Rapid stabilization of a degenerate parabolic equation

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5. Exponential stability

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The degenerate parabolic equation

In general, we consider the following degenerate parabolic equation.

$$\begin{cases} \partial_t u & = (x^{\alpha} \partial_x u)_x, & (t,x) \in \mathbb{R}^+ \times (0,1), \\ u(t,0) & = 0, \ u(t,1) = U(t), & t \in \mathbb{R}^+, \\ u(0,x) & = u_0(x), & x \