Initial data identification for Hamilton-Jacobi equations

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joint work with
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Time-evolution equation:

$$\begin{cases} \partial_t u = A(u), & t > 0 \\ u(0) = u_0. \end{cases}$$

Assumption: well-posedness

- Existence
- Uniqueness
- Continuity w.r.t. u_0.

Problem: If we are given $u(T)$ for some $T > 0$, can we deduce u_0?

Issues:

- Existence of at least a compatible initial condition.
- Uniqueness of inverse designs.
- What if the observation $u(T)$ is noisy?

Hamilton-Jacobi equation

$$\begin{cases} \partial_t u + H(x, \nabla_x u) = 0, & (0,T) \times \mathbb{R}^N \\ u(0,\cdot) = u_0(\cdot) \in \mathrm{Lip}(\mathbb{R}^N) \end{cases}$$

$$H \in C^2(\mathbb{R}^{2N}) \quad H_{pp}(x,p) \geq c\, I_N.$$

Crandall-Lions, 1980's:

Well-posedness in the sense of viscosity solutions

$$\begin{array}{rccc} S_T^+ : & \mathrm{Lip}(\mathbb{R}^N) & \longrightarrow & \mathrm{Lip}(\mathbb{R}^N) \\ & u_0 & \longmapsto & u(T,\cdot) \end{array}$$

Given a function $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, construct u_0 such that

$$S_T^+ u_0(x) \approx u_T(x) \qquad \forall x \in \mathbb{R}^N.$$

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- **Admissible observations:** Characterize the image of the operator S_T^+:

$$\mathcal{R}_T := \left\{ u_T \in \mathrm{Lip}(\mathbb{R}^N) \quad \text{s.t.} \quad \exists u_0 \in \mathrm{Lip}(\mathbb{R}^N) \text{ satisfying } S_T^+ u_0 = u_T \right\}.$$

1. Backward-forward criterion.
2. Geometric criterion (semiconcavity condition).

$$\begin{cases} \partial_t u + H(x, \nabla_x u) = 0, & (0,T) \times \mathbb{R}^N \\ u(0,\cdot) = u_0(\cdot) \in \mathrm{Lip}(\mathbb{R}^N) \end{cases}$$

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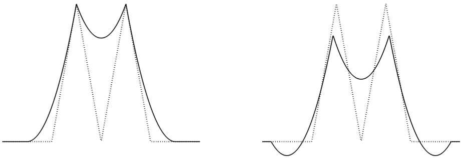
- **Closest admissible function:**

1. Backward-forward projection (semiconcave envelope)
2. L^2–projection



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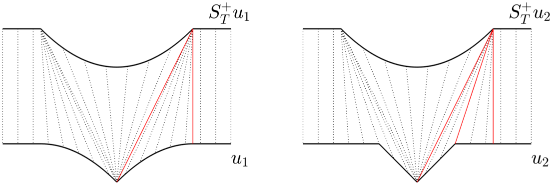
- **Initial-condition reconstruction:** Lack of backward uniqueness



Lack of backward uniqueness

Consider the one-dimensional Hamilton-Jacobi equation

$$\partial_t u + \frac{(\partial_x u)^2}{2} = 0.$$

Lack of backward uniqueness due to the loss of regularity.

The viscosity solutions with initial condition u_1 and u_2 coincide at time $t = T$.

After time T, both solutions are indistinguishable.

Characterization of the admissible set

Admissible targets

For a time horizon $T > 0$ and a given target $u_T \in \text{Lip}(\mathbb{R}^N)$, let us define

$$I_T(u_T) := \left\{ u_0 \in \text{Lip}(\mathbb{R}^N); \text{ such that } S_T^+ u_0 = u_T \right\}.$$

Goal: Give necessary and sufficient conditions for $I_T(u_T) \neq \emptyset$.

The **natural candidate** is obtained by reversing the time in the (HJ) equation, considering u_T as terminal condition.

We need a notion of solution for which the terminal value problem

$$\begin{cases} \partial_t u + H(x, \nabla_x u) = 0, & (0, T) \times \mathbb{R}^N \\ u(T, \cdot) = u_T(\cdot) \in \text{Lip}(\mathbb{R}^N) \end{cases}$$

is well-posed.

