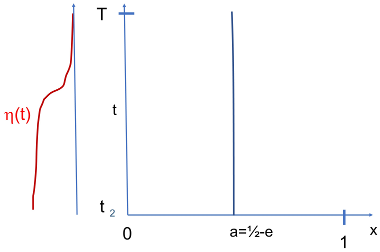


$\eta(t)$ and $a(t)$

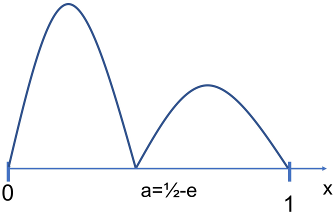


$|\psi(t_1, x)|^2$

Third subinterval (t_2, T) Take $a(t,x) = \eta(t)\delta_{a_1}(x)$ with $a_1 = 1/2 - \varepsilon$ and $\eta(t)$ descending slowly from the large value η_M to zero (adiabatic regime)



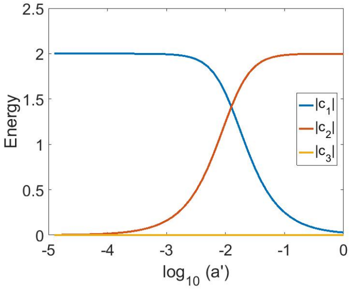
$\eta(t)$ and $a(t)$



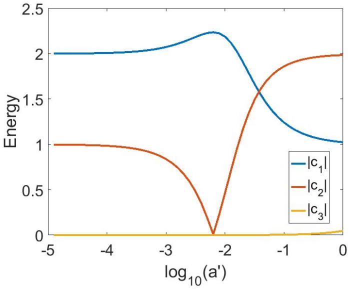
$|\psi(t_1, x)|^2$

Then $\psi(x,T) = c_1(T)\varphi_1 + c_2(T)\varphi_2 + O(\varepsilon)$ with $|c_1(T)| = 1$ and $|c_2(T)| = 2$

Remark: The permutation of energy in the modes is performed in step 2. This is obtained with a rapid nonadiabatic movement of the Dirac support $a(t)$. Sufficiently slow movements will produce an adiabatic regime with no permutation of energy. Intermediate situations are also possible



$$\psi_0(x) = 2\phi_1(x,0)$$



$$\psi_0(x) = 2\phi_1(x,0) + \phi_2(x,0)$$

Numerical approximation: Spectral method

Take $X = L^2(0,1)$, $\mathbf{a} = (a_1, ..., a_K)$ and consider the associated eigenpairs of $A^{\eta,\mathbf{a}}$

$$(\lambda_k(t), \phi_k(x,t)), \quad k \geq 1.$$

Consider also the eigenpairs of the Dirichlet Laplacian

$$(\mu_k(t), w_k(x,t)), \quad k \geq 1.$$

Define

$$X^N = span\{w_k\}_{k=1}^N, \qquad P^N : X \to X_N.$$

Discrete problem: Find $\psi_N(t) \in X_N$ such that,

$$\begin{cases} i\partial_t \psi_N = P^N A^{\eta,\mathbf{a}}(t)\psi_N, & t > 0 \\ \psi_N(0) = P^N \psi^0. \end{cases}$$

Theorem

Assume that a and η satisfy the hypotheses to guarantee the existence of a solution $\psi \in C([0,T]; H_0^1)$ with initial data $\psi^0 \in H_0^1$. Let ψ_N be the solution of the corresponding finite dimensional approximation. Then, for $t \in [0,T]$,

$$\|\psi(t) - \psi_N(t)\|_{L^2} \leq \left(1 + 2T\frac{\eta_M}{\pi}\right)\frac{\sqrt{\eta_M}}{\sqrt{3}\sqrt{N}}\|\psi(t)\|_{L^\infty((0,T);H_0^1)},$$

where $\eta_M = \max_{t\in[0,T]} \eta(t)$.

Remark The estimate depends on η and T that are large. Therefore it requires N large.

The numerical scheme is convergent but the estimates in the theorem are somehow useless in the simulation of the control since

1. The adiabatic regime requires $T >> 1$
2. In step 2 we require $\eta >> 1$

In practice, this means that the parameters η, T and N must be carefully chosen.

