Controllability properties of a fractional diffusion equation

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Controllability

- $(X, \langle\, , \rangle_X)$, $(U, \langle\, , \rangle_U)$ Hilbert spaces
- $A : D(A) \subset X \to X$ unbounded operator in X which generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ in X.
- $B \in \mathcal{L}(U, X)$ be a bounded linear operator.
- Given $T > 0$, $z^0 \in X$ and $u \in L^2(0, T; U)$ there exists a unique weak solution $z \in \mathcal{C}([0, T]; X)$ of the equation:

$$\begin{cases} z'(t) = Az(t) + Bu(t) & t \in (0, T) \\ z(0) = z^0. \end{cases} \tag{1}$$

The term $Bu(t)$ represents a control mechanism introduced in the system to change its dynamics. u is the control.

Controllability

- Equation (1) is *exactly controllable in time T* if, for any $z^0, z^1 \in X$, there exists $u \in L^2(0,T;U)$ such that the solution of (1) verifies

$$z(T) = z^1. \tag{2}$$

- Equation (1) is *null-controllable in time T* if, for any $z^0 \in X$, there exists $u \in L^2(0,T;U)$ such that the solution of (1) verifies

$$z(T) = 0. \tag{3}$$

- Equation (1) is *approximately controllable in time T* if, for any $z^0, z^1 \in X$ and $\varepsilon > 0$, there exists $u \in L^2(0,T;U)$ such that the solution of (1) verifies

$$\|z(T) - z^1\|_X \leq \varepsilon. \tag{4}$$

Problem of moments

- The spectrum of A^* is given by a family of eigenvalues $(\lambda_n)_{n\in\mathcal{N}} \subset \mathbb{C}$ and there exists a sequence of eigenvectors $(\Phi_n)_{n\in\mathcal{N}} \subset D(A^*)$ which is an orthonormal basis of X.

Theorem

Equation (1) *is null-controllable if and only if for any*

$$z^0 = \sum_{n\in\mathcal{N}} z_n^0 \Phi_n \in X,$$

there exists $u \in L^2(0,T;U)$ *such that*

$$\int_0^T \langle u(t), B^*\Phi_n\rangle_U \, e^{-\overline{\lambda}_n t}\, \mathrm{dt} = -\mathrm{z}_\mathrm{n}^0 \qquad (\mathrm{n} \in \mathcal{N}). \tag{5}$$

Relations (5) represent a problem of moments.

Problem of moments

H. O. Fattorini and D. L. Russell, *Exact controllability theorems for linear parabolic equation in one space dimension*, Arch. Rat. Mech. Anal., '71.

Through the *problem of moments* the controllability property is reduced to the study of **existence and the properties (the norm)** of a biorthogonal sequence $(\theta_m)_{m\in\mathbb{N}}$ to $\left(B^*\Phi_n e^{-\lambda_n t}\right)_{n\in\mathbb{N}}$ in $L^2(0,T;U)$:

$$\langle B^*\Phi_n e^{-\lambda_n t}\,,\,\theta_m(t)\rangle_{L^2(0,T;U)} = \delta_{nm} \qquad (n,m\in\mathbb{N}). \tag{6}$$

- An element θ_m is a control! It controls an initial data with one mode only.
- If a biorthogonal sequence $(\theta_n)_{n\in\mathbb{N}}$ exists then a control u is given (formally) by

$$u(t) = \sum_{m\in\mathbb{N}} (-z_n^0)\theta_m(t). \tag{7}$$

On the biorthogonal sequence

- A necessary condition for the existence of a biorthogonal sequence $(\theta_m)_{m\in\mathbb{N}}$ to $\left(B^*\Phi_n e^{-\lambda_n t}\right)_{n\in\mathbb{N}}$ in $L^2(0,T;U)$ is **the minimality of the set $\left(B^*\Phi_n e^{-\lambda_n t}\right)_{n\in\mathbb{N}}$ in $L^2(0,T;U)$ (this means that each element of the set lies outside the closed linear span of the others).**
- To show the existence of a control $u\in L^2(0,T;U)$ we have to show the (absolute) convergence of the series which defines the control u. For this we need **estimates of the norms** $\|\theta_m\|_{L^2(0,T;U)}$ to ensure that