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The vanishing viscosity limit

Viscosity solutions (Crandall-Lions, 1980's)

For any $u_0 \in \mathrm{Lip}(\mathbb{R}^N)$, the viscosity solution $u(t,x)$ is the pointwise limit as $\varepsilon \to 0^+$ of $u_\varepsilon(t,x)$, the solution to the semilinear parabolic problem

$$\begin{cases} \partial_t u_\varepsilon - \varepsilon \Delta u_\varepsilon + H(x, \nabla_x u_\varepsilon) = 0, & (t,x) \in (0,T) \times \mathbb{R}^N \\ u_\varepsilon(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{VHJ}$$

Backward viscosity solutions

For any $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, the backward viscosity solution $v(t,x)$ is the pointwise limit as $\varepsilon \to 0^+$ of $v_\varepsilon(t,x)$, the solution to the semilinear parabolic problem

$$\begin{cases} \partial_t v_\varepsilon + \varepsilon \Delta v_\varepsilon + H(x, \nabla_x v_\varepsilon) = 0, & (t,x) \in (0,T) \times \mathbb{R}^N \\ v_\varepsilon(T,x) = u_T(x), & x \in \mathbb{R}^N. \end{cases} \tag{BVHJ}$$

Remark:

- The backward viscosity solution is not a viscosity solution.
- Same definition as viscosity solution with the inequalities reversed.

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Backward viscosity solutions

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Remark:

- The backward viscosity solution is not a viscosity solution.
- Same definition as viscosity solution with the inequalities reversed.

We can then define the **(forward) viscosity operator** as

$$\begin{array}{rccl} S_T^+ : & \mathrm{Lip}(\mathbb{R}^N) & \longrightarrow & \mathrm{Lip}(\mathbb{R}^N) \\ & u_0 & \longmapsto & S_T^+ u_0 := u(T, \cdot) \end{array}$$

where $u(T, \cdot)$ is the unique viscosity solution to (HJ) at time $t = T$ with initial condition;

and we also define the **backward viscosity operator** as

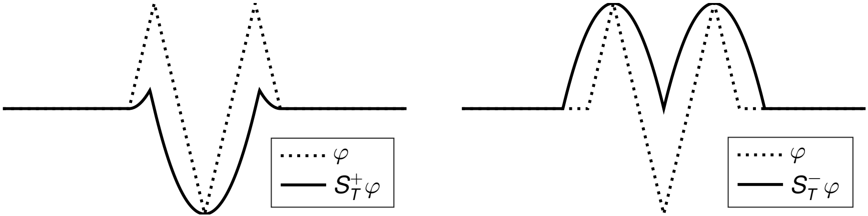
$$\begin{array}{rccl} S_T^- : & \mathrm{Lip}(\mathbb{R}^N) & \longrightarrow & \mathrm{Lip}(\mathbb{R}^N) \\ & u_T & \longmapsto & S_T^- u_T := v(0, \cdot) \end{array}$$

where $v(0, \cdot)$ is the unique backward viscosity solution to (HJ) at time $t = 0$ with terminal condition u_T.

Forward and backward viscosity solutions

Property

1. For any $\varphi \in \mathrm{Lip}(\mathbb{R}^N)$ and any $T > 0$, the function $S_T^+\varphi(x)$ is **semiconcave**.
2. For any $\varphi \in \mathrm{Lip}(\mathbb{R}^N)$ and any $T > 0$, the function $S_T^-\varphi(x)$ is **semiconvex**.



Definition

Let $f \in \mathrm{Lip}(\mathbb{R}^N)$:

- We say that f is **semiconcave** if $\exists C > 0$ such that $D^2 f \leq C\, I_N$ in $\mathbb{R}^N$.
- We say that f is **semiconvex** if $\exists C > 0$ such that $D^2 f \geq -C\, I_N$ in $\mathbb{R}^N$.

Theorem

Let $u_T \in \mathrm{Lip}(\mathbb{R}^N)$ and $T > 0$. Then $I_T(u_T) \neq \emptyset$ if and only if $S_T^+(S_T^- u_T) = u_T$.

- It is clear that, in order to be reachable, the target u_T **needs to be semiconcave**.
- Being semiconcave is however **not sufficient** for being reachable.
- The optimal semiconcavity condition for reachability depends on the Hamiltonian H and the time-horizon T.

