Existence, weak-strong uniqueness, and long-time behavior for fractional cross-diffusion systems in a bounded domain

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Classical vs cross-diffusion models in biology

Let $u_i := u_i(t,x) \geq 0$ be space density of a species i (for $i = 1, \ldots, n$) at time $t \geq 0$ and consider $U = (u_1, \ldots, u_n)$. E.g. *wildlife populations*.

Classical reaction-diffusion model:

$$\partial_t U - \Delta[DU] = F(U),$$

with $F : \mathbb{R}_+^n \longrightarrow \mathbb{R}^n$ and $D = \operatorname{diag}(d_1, \ldots, d_n)$ a positive diagonal matrix.

Interactions between individuals of different species affect the *growth rate* of the populations (reaction term).

Reaction-cross-diffusion model:

$$\partial_t U - \Delta[A(U)] = F(U),$$

with $F : \mathbb{R}_+^n \longrightarrow \mathbb{R}^n$ and $A : \mathbb{R}_+^n \longrightarrow (\mathbb{R}_+^*)^n$.

Interactions between individuals of different species affect the *growth rate* (reaction term) and the *spreading* of the populations (cross-diffusion).

Example: Shigesada-Kawasaki-Teramoto system (1979)

- $t \geq 0$: time; $x \in \Omega$: space; ($\Omega \subset \mathbb{R}^d$: environment);
- $u = u(t,x) \geq 0$: density of first species at time $t \geq 0$;
- $v = v(t,x) \geq 0$: density of second species at time $t \geq 0$.

SKT system

$$\begin{cases} \partial_t u - \Delta\left(d_1 u + d_\alpha u^2 + d_\beta uv\right) = u\left(r_1 - r_a u - r_b v\right), & t > 0,\ x \in \Omega, \\ \partial_t v - \Delta\left(d_2 v + d_\gamma v^2 + d_\delta uv\right) = v\left(r_2 - r_c v - r_d u\right), & t > 0,\ x \in \Omega, \\ \nu \cdot \nabla u = \nu \cdot \nabla v = 0, & x \in \partial\Omega. \end{cases}$$

Interpretation:

- $d_i > 0$: classical diffusion;
- $d_\alpha,\ d_\gamma \geq 0$: self-diffusion;
- $d_\beta,\ d_\delta \geq 0$: cross-diffusion;
- $r_i > 0$ intrinsic growth rate; $r_a,\ r_c > 0$: intraspecific competition; $r_b,\ r_d > 0$: interspecific competition.

NB: cross/self-diffusion $\rightarrow$ repulsive effect due to competition.

Motivation for fractional diffusion: Brownian motion vs Lévy jump process

- [Viswanathan et al., Nature 1996]: Lévy flight search patterns of wandering albatrosses.
- [Massaccesi-Valdinoci, J. Math. Bio. 2017]: Is a nonlocal diffusion strategy convenient for biological populations in competition?

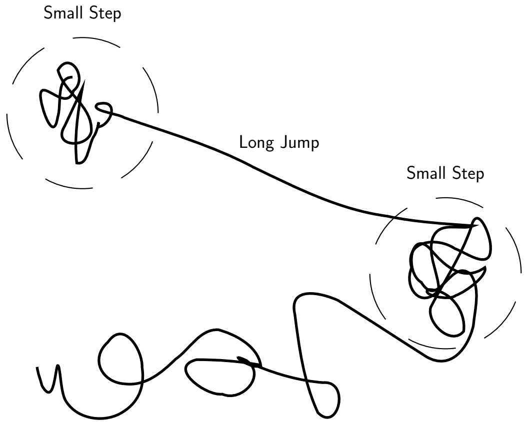

Figure: *Brownian motion* (standard diffusion) vs *Lévy jumps* (fractional diffusion).