$$\sum_{m\in\mathbb{N}} \left|z_n^0\right| \|\theta_m\|_{L^2(0,T;U)} < \infty.$$

Diffusion equation

Diffusion occurs when particles spread freely. They move from a region where they are in high concentration to a region where they are in low concentration. The word diffusion derives from the Latin word, diffundere, which means "to spread way out". The usual one-dimensional diffusion equation reads as follows:

$$z_t(t,x) - Dz_{xx}(t,x) = 0. \tag{8}$$

- It has been proposed by Joseph Fourier in 1822 as model for the temperature distribution in materials.
- Adolf Fick has used it in 1855 to describe the process of equalization of the concentration in a medium with an initially non-homogeneous distribution of some substance.

In both cases it is assumed that the flux is proportional to the gradient of the temperature (or concentration).

Normal diffusion

Diffusion is a result of random (Brownian) motion of the molecules or atoms. In 1905 Albert Einstein determined how far a Brownian particle travels in a given time interval by considering the collective motion of Brownian particles. He deduced that the square of the displacement of a particle at some time relative to the position of the particle at zero time, averaged over many particles (the mean squared displacement - MSD - $\langle r^2 \rangle$), is proportional to t:

$$\langle r^2 \rangle \sim t.$$

This is a consequence of the fact that the fundamental solution of (8) is given by

$$\frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}.$$

Normal diffusion

- The square of the mean squared displacement $\sqrt{\langle r^2 \rangle}$ measures how fast the particles are spreading. Einstein relation says that it is proportional to $\sqrt{t}$.
- Also, we can measure the velocity of spreading by using the quantities named full width at half maximum (FWHM) or full width at tenth maximum (FWTM) which is proportional to $\sqrt{t}$. The diffusion packet initially concentrated at a point takes later the Gaussian form which width grows in time as $\sqrt{t}$.

This type of diffusion is called *normal diffusion.*

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This type of diffusion is called *normal diffusion.*

In nature we can also find *other types of diffusion!!!*

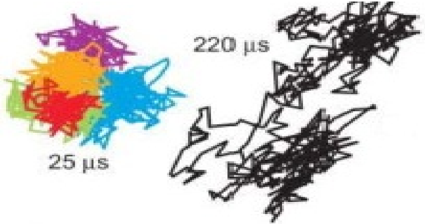
Anomalous diffusion

Anomalous diffusion is a diffusion process in which the spreading of particles is proportional with t^β and $\beta \neq 1/2$.

- Superdiffusion ($\beta > 1/2$) or higher diffusion rate. The particles spreading in a turbulent atmosphere or the movements of some animal populations (albatross, plankton, spider monkey).
- Subdiffusion ($\beta < 1/2$) or slower propagation of the concentration front. Describes many physical scenarios, most prominently within crowded systems, for example protein diffusion within cells cytoplasm, or diffusion through porous media.

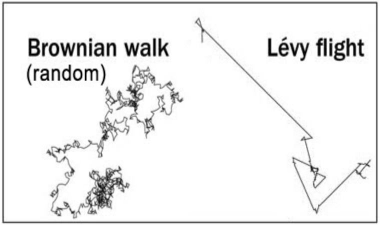
Subdiffusion

Biology contains a wealth of subdiffusive phenomena, such as the way some that proteins diffuse across cell membranes. Akihiro Kusumi and co-workers at Nagoya University in Japan performed experiments in which they tracked a single protein molecule in the plasma membrane of live cells. Fluorescent-molecule video imaging revealed that the molecules spend relatively long times trapped between nanometre-sized compartments in the cytoplasm. This, claimed Kusumi and co-workers, was the origin of the anomalous subdiffusion.