Semiconcavity condition

Let $H(x,p)$ satisfy

$$H \in C^2(\mathbb{R}^{2N}), \qquad \text{and} \qquad H_{pp}(x,p) \geq c\, I_N, \quad \forall (x,p) \in \mathbb{R}^{2N}.$$

Then, for any $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, we have

$$I_T(u_T) \neq \emptyset \qquad \text{implies} \qquad D^2 u_T(x) \leq \frac{1}{cT} I_N \quad \text{in the viscosity sense,}$$

or equivalently, the function $x \longmapsto u_T(x) - \dfrac{\|x\|^2}{2cT}$ is concave.

Theorem

- If $N = 1$, then $I_T(u_T) \neq \emptyset$ if and only if

$$\partial_{xx} u_T(x) - \frac{1}{T\, H_{pp}(\partial_x u_T(x))} \leq 0, \qquad \text{in } \mathbb{R}.$$

- If $N \geq 1$, and $H(p) = \dfrac{\langle Ap, p\rangle}{2}$ for some matrix $A > 0$, then $I_T(u_T) \neq \emptyset$ if and only if

$$D^2 u_T(x) - \frac{A^{-1}}{T} \leq 0, \qquad \text{in } \mathbb{R}^N.$$

Projections on the admissible set

Projection on the Reachable Set

When the given target is not reachable, we can consider the problem of “approximating” it by means of a reachable target.

Definition

For any $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, the function

$$u_T^* := S_T^+(S_T^- u_T)$$

satisfies $I_T(u_T^*) \neq \emptyset$.

We call u_T^* the **backward-forward projection of u_T onto the set of reachable targets**.

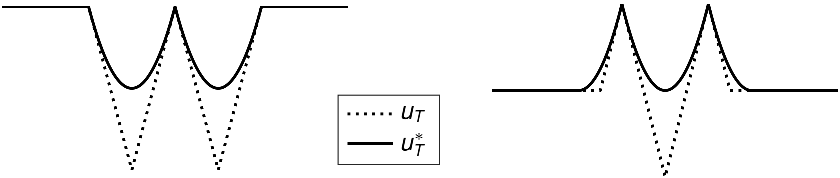
$$\begin{tikzcd}
S_T^- u_T \arrow[rr, bend left, "S_T^+"] & & S_T^+(S_T^- u_T) = u_T^* \\
& & u_T \arrow[ull, bend left, "S_T^-"]
\end{tikzcd}$$

Theorem

Let $N = 1$ and let H be a $C^2(\mathbb{R})$ uniformly convex Hamiltonian. Then, for any $u_T \in \mathrm{Lip}(\mathbb{R})$, the function $u_T^* = S_T^+(S_T^- u_T)$ is the viscosity solution to the obstacle problem

$$\min\left\{ v - u_T, \; -\partial_{xx} v + \frac{[H''(\partial_x v)]^{-1}}{T} \right\} = 0.$$

For any given target $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, the function $u_T^* = S_T^+(S_T^- u_T)$ is the **smallest reachable target bounded from below by u_T**.



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Let

$$H(p) = \frac{\langle Ap, p\rangle}{2} \quad \text{for some matrix } A > 0.$$

Then, for any $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, the function $u_T^* = S_T^+(S_T^- u_T)$ is the viscosity solution to the obstacle problem

$$\min\left\{ v - u_T,\ -\lambda_N\left[D^2 v - \frac{A^{-1}}{T}\right]\right\} = 0. \tag{1}$$

- Here, $D^2 v$ denotes the Hessian matrix of v; and for a symmetric matrix X, $\lambda_N[X]$ denotes its greatest eigenvalue.
- Observe that for T large, equation (1) is an approximation of the equation for the **concave envelope** of u_T

$$\min\left\{ v - u_T,\ -\lambda_N\left[D^2 v\right]\right\} = 0.$$

- For any $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, we call u_T^* the $\frac{A^{-1}}{T}$**–semiconcave envelope** of u_T in $\mathbb{R}^N$.