Fractional cross-diffusion system

We consider the following *fractional cross-diffusion system* modeling the dynamics of multi-species populations:

$$\partial_t u_i = \operatorname{div} \sum_{j=1}^{n} a_{ij} u_i \nabla(-\Delta)^{-\frac{1-\beta}{2}} u_j, \qquad (t,x) \in (0,\infty) \times \Omega, \tag{1}$$

$$u_i(0,x) = u_{0,i}(x), \qquad x \in \Omega, \tag{2}$$

$$\nu \cdot \nabla(-\Delta)^{-\frac{1-\beta}{2}} u_i(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \partial\Omega, \tag{3}$$

for $i = 1, \ldots, n$. Here $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary, ν is the outward normal vector to $\partial\Omega$, $d \geq 2$, $0 < \beta < 1$, $a_{ij} \geq 0$ for $i, j = 1, \ldots, n$ are constants.

We define the *inverse (spectral) fractional Laplacian* $(-\Delta)^{-s}$, $0 < s < 1$, via its *spectral decomposition* as follows:

$$(-\Delta)^{-s}\phi = \sum_{k=1}^{\infty} \lambda_k^{-s} (\psi_k, \phi)_{L^2} \psi_k, \qquad \forall \phi \in H^2(\Omega),$$

where

$$-\Delta\psi_k = \lambda_k \psi_k \quad \text{in } \Omega, \qquad \nu \cdot \nabla\psi_k = 0 \quad \text{in } \partial\Omega, \qquad k \geq 1.$$

Assumptions on the matrix $(a_{ij})_{i,j=1,\dots,n}$

- *Detailed balance condition:*

$$\exists \pi_1, \dots, \pi_n > 0 : \quad \pi_i a_{ij} = \pi_j a_{ji}, \quad i, j = 1, \dots, n;$$

- Spectrum is entirely contained in the half-plane $\{z \in \mathbb{C} : \Re(z) > 0\}$.

Under these assumptions, the matrix $\pi_i a_{ij}$ is **symmetric and positive definite**.

Furthermore, system (1)–(3) admits the following **energy functional**:

$$H[u] = \int_\Omega \sum_{i=1}^n \pi_i u_i \log u_i \,\mathrm{d}x.$$

which gives the energy dissipation estimate

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} H[u] + \alpha \int_\Omega \sum_{i=1}^n |\nabla(-\Delta)^{-\frac{1-\beta}{4}} u_i|^2 \,\mathrm{d}x \le 0, \qquad t > 0,}$$

where $\alpha > 0$ is the smallest eigenvalue of the matrix $(\pi_i a_{ij})_{i,j=1,\dots n}$.

Standard cross-diffusion:

- [Jüngel, Nonlinearity 2015], [Chen-Jüngel, JDE 2021], etc. etc.
- [Daus-Tang, Appl. Math. Lett. 2019]: convergence to equilibrium in bounded domains.

Fractional cross-diffusion: In $\mathbb{R}^d$, the system

$$\begin{cases} \partial_t u_i + \sigma_i(-\Delta)^{\alpha} u_i - \operatorname{div} \sum_{j=1}^{n} a_{ij} u_i \nabla(-\Delta)^{-\frac{1-\beta}{2}} u_j = 0, & (t,x) \in (0,\infty)\times\mathbb{R}^d, \\ u_i(0,x) = u_{0,i}(x), & x \in \mathbb{R}^d \end{cases}$$

has been derived in [Daus-Ptashnyk-Raithel, JDE 2022] as the many-particle limit of the following *interacting particle system driven by Lévy noise*:

$$\mathrm{d}X_i^{k,N}(t) = -\sum_{j=1}^{n} \frac{1}{N} \sum_{\ell=1}^{N} a_{ij} \nabla(-\Delta)^{(\beta-1)/2} V_N \left(X_i^{k,N}(t) - X_j^{\ell,N}(t) \right) \mathrm{d}t + \sqrt{2\sigma_i}\, \mathrm{d}L_i^k(t),$$

where $i = 1, \ldots, n$ and $k = 1, \ldots, N$, $X_i^{k,N}(t)$ is the position of the k th particle of species i at time t, V_N is a potential function, and L_i^k is a Lévy process of index $\alpha \in (0,1)$.