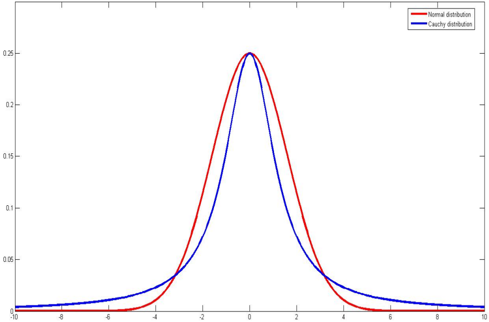


Superdiffusion

A possibility for anomalous diffusion is that the random walker remains in motion without changing direction for a time that follows a Lévy distribution with a polynomially decaying tail.
In this case the step lengths and the waiting times have a broad distribution. Such "Lévy flight" correspond to a process in which the mean-squared displacement grows faster than it does in normal diffusion. Such processes are therefore termed superdiffusive.



Anomalous diffusion



The fundamental solution of (8) ($\beta = 1/2$) is: $\frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$.

For $\beta = 1$ we have a different fundamental solution: $\frac{Dt}{\pi(x^2+(Dt)^2)}$.

Anomalous diffusion

"Our picture of diffusion 100 years after Einstein published his groundbreaking papers has clearly become much broader, quite literally.[...] But the clear picture that has emerged over the last few decades is that although these phenomena are called anomalous, they are abundant in everyday life: anomalous is normal!"

(J. Klafter and I. M. Sokolov, *Anomalous diffusion spreads its wings*, Phys. World 18 (2005), 29-32.)

Anomalous diffusion

- M. Weiss, M. Elsner, F. Kartberg and T. Nilsson, *Anomalous Subdiffusion Is a Measure for Cytoplasmic Crowding in Living Cells*, Biophysical Journal 87 (2004), 3518-3524.
- J. Klafter and I. M. Sokolov, *Anomalous diffusion spreads its wings*, Phys. World 18 (2005), 29-32.
- L. F. Richardson *Atmospheric diffusion shown on a distance-neighbour graph*, Proceedings of the Royal Society of London A, 110(1926), 709-737.
- E. R. Weeks and H. L. Swinney *Anomalous diffusion resulting from strongly asymmetric random walks*, Phys. Rev. E, 57(1998), 4915-4920.
- G. Zumofen, J. Klafter and A. Blumen, *Trapping aspects in enhanced diffusion*, J. Stat. Phys., 65(1991), 991-1013.

Mathematical models (I)

Several models have been proposed to describe anomalous diffusion processes.

$$\partial_{t,0+}^{\alpha} z - \partial_{xx}^{2} z = 0,$$

where $\partial_{t,0+}^{\alpha}$ is the left-hand side Caputo derivative at 0:

$$\partial_{t,0+}^{\alpha} g = \int_0^t \frac{g'(s)}{(t-s)^{\alpha}} \, \mathrm{ds}.$$

Q. Lü, E. Zuazua, *On the lack of controllability of fractional in time ODE and PDE*, Mathematics of Control, Signals, and Systems, 28 (2016), 10.

$$z_t - \partial_t^{1-\alpha}(\partial_{xx}^2) z = 0.$$

Mathematical models (II)

Instead of considering fractional derivative in time, we can take the fractional spatial derivative:

$$z_t + (-\partial_{xx}^2)^\alpha z = 0.$$

How do we define the fractional operator?

Mathematical models (III)

For $0 < s < 1$, the fractional Laplacian of order s, $(-\Delta)^s$, in $\mathbb{R}^d$ can be defined on functions $f : \mathbb{R}^d \to \mathbb{R}$ as a Fourier multiplier given by the formula

$$\mathcal{F}((-\Delta)^s f))(\xi) = |\xi|^{2s}\mathcal{F}(f)(\xi),$$

where the Fourier transform $\mathcal{F}(f)$ of $f : \mathbb{R}^d \to \mathbb{R}$ is given by

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x)e^{-i\,x\cdot\xi}\,\mathrm{dx}.$$

More concretely, the fractional Laplacian can be written as a singular integral operator defined by

$$(-\Delta)^s f(x) = c_{d,s}\int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+2s}}\,\mathrm{dy},$$

where $c_{d,s} = \frac{4^s\Gamma(d/2+s)}{\pi^{d/2}|\Gamma(-s)|}$.

Mathematical models (IV)

The fractional Laplacian is a pseudo-differential operator with symbol $\|\xi\|^{2\alpha}$. We remark that on $\mathbb{R}^d$ the two definitions (as a singular integral and through the Fourier transform) are all equivalent; this, however, is not true anymore when working on open subsets of $\mathbb{R}^d$ the main reason being the non-locality of the operator.