Theorem

Let

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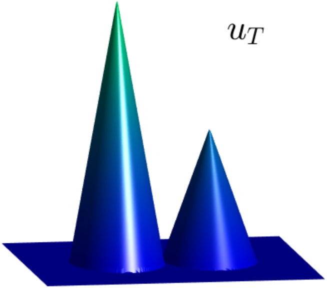

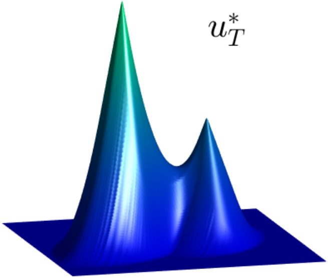

L^2-projection onto the admissible set

The L^2-projection

Given $u_T \in \mathrm{Lip}(\mathbb{R}^N)$ and $T > 0$,

$$\underset{\varphi_T \in \mathcal{R}_T}{\text{minimize}}\ \mathcal{F}_T(\varphi_T) := \|\varphi_T - u_T(\cdot)\|^2_{L^2(\mathbb{R}^N)},$$

where

$$\mathcal{R}_T := \{\varphi \in \mathrm{Lip}(\mathbb{R}^N)\ ; \quad \exists u_0 \in \mathrm{Lip}(\mathbb{R}^N) \text{ s.t. } S_T^+ u_0 = \varphi_T\}.$$

Issue: In many cases, the set $\mathcal{R}_T$ is non-convex, and its characterization does not allow to implement standard optimization algorithms.

Optimal control problem

Given $u_T \in \mathrm{Lip}(\mathbb{R}^N)$ and $T > 0$,

$$\underset{u_0 \in \mathrm{Lip}(\mathbb{R}^N)}{\text{minimize}}\ \mathcal{J}_T(u_0) := \|S_T^+ u_0 - u_T(\cdot)\|^2_{L^2(\mathbb{R}^N)}.$$

Issue: We need to compute derivatives of $\mathcal{J}_T(\cdot)$.

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The gradient of $\mathcal{J}_T$:

For any $u_0 \in \mathrm{Lip}(\mathbb{R}^N)$ the gradient of $\mathcal{J}_T$ at u_0 is the following linear functional:

$$w \in \mathrm{Lip}(\mathbb{R}^N) \longmapsto \partial_w \mathcal{J}_T(u_0) = 2 \int_{\mathbb{R}^N} \left(S_T^+ u_0(x) - u_T(x) \right) \partial_w S_T^+ u_0(x) dx,$$

where $\partial_w S_T^+ u_0(x)$ is the directional Gâteaux derivative of the operator S_T^+ with respect to u_0:

$$\partial_w S_T^+ u_0(\cdot) = \lim_{\delta \to 0^+} \frac{S_T^+(u_0 + \delta w) - S_T^+ u_0}{\delta}. \tag{2}$$

Theorem

For any $u_0, w \in \mathrm{Lip}(\mathbb{R}^N)$, the limit in (2) exists in $L^1_{loc}(\mathbb{R}^N)$, and is linear with respect to w.

Theorem

Let $N = 1$ or $N \geq 1$ and $H(x,p) = \|p\|^2 + f(x)$. For any $u_0, w \in \mathrm{Lip}(\mathbb{R}^N)$, the gradient of the functional $\mathcal{J}_T$ at $u_0 \in \mathrm{Lip}(\mathbb{R}^N)$ can also be given as

$$w \in \mathrm{Lip}(\mathbb{R}^N) \longmapsto \partial_w \mathcal{J}_T(u_0) = 2 \int_{\mathbb{R}^N} w(x)\pi(0,x)dx,$$

where $\pi(0,\cdot)$ is the solution at time 0 of the backward conservative transport equation

$$\begin{cases} \partial_t \pi + \mathrm{div}(a(t,x)\pi) = 0 & (0,T) \times \mathbb{R}^N \\ \pi(T) = S_T^+ u_0 - u_T & \mathbb{R}^N, \end{cases} \tag{3}$$

where $a(t,x) = H_p(x, \nabla u(t,x))$.

Remark:

- The proof utilizes the well-posedness of the backward transport equation (3), which relies on the following one-sided-Lipschitz estimate

$$\langle a(t,y) - a(t,x),\ y - x \rangle \leq \alpha(t)|y-x|^2$$

which only holds if $N = 1$ or if $N \geq 1$ and H is quadratic in p.

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where $a(t,x) = H_p(x, \nabla u(t,x))$.

Remark:

- Due to the low regularity of the transport coefficient $a(t,x)$, the solution $\pi(t,x)$ to (3) at time $t = 0$ is a Radon measure, which might be discontinuous with respect to the Lebesgue measure.

Theorem

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where $a(t,x) = H_p(x, \nabla u(t,x))$.