Brief literature review

- [Daus-Ptashnyk-Raithel, JDE 2022]: global-in-time existence of strong solutions for sufficiently small initial data in $H^s\left(\mathbb{R}^d\right)$ (with $s > d/2$) in the regime $2\alpha > \beta + 1$, i.e. *when self-diffusion dominates cross-diffusion*;
- [Jüngel-Zamponi, JDE 2022]: global-in-time existence of weak solutions without any smallness assumption on the initial data and for all $\alpha, \beta \in (0, 1)$

Our aims

We study the problem of existence of weak solutions in a *bounded domain* and use relative-entropy methods to analyze weak-strong uniqueness and long-time behavior.

Main results: Existence

Theorem (Global existence of weak solutions)

Let us assume that the initial data $u_0 : \Omega \to [0,\infty)^n$ *satisfies* $\int_\Omega u_0 |\log u_0| \,\mathrm{d}x < \infty$ *for* $i = 1, \dots, n$ *(finite entropy).*

Then system (1)–(3) *has a weak solution* $u : (0,\infty) \times \Omega \to \mathbb{R}^n$ *with non-negative components satisfying*

$$u_i \in L^\infty(0,\infty; L^1(\Omega)), \quad \nabla(-\Delta)^{-\frac{1-\beta}{4}} u_i \in L^2(0,\infty; L^2(\Omega)),$$
$$\partial_t u_i \in L^r(0,\infty; W^{1,r'}(\Omega)'),$$

for $i = 1, \dots, n$*, with* $r > 1$ *suitable and* $r' = r/(r-1)$*, and*

$$\int_0^\infty \langle \partial_t u_i, \phi \rangle \,\mathrm{d}x + \sum_{j=1}^n a_{ij} \int_0^\infty \int_\Omega \nabla\phi \cdot u_i \nabla(-\Delta)^{-\frac{1-\beta}{2}} u_j \,\mathrm{d}x\,\mathrm{d}t = 0, \quad i = 1, \dots n,$$

$$u_i(t,\cdot) \to u_{0,i} \quad \text{strongly in } C^0([0,T], W^{1,r'}(\Omega)') \text{ as } t \to 0, \quad i = 1, \dots n,$$

for every $\phi \in L^{r'}(0,\infty; W^{1,r'}(\Omega))$*, where* $\langle \cdot, \cdot \rangle$ *is the duality product between* $W^{1,r'}(\Omega)'$ *and* $W^{1,r'}(\Omega)$*.*

Theorem (Weak-strong uniqueness)

Let us assume that (1)–(3) *has a weak solution* v *satisfying*

$$\exists q_2 > \frac{2d}{1+\beta} : \quad \nabla \log v_i \in L^{q_1}(0,T;L^{q_2}(\Omega)), \quad q_1 = \frac{2(d+1+\beta)}{1+\beta-2d/q_2}, \quad i = 1,\dots,n,$$

for every $T > 0$. *Then every weak solution* u *to* (1)–(3) *(in the sense of Theorem 1) coincides with* v *a.e. in* Ω, *for* $t > 0$.

Theorem (Long-time behaviour)

Let us assume that Ω *is connected. Then the weak solutions to* (1)–(3) *(in the sense of Theorem 1) converge as* $t \to \infty$ *strongly in* $L^1(\Omega)$ *towards* $u^\infty \equiv ⨍_\Omega u_0 \, \mathrm{d}x$ *with an exponential decay rate.*

We define the *truncated fractional Laplacian*

$$(-\Delta)_N^{-s} u = \sum_{i=1}^{N} \lambda_i^{-s} (\psi_i, u)_{L^2(\Omega)} \psi_i, \quad u \in L^2(\Omega)$$

and the *zero-average quadratic correction*

$$g_\rho[u](x) = u(x)(W_\rho * (\mathbb{1}_\Omega u))(x) - \fint_\Omega u(y)(W_\rho * (\mathbb{1}_\Omega u))(y)\,\mathrm{d}y, \quad u \in L^2(\Omega),$$

where W_ρ is a mollifier with the properties

$$W_\rho(x) = \rho^{-d} W_1(x/\rho) \quad \text{for } x \in \mathbb{R}^d, \quad \rho > 0,$$
$$W_1 \in C_c^0(\mathbb{R}^d), \quad W_1 \geq 0 \text{ in } \mathbb{R}^d, \quad \|W_1\|_{L^1(\mathbb{R}^d)} = 1.$$

NB: $W_\rho * u \to u$ a.e. in $\mathbb{R}^d$, and $g_\rho : L^2(\Omega) \to L_0^2(\Omega)$ is continuous.