In the literature we find different notions of the fractional Laplacian in a bounded domain which are not equivalent:

- As singular integral operator. This definition generalizes the one in $\mathbb{R}^d$.
- As spectral fractional operator. This operator consists in the $\alpha-$th power of the Laplacian $-\Delta$, obtained by using its spectral decomposition.

Mathematical models (V)

Let $\Omega \subset \mathbb{R}$ be an arbitrary open set and, for $0 < \alpha < 1$, let us introduce the space

$$\mathcal{L}^1(\Omega) = \left\{ u : \Omega \to \mathbb{R} \, : \, \int_\Omega \frac{|u(x)|}{(1+|x|)^{1+2\alpha}} \, \mathrm{dx} < \infty \right\}.$$

Then, for $u \in \mathcal{L}^1(\Omega)$, we restrict the kernel of the fractional Laplacian to Ω and we define the operator

$$\begin{aligned} (-\Delta)^\alpha u(x) &= c_\alpha P.V. \int_\Omega \frac{u(x)-u(y)}{|x-y|^{1+2\alpha}} \, \mathrm{dy} \\ &= c_s \lim_{\varepsilon \to 0^+} \int_{y \in \Omega, \; |x-y| > \varepsilon} \frac{u(x)-u(y)}{|x-y|^{1+2\alpha}} \, \mathrm{dy} \qquad (x \in \Omega). \end{aligned}$$

Umberto BICCARI, On the controllability of Partial Differential Equations involving non-local terms and singular potentials, PhD Thesis (2016).

Mathematical models (VI)

Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ be the one-dimensional torus.

For $\alpha > 0$ let $(-\partial_{xx})^\alpha : D((-\partial_{xx})^\alpha) \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denote the fractional power of the Laplace operator on $\mathbb{T}$,

$$D((-\partial_{xx})^\alpha) = \left\{ u(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx} \,\middle|\, \sum_{n\in\mathbb{Z}} |a_n|^2 \, |n|^{4\alpha} < \infty \right\},$$

$$u(x) = \textstyle\sum_{n\in\mathbb{Z}} a_n e^{inx} \longrightarrow (-\partial_{xx})^\alpha u(x) = \textstyle\sum_{n\in\mathbb{Z}} a_n \, |n|^{2\alpha} \, e^{inx}.$$

Our equation and the controllability problem

Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ be the one-dimensional torus. For $\alpha > 0$ we consider the controlled linear one-dimensional equation with anomalous diffusion

$$\begin{cases} z_t(t,x) + (-\partial_{xx})^\alpha z(t,x) = f(x)\, u(t) & x \in \mathbb{T},\ t > 0, \\ z(0,x) = z^0(x) & x \in \mathbb{T}, \end{cases} \tag{9}$$

where f is a given function from $L^2(\mathbb{T})$ supported in some subset ω of $\mathbb{T}$ and u stands for the control input and depends only on t ("lumped" or "biliniar" control).

In the absence of the control u, the solutions of (9) decay exponentially as $t \to \infty$ in, say, L^2. When $\alpha = 1$ we recover the classical heat equation. For $\alpha > 1$ the equation is sub-diffusive whereas for $\alpha \in (0,1)$ it is super-diffusive. Equation (9) is of parabolic type for any $\alpha > 0$.

Given $T > 0$ we say that (9) is **null-controllable in time** T if, for each $z^0 \in L^2(\mathbb{T})$, there exists a control $u \in L^2((0,T) \times \mathbb{T})$ such that the corresponding solution of (9) verifies

$$z(T,x) = 0. \tag{10}$$

The case $\alpha > \frac{1}{2}$

When $\alpha > 1/2$ and the control profile f satisfies

$$\begin{aligned} &\int_{\mathbb{T}} f(x)e^{inx}\,\mathrm{dx} \neq 0 \quad (n \in \mathbb{Z}), \\ &\liminf_{|n|\to\infty} \left(\left| \int_{\mathbb{T}} f(x)e^{inx}\,\mathrm{dx} \right| e^{\eta\lambda_n} \right) > 0 \quad (\eta > 0), \end{aligned} \tag{11}$$

system (9) is null controllable for an arbitrarily short time and with smooth time-dependent controls u.