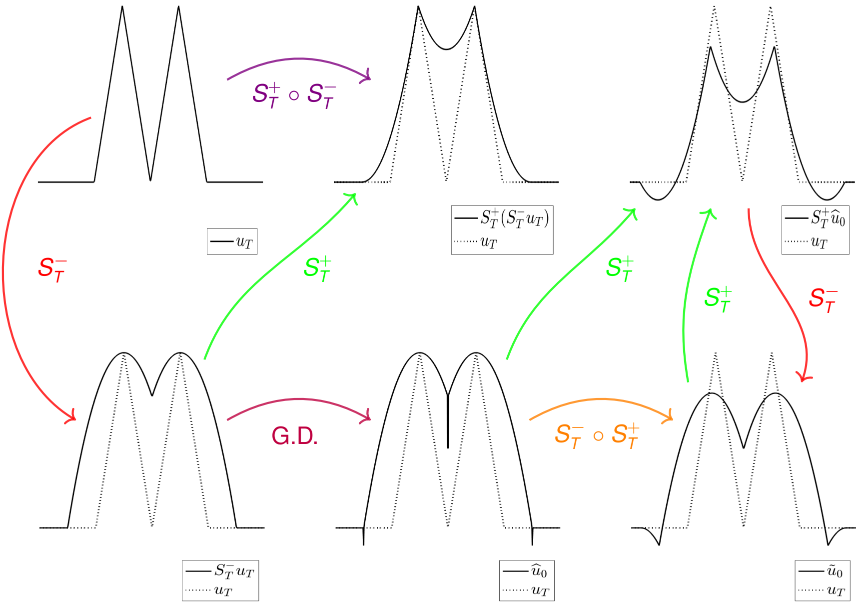
Remark:

- Gradient Descent Algorithm:

$$u_{i+1}(\cdot) = u_i - \delta_i \tilde{w}(\cdot),$$

where $\tilde{w}(\cdot)$ is a Lipschitz approximation of $\pi(0,\cdot)$.

L^2-projection onto the admissible set



Initial data reconstruction

Let $u_T \in \mathrm{Lip}(\mathbb{R}^N)$, and let u_T^* be a projection of u_T onto $\mathcal{R}_T$.

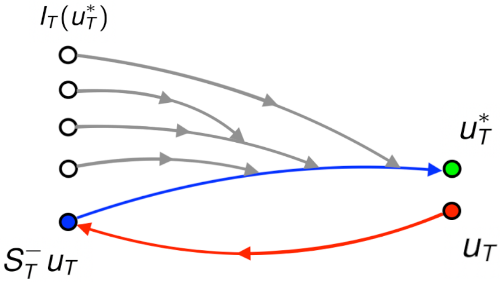
Goal:

Construct all the initial conditions $u_0 \in \mathrm{Lip}(\mathbb{R}^N)$ satisfying $S_T^+ u_0 = u_T^*$.

Let us recall the definition of the set

$$I_T(u_T^*) := \left\{ u_0 \in \mathrm{Lip}(\mathbb{R}^N);\ \text{such that } S_T^+ u_0 = u_T^* \right\}$$

of inverse designs for u_T.



Characterization of $I_T(u_T^*) := \{u_0 \in \text{Lip}(\mathbb{R}^N) \text{ s.t. } S_T^+ u_0 = u_T^*\}$.

Step 1: Set the initial condition given by the backward viscosity solution

$$\tilde{u}_0(x) := S_T^- u_T^*(x) \in I_T(u_T^*).$$

- First of all, we can prove that, for any $u_0 \in \text{Lip}(\mathbb{R}^N)$, it holds that

$$S_T^- \circ S_T^+ u_0(x) \leq u_0(x) \qquad \forall x \in \mathbb{R}^N.$$

- Then, we can deduce that, if $u_0 \in I_T(u_T)$, i.e.

$$S_T^+ u_0(x) = u_T^*(x) \qquad \forall x \in \mathbb{R}^N,$$

then

$$u_0(x) \geq S_T^- \circ S_T^+ u_0(x) = S_T^- u_T^*(x) = \tilde{u}_0(x) \qquad \forall x \in \mathbb{R}^N.$$

This proves that $\tilde{u}_0$ is the smallest initial condition in $I_T(u_T^*)$.

Characterization of $I_T(u_T^*) := \{u_0 \in \mathrm{Lip}(\mathbb{R}^N) \text{ s.t. } S_T^+ u_0 = u_T^*\}$.