We write the solution ψ as linear combination of eigenfunctions,

$$\psi(t,x) = \sum_{k=1}^{\infty} \hat{\psi}_k(t)\phi_k(t,x), \qquad \hat{\psi}_k(t) = \int_0^1 \psi(t,x)\overline{\phi_k(t,x)}\,dx.$$

We prove the following:

Step 1

$$\|\psi(t) - \psi_N(t)\|_{L^2} \leq \left(1 + 2T\frac{\eta_M}{\pi}\right) \|(I - P^N)\psi\|_{L^\infty((0,T);H_0^1)},$$

Step 2

$$\|(I - P^N)\phi_k(t,\cdot)\|_{H_0^1} \leq \frac{\sqrt{2}}{\pi}\frac{\sqrt{\eta(t)}}{\sqrt{N}}\sqrt{\lambda_k - \nu_1}, \qquad \forall k \in \mathbb{N}^*.$$

Matrix formulation

$$\psi_N(t,x) = \sum_{n=1}^{N} \hat{\psi}_n(t)\sqrt{2}\sin(n\pi x).$$

$$\begin{cases} \Psi' = (D + P(t))\Psi, \\ \Psi(0) = \Psi^0. \end{cases}$$

where $P(t) = -2i\eta(t)\ \sum_{j=1}^{J} s_j \otimes s_j$ for $s_j = (\sin(k\pi a_j(t)))_{k=1}^{N}$ and

$$\Psi = \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \dots \\ \hat{\psi}_N \end{pmatrix}, \quad D = -i\pi^2 \begin{pmatrix} 1^2 & 0 & \dots & 0 \\ 0 & 2^2 & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & N^2 \end{pmatrix}.$$

Time discretization

we propose a midpoint implicit method in time:

$$\frac{\Psi^{k+1} - \Psi^k}{\Delta t} = \left(D + P\left(\frac{t_{k+1} + t_k}{2}\right)\right) \frac{\Psi^{k+1} + \Psi^k}{2}, \quad k = 0, ..., K-1,$$

that is second order accurate and convergent as long as $\|P''(t)\|$ is bounded in $t \in [0, T]$. Another advantage of this method, especially for long time simulations, is that it conserves the L^2-norm since D and P are purely imaginary matrixes

$$\|\Psi^K\|^2 = \|\Psi^0\|^2.$$

Experiments: Example considered at the beginning

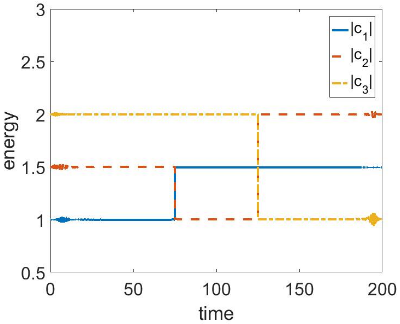
Assume that we want to permute the 'energy' of the first three modes:

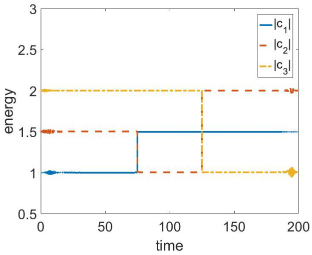
$$\psi^0 = \varphi_1 + \tfrac{3}{2}\varphi_2 + 2\varphi_3$$
$$\psi^f = \tfrac{3}{2}\varphi_1 + 2\varphi_2 + \varphi_3$$

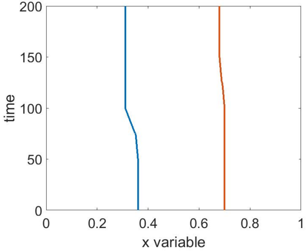
The control will produce a solution for which

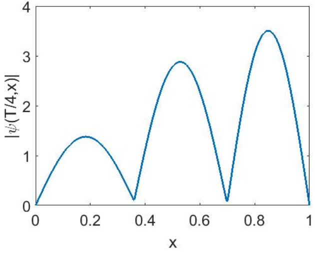
$$\psi(x,T) = c_1(T)\varphi_1 + c_2(T)\varphi_2 + c_3(T)\varphi_3 + O(\varepsilon),$$

where $|c_1(T)| \sim 3/2$, $|c_2(T)| \sim 2$ and $|c_3(T)| \sim 1$.





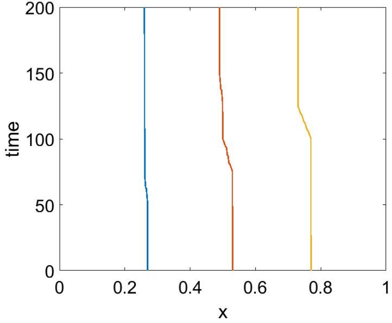




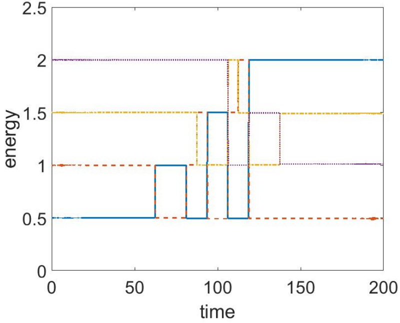
Trajectories of the Diracs

Solution after including the Diracs.

Example 2: permutation of 4 modes



Trajectories of the Diracs



energy jumps of the Fourier modes

1. For simplicity we have focused on permutations of energy states. However, the technique can be adapted to any redistribution of the energy in a finite number of Fourier coefficients.
2. Probably the paths of Diracs (controls) can be optimized.
3. The idea can be adapted to higher dimensions, at least in simple situations.