H. O. Fattorini and D. L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension*, **Arch. Ration. Mech. Anal., 43 (1971), 272-292.**

Proof for the case $\alpha > \frac{1}{2}$

The proof is based on the fact that the null-control problem may be rewritten as a *problem of moments* of the following form:
Find $u \in L^2(0,T)$ such that

$$\int_0^T u(t) e^{|n|^{2\alpha} t} \, \mathrm{dt} = \beta_n, \tag{12}$$

where β_n depends on the Fourier coefficients of the initial data to be controlled and of the profile f.

This result may be proved by means of a construction and careful evaluation of the norm of a biorthogonal sequence $(\theta_m)_{m\geq 1}$ to the family of exponential functions $\{e^{\lambda_n t}\}_{n\geq 1}$, with $\lambda_n = |n|^{2\alpha}$.

The result is based on the Müntz theorem.

Exponential Functions

The Müntz theorem is a basic result of approximation theory, proved by Herman Müntz in 1914. Roughly speaking, the theorem shows to what extent the Weierstrass theorem on polynomial approximation can have holes dug into it, by restricting certain coefficients in the polynomials to be zero.

Theorem

Let $\Lambda = (\lambda_n)_{n\geq 0}$, $0 = \lambda_0 < \lambda_1 < ...$, be an increasing sequence of non-negative real numbers. Then $\mathcal{S}(\Lambda) =$ Span $\left\{ x^{\lambda_n} \,\middle|\, n = 0, 1, ... \right\}$, the Müntz space associated to Λ, is a dense subset of $C[0,1]$ if and only if

$$\sum_{n\geq 1} \frac{1}{\lambda_n} = \infty. \tag{13}$$

In the context of system (9) when $\lambda_n = |n|^{2\alpha}$, condition (11) is verified if and only if $\alpha > 1/2$.

Construction of a biorthogonal sequence

We consider a family of exponential functions $\left(e^{-\lambda_n t}\right)_{n\in\mathbb{Z}}$ in $L^2(0,T)$. In order to construct a family of biorthogonals $(\theta_m)_{m\in\mathbb{Z}}$ we define a family $(\Psi_m)_{m\in\mathbb{Z}}$ of entire functions with the following properties.

H1. There exist two positive constants A and B such that

$$|\Psi_m(z)| \leq Ae^{B|z|} \qquad (z \in \mathbb{C}),$$

i. e. Ψ_m is an entire function of exponential type at most B.

H2. For any $m \in \mathbb{Z}$ we have that $\Psi_m \in L^2(\mathbb{R})$.

H3. For every m, $n \in \mathbb{Z}$ we have that $\Psi_m(i\overline{\lambda}_n) = \delta_{mn}$.

Paley-Wiener Theorem

Theorem (**Paley-Wiener, 1934**)

Let $\psi(z)$ *be an entire function such that*

$$|\psi(z)| \leq A e^{B|z|},$$

for positive constants A *and* B *and all values of* z*, and*

$$\int_{\mathbb{R}} |\psi(x)|^2 \, \mathrm{d}x < \infty.$$

Then, there exists a function ϕ *in* $L^2[-B, B]$ *such that*

$$\psi(z) = \int_{-B}^{B} \phi(t) e^{izt} \, \mathrm{d}t.$$

Construction of a biorthogonal sequence

By using Paley-Wiener Theorem, we can provide an explicit biorthogonal sequence.