We then consider the following approximated problem (*three-level approximation*):

$$\begin{cases} \partial_t u_i^{(\rho,N,\kappa)} - \kappa \Delta u_i^{(\rho,N,\kappa)} + \kappa g_\rho[u_i^{(\rho,N,\kappa)}] \\ \quad = \operatorname{div}\left(\sum_{j=1}^n a_{ij} (u_i^{(\rho,N,\kappa)})_+ \nabla (-\Delta)_N^{(\beta-1)/2} u_j^{(\rho,N,\kappa)} \right), & t>0, x \in \Omega, \\ \kappa \nu \cdot \nabla u_i^{(\rho,N,\kappa)} + \sum_{j=1}^n a_{ij} \nu \cdot \nabla (-\Delta)_N^{(\beta-1)/2} u_j^{(\rho,N,\kappa)} = 0, & t>0, \ x \in \partial\Omega, \\ u_i^{(\rho,N,\kappa)}(0,x) = u_{0,i}(x), & x \in \Omega, \end{cases}$$

where $z_+ = \max(0,z)$ denotes the positive part of $z \in \mathbb{R}$.

Key a priori estimates:

$$\begin{aligned} \|u_i\|_{L^\infty(0,\infty;L^1(\Omega))} + \|\nabla(-\Delta)^{(\beta-1)/2} u_i\|_{L^2(0,\infty;L^{d+\beta-1}(\Omega))} &\le C, \\ \exists p^*, q^* > 1: \quad \|u_i \nabla(-\Delta)^{(\beta-1)/2} u_j\|_{L^{q^*}(0,T;L^{p^*}(\Omega))} &\le C, \\ \exists p > 1: \quad \|\partial_t u_i\|_{L^p(0,\infty;W^{1,p'}(\Omega)')} &\le C, \end{aligned}$$

for $i,j = 1,\dots n$.

□

We consider the following **relative entropy** between u and v:

$$H[u|v] = H[u] - H[v] = \sum_{i=1}^{n} \pi_i \int_\Omega \left(u_i \log \frac{u_i}{v_i} - u_i + v_i \right) \mathrm{d}x.$$

Differentiating it in time, we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} & H[u|v] \\
&= \sum_{i=1}^{n} \pi_i \int_\Omega \left(\left(\log \frac{u_i}{v_i} \right) \partial_t u_i + \left(1 - \frac{u_i}{v_i} \right) \partial_t v_i \right) \mathrm{d}x \\
&= - \sum_{i,j=1}^{n} \pi_i a_{ij} \int_\Omega \left(u_i \nabla \log \frac{u_i}{v_i} \cdot \nabla (-\Delta)^{-\frac{1-\beta}{2}} u_j - v_i \nabla \frac{u_i}{v_i} \cdot \nabla (-\Delta)^{-\frac{1-\beta}{2}} v_j \right) \mathrm{d}x \\
&= - \sum_{i,j=1}^{n} \pi_i a_{ij} \int_\Omega (-\Delta)^{-\frac{1-\beta}{4}} \nabla (u_i - v_i) \cdot (-\Delta)^{-\frac{1-\beta}{4}} \nabla (u_j - v_j) \, \mathrm{d}x \\
&\quad + \sum_{i,j=1}^{n} \pi_i a_{ij} \int_\Omega \frac{u_i - v_i}{v_i} \nabla v_i \cdot \nabla (-\Delta)^{-\frac{1-\beta}{2}} (u_j - v_j) \, \mathrm{d}x,
\end{aligned}$$

where we integrated by parts and used $\pi_i a_{ij} = \pi_j a_{ji}$.