Step 2: Let us define the following subset[1] of $\mathbb{R}^N$:

$$X_T(u_T^*) := \left\{ z - T\, H_p(\nabla u_T^*(z));\ \forall z \in \mathbb{R}^N \text{ such that } u_T^*(\cdot) \text{ is differentiable at } z \right\}.$$

Using the optimality condition for the associated problem in calculus of variations we can prove that

$$S_T^+ u_0(x) \leq u_T^*(x)\ \forall x \in \mathbb{R}^N \quad \text{if and only if} \quad u_0(x) \leq \tilde{u}_0(x)\ \forall x \in X_T(u_T^*).$$

Combining this with Step 1, it we obtain that all the initial conditions $u_0 \in I_T(u_T^*)$ coincide with $\tilde{u}_0$ in the set $X_T(u_T^*)$.

[1] (Here we use that H is x-independent. For x-dependent Hamiltonians the expression for $X_T(u_T^*)$ is very lengthy.)

Theorem

Let $u_T \in \mathrm{Lip}(\mathbb{R}^N)$ be such that $I_T(u_T) \neq \emptyset$ and set the function $\tilde{u}_0 := S_T^- u_T$. Then, for any $u_0 \in \mathrm{Lip}(\mathbb{R}^N)$, the two following statements are equivalent:

(i) $u_0 \in I_T(u_T)$;

(ii) $u_0(x) \geq \tilde{u}_0(x),\ \forall x \in \mathbb{R}^N \quad \text{and} \quad u_0(x) = \tilde{u}_0(x),\ \forall x \in X_T(u_T)$,

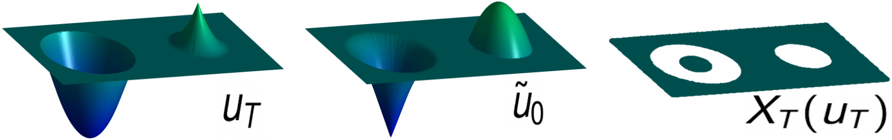
where $X_T(u_T)$ is the subset of $\mathbb{R}^N$ given by

$$X_T(u_T) := \left\{ z - T\, H_p(\nabla u_T(z));\ \forall z \in \mathbb{R}^N \text{ such that } u_T(\cdot) \text{ is differentiable at } z \right\}.$$

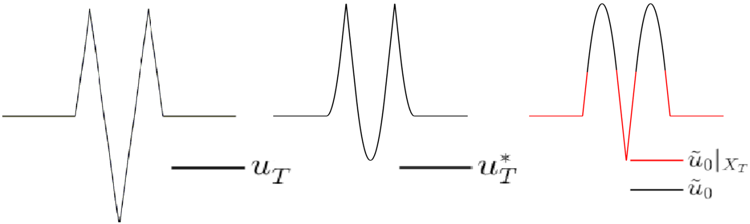
We can write $I_T(u_T)$ in the following way

$$I_T(u_T) = \left\{ \tilde{u}_0 + \varphi\, ;\ \varphi \in \mathrm{Lip}(\mathbb{R}^n) \text{ such that } \varphi \geq 0 \text{ and } \mathrm{supp}(\varphi) \subset \mathbb{R}^n \setminus X_T(u_T) \right\}.$$

Observe that $I_T(u_T)$ is a convex cone with $\tilde{u}_0$ as vertex.



Initial data reconstruction



Remarks:

- If $X_T(u_T) = \mathbb{R}^N$, then $I_T(u_T) = \{\tilde{u}_0\}$. It is the case of solutions that are differentiable everywhere in $[0, T] \times \mathbb{R}^N$.
- If $X_T(u_T)$ is a proper subset of $\mathbb{R}^N$, there is no backward uniqueness. We cannot uniquely determine the initial datum.
- In any case, the initial datum is uniquely determined in $X_T(u_T)$, while in $\mathbb{R}^N \setminus X_T(u_T)$ we only have a lower bound. The information in $\mathbb{R}^N \setminus X_T(u_T)$ is partially lost at time T.

Thank you for your attention!!!

References:

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- C. Esteve-Yagüe and Enrique Zuazua, *Differentiability with respect to the initial condition for Hamilton-Jacobi equations*, to appear in SIAM J. Math. Anal.