Proposition

Let $(\Psi_m)_{m\in\mathbb{Z}}$ *be a family of entire functions such that hypotheses* H1–H3 *hold. There exists a sequence* $(\theta_m)_{m\in\mathbb{Z}}$ *biorthogonal to the family* $\left(e^{\lambda_n t}\right)_{n\in\mathbb{Z}}$ *in* $L^2(-B,B)$ *given by the following expression*

$$\theta_m(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \Psi_m(x)e^{ixt}\,\mathrm{d}x. \tag{14}$$

Moreover, we have that

$$\|\theta_m\|_{L^2(-B,B)} = \sqrt{2\pi}\|\Psi_m\|_{L^2(\mathbb{R})}. \tag{15}$$

R. E. A. C. Paley and N. Wiener, *Fourier Transforms in Complex Domains*, **AMS Colloq. Publ., Vol. 19, New York, 1934.**

The case $\alpha \leq \frac{1}{2}$

Theorem

Let $0 < \alpha \leq 1/2$ *and suppose that the Fourier coefficients of* f *satisfy (11). Then any nontrivial initial state* $z^0 = \sum_n z_n^0 e^{inx}$ *with the property that for any* $\mu > 0$ *there exists a constant* $C_\mu > 0$ *such that*

$$|z_n^0| \leq C_\mu e^{\mu |n|^{2\alpha}} \quad (n \in \mathbb{Z}) \tag{16}$$

cannot be driven to zero in time $T > 0$ *by means of a control* $u \in L^2(0, T)$, *whatever* $T > 0$ *is.*

S. M. and E. Zuazua: *On the controllability of a fractional order parabolic equation*, **SIAM J. Control Optim. 44 (2006), no. 6, 1950-972.**

L. Miller, *On the controllability of anomalous diffusions generated by the fractional Laplacian*, **Mathematics of Control, Signals, and Systems, 18 (2006), 260-271.**

Exponential Functions

Since, for $\alpha \leq \frac{1}{2}$, we have that

$$\sum_{n\geq 1} \frac{1}{\lambda_n} = \infty,$$

Müntz theorem implies that $\mathcal{S}(\Lambda) =$ Span $\left\{x^{\lambda_n} \,\middle|\, n = 0, 1, ...\right\}$ is complete in $C[0, T]$, for any $T > 0$.

This implies that the set of exponential functions $\left(e^{\lambda_n t}\right)_{n\geq 1}$ is not minimal in $L^2(0, \infty)$ and, consequently, there is no biorthogonal sequence to $\left(e^{\lambda_n t}\right)_{n\geq 1}$ in $L^2(0, T)$.

From the controllability point of view, this implies that equation (9) is not spectrally controllable if $\alpha \leq 1/2$.

Similar cases

- **L. Rosier, P. Rouchon**, *On the exact controllability of a wave equation with structural damping*, **Internat. J. Tomogr. Statist., 5 (2007), 79-84.**
- **F. Chavez-Silva, L. Rosier, E. Zuazua**, *Null controllability of a system of viscoelasticity with a moving control*, **J. Math. Pures Appl., 101 (2014) 198-222.**
- **P. Martin, L. Rosier, P. Rouchon**, *Null controllability of the structurally damped wave equation with moving control*, **SIAM J. Control Optim., Vol. 51 (2013), No. 1, pp. 660-684.**
- **S.M**, *On the controllability of the linearized Benjamin-Bona-Mahony equation*, **SIAM J. Control Optim. 39 (2001), 1677-1696.**
- **L. Rosier, B.-Y. Zhang**, *Unique continuation property and control for the Benjamin-Bona-Mahony equation on a periodic domain,* **Journal of Differential Equations 254 (2013) 141-178.**

Controllability with moving control

For $\alpha \in (0, \frac{1}{2})$ and $c > 0$ we consider the controlled linear one-dimensional equation with super-diffusion and moving control

$$\begin{cases} z_t(t,x) + (-\partial_{xx})^{\alpha} z(t,x) = f(x-ct)\, u(t) & x \in \mathbb{T},\ t > 0, \\ z(0,x) = z^0(x) & x \in \mathbb{T}, \end{cases} \tag{17}$$

where, as before, f is a given function from $L^2(\mathbb{T})$ supported in some subset ω of $\mathbb{T}$ and u stands for the control input.