Using the fact that $\pi_i a_{ij}$ is positive definite with smallest eigenvalue is $\alpha > 0$, the Sobolev embedding theorem, Hölder and Young's inequalities, we deduce (after lots of computations...)

$$\frac{\mathrm{d}}{\mathrm{d}t} H[u|v] + \frac{\alpha}{4} \sum_{i=1}^{n} \int_{\Omega} \left|(-\Delta)^{-\frac{1-\beta}{4}} \nabla(u_i - v_i)\right|^2 \mathrm{d}x \leq C \sum_{i=1}^{n} \|\nabla \log v_i\|_{L^{q_2}}^{q_1} \|u_i - v_i\|_{L^1}^2,$$

with $q_1 = 2\frac{d+1+\beta}{1+\beta-2d/q_2}$, $q_2 = \frac{2q}{q-2} > \frac{2d}{1+\beta}$.

Csiszár-Kullback-Pinsker inequality then yields

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} H[u|v] + \frac{\alpha}{4} \sum_{i=1}^{n} \int_{\Omega} \left|(-\Delta)^{-\frac{1-\beta}{4}} \nabla(u_i - v_i)\right|^2 \mathrm{d}x \leq C \sum_{i=1}^{n} \|\nabla \log v_i\|_{L^{q_2}}^{q_1} H[u|v].}$$

We conclude the proof by applying Gronwall's lemma. □

Sketch of the convergence to equilibrium

By similar computations as before, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} H[u|u^\infty] \le -\alpha \int_\Omega \sum_{i=1}^n |\nabla(-\Delta)^{-\frac{1-\beta}{4}} u_i|^2 \,\mathrm{d}x,$$

where $H[u|u^\infty] = H[u] - H[u^\infty]$ is the relative entropy.

Applying a Poincaré-type inequality, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t} H[u|u^\infty] \le -C \int_\Omega \sum_{i=1}^n |u_i - u_i^\infty|^2 \,\mathrm{d}x,$$

which yields

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} H[u|u^\infty] \le -C H[u|u^\infty], \quad t > 0.}$$

Then, by Gronwall's inequality, we deduce

$$H[u(t)|u^\infty] \le e^{-Ct} H[u_0|u^\infty], \quad t > 0,$$

which implies the **strong convergence of** $u(t,\cdot)$ **towards the equilibrium** u^∞ **in** $L^1(\Omega)$ **with exponential rate** via a Csiszár-Kullback-Pinsker's inequality. □

Open problems and future work

- **Intermediate asymptotics and self-similar profiles**: not clear even for standard cross-diffusion in $\mathbb{R}^d$ (but cf. [Corrias-Escobedo-Matos, JDE 2014] for the *2D parabolic Keller-Segel chemotaxis system*);
- **Stabilization and controllability problems:** not fully clear, even for the porous medium equation –
  - [Coron-Diáz-Drici-Mignazzini, Chin. Ann. Math. 2013]: null controllability via return method; but avoiding free boundary formation;
  - [Geshkovski, ESAIM:COCV 2020]: controllability of pressure and its free boundary to the non-trivial Barenblatt profile, but for a "perturbed" PME with cut-off in the nonlinear term.
- **Partial Hölder regularity for fractional cross diffusion:** cf. [Braukhoff-Raithel-Zamponi, JMPA 2022] for SKT system (entropy dissipation inequalities instead of energy estimates in the Campanato iteration);
- **Interface propagation estimates:** see [Fischer, SIMA 2013] for Keller-Segel (which could also be improved by using the Stampacchia-type argument introduced in [De Nitti-Fischer, CPDE 2022]);
- **Many-particle limit in a bounded domain:** extending the results in [Daus-Ptashnyk-Raithel, JDE 2022] taking into account suitable boundary behavior for the interacting particle system (cf. [Garbaczewski-Stephanovich, Phys. Rev. E 2019]).

Thank you for your attention!

[1] N. De Nitti, N. Zamponi, and E. Zuazua. Fractional cross-diffusion in a bounded domain: existence, weak-strong uniqueness, and long-time asymptotics. *In preparation*, 2022.