Given $T > 0$ we say that (17) is **null-controllable in time** T if, for each $z^0 \in L^2(\mathbb{T})$, there exists a control $u \in L^2((0,T) \times \mathbb{T})$ such that the corresponding solution of (17) verifies

$$z(T,x) = 0. \tag{18}$$

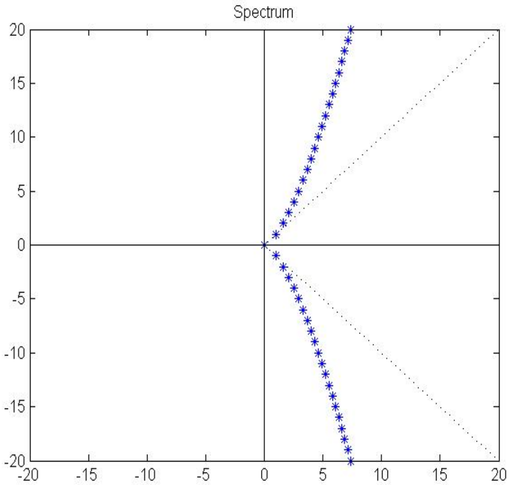
The corresponding moment problem to the null-controllability problem (17)-(18) is the following:

Find $u \in L^2(0,T)$ such that

$$\int_0^T u(t)e^{(|n|^{2\alpha}+icn)t}\,\mathrm{d}t = \beta_n \qquad (n \in \mathbb{Z}), \qquad (19)$$

where β_n depends on the Fourier coefficients of the initial data z^0 to be controlled.

Notice that the exponents changed from $|n|^{2\alpha}$ (purely real) to $|n|^{2\alpha} + icn$ (complex). Our problem does not have anymore a parabolic nature.



We pass from a parabolic type spectrum to a mostly hyperbolic one. This situation has not been much treated in the literature.

Complex exponential functions

A first question is related to the completeness of exponential functions $\left(e^{-\lambda_n t}\right)_{n\geq 1}$ in $L^2(0,\infty)$. The Müntz theorem has been generalized to complex exponents by Otto Szász (1884-1952) in 1916.

Theorem

Let $\Lambda = (\lambda_n)_{n\geq 0}$ *be complex numbers such that* $\lambda_0 = 0$ *and* $\liminf_{n\to\infty} \Re(\lambda_n) > 0$*. Then the space Span* $\left\{x^{\lambda_n} \,\middle|\, n = 0, 1, ...\right\}$ *is a dense subset of* $C[0,1]$ *if and only if*

$$\sum_{n\geq 1} \frac{|\Re(\lambda_n)|}{1+|\lambda_n|^2} = \infty. \tag{20}$$

In our case $\alpha < \frac{1}{2}$ we have that:

$$\sum_{n\geq 1} \frac{|\Re(\lambda_n)|}{1+|\lambda_n|^2} = \sum_{n\geq 1} \frac{n^{2\alpha}}{1+n^2+n^{2\alpha}} < \infty.$$

The case $\alpha = \frac{1}{2}$

For $\alpha = \frac{1}{2}$, the family of exponential functions $\left(e^{-\lambda_n t}\right)_{n\in\mathbb{Z}}$ is complete in $L^2(0,a)$, for any $a > 0$. Indeed, since

$$\sum_{n\in\mathbb{Z}} \frac{\Re(\lambda_n)}{1+|\lambda_n|^2} = \infty, \tag{21}$$

the completeness is a consequence of the Szász Theorem.
From the controllability point of view, it follows that (17) is not spectrally controllable if $\alpha = \frac{1}{2}$.

A biorthogonal sequence

Theorem

Let $T > \frac{2\pi}{c}$ and $\epsilon < \frac{T}{2} - \frac{\pi}{c}$. There exists a biorthogonal sequence $(\theta_m)_{m\in\mathbb{Z}}$ to the family of complex exponential functions $\left(e^{\lambda_n t}\right)_{n\in\mathbb{Z}}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ with the property that

$$\|\theta_m\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} \leq C \exp\left(\left(\frac{\pi}{c} + \epsilon\right) \Re(\lambda_m)\right) \qquad (m \in \mathbb{Z}). \tag{22}$$

We construct a family $(\Psi_m)_{m\in\mathbb{Z}}$ of entire functions of exponential type, belonging to L^2 on the real axis, such that $\Psi_m(\lambda_n) = \delta_{mn}$. The inverse Fourier transforms of $(\Psi_m)_{n\in\mathbb{Z}}$ gives us the biorthogonal. Each function Ψ_m is obtained from a Weierstrass type product P_m multiplied by a function M_m with a suitable behavior on the real axis in order to ensure the L^2 property of Ψ_m on the real axis. Finally, Plancherel's Theorem estimates the norms $\|\theta_m\|_{L^2(\mathbb{R})}$ from the evaluation of $\|\Psi_m\|_{L^2(\mathbb{R})}$.

The controllability result [C. Niță, S.M., SICON 2022]

Theorem

Let $\alpha \in \left(0, \frac{1}{2}\right)$, $c > 0$ and $f \in L^2(\mathbb{T})$ be a function which verifies (11). *For any $T > \frac{2\pi}{c}$ and $z^0 \in L^2(\mathbb{T})$, there exists a control $u \in L^2(0,T)$ which leads to zero the solution z of* (17) *at time T.*

Proof : Let $T > \frac{2\pi}{c}$ and $z^0 = \sum_{n\in\mathbb{Z}} z_n^0 e^{inx}$. We construct a control $u \in L^2(0,T)$ of (17) as follows

$$u(t) = -\sum_{n\in\mathbb{Z}} \frac{e^{-\Re(\lambda_m)\frac{T}{2}}}{\widehat{f}_m} z_n^0 \theta_m \left(t - \frac{T}{2}\right) \qquad (t \in (0,T)), \qquad (23)$$

From the properties of the biorthogonal sequence $(\theta_m)_{m\in\mathbb{Z}}$ and $(\widehat{f}_m)_{m\in\mathbb{Z}}$, it is easy to see that u is well defined in $L^2(0,T)$ and it is a control for (17).

The case $\alpha = \frac{1}{2}$ (Cauchy diffusion)

Theorem

Let $\alpha = \frac{1}{2}$ and $f \in L^2(0,1)$ verifying (11). Equation (9) is approximately controllable in any time $T > 0$.

Theorem

Let $\alpha = \frac{1}{2}$ and $f \in L^2(0,1)$ verifying (9). There exist positive constants C_1, C_2 and T_0 such that, for any $\varepsilon > 0$, $T > T_0$ and $u^0 \in L^2(0,1)$, one can find an approximate control g_ε of (9) with the property that

$$\|g_\varepsilon\|_{L^2(0,T)} \leq C_1 \left(1 + C_2 \|f\|_{L^2(0,1)}\right)^{\frac{T_0}{T-T_0}} \left\|u^0\right\|_{L^2(0,1)}^{\frac{T}{T-T_0}} \left(\frac{2}{\varepsilon}\right)^{\frac{T_0}{T-T_0}}. \tag{24}$$

Ideea of the proof

- For each $T > 0$ there exists a biorthogonal sequence $(\theta_k(T, \cdot))_{1\leq k\leq N}$ to the family $(\exp(-\lambda_j t))_{1\leq j\leq N}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ with the following property

$$\|\theta_k(T, \cdot)\|_{L^2\left(-\frac{T}{2}, \frac{T}{2}\right)} \leq \frac{32\sqrt{2}}{T^2} \exp\left[\omega_0\left(T\right) N + \frac{T}{2}\lambda_k\right] \quad (1 \leq k \leq N).$$

- The control

$$g_N(t) = \sum_{k=1}^{N} -\frac{u_k^0}{f_k} \exp(-\lambda_k T)\theta_k\left(T, \frac{T}{2} - t\right) \qquad (t \in (0, T))$$

ensures that the solution u verifies $\Pi_N u\left(\frac{T}{2}, \cdot\right) = 0$.

- We define $g_\varepsilon \in L^2(0, T)$ by

$$g_\varepsilon(t) = \begin{cases} g_N(t) & t \in \left[0, \frac{T}{2}\right], \\ 0 & t \in \left(\frac{T}{2}, T\right], \end{cases}$$

where $N = N(\varepsilon) \in \mathbb{N}^*$ is such that $\|u(t, \cdot)\|_{L^2} \leq \varepsilon$.

Perspective and open problems

- The case $\alpha = \frac{1}{2}$ (Cauchy diffusion).
- Other models for the anomalous diffusion with fractional derivatives.
- Space dimension $d > 1$.

The end

Thank you very much for your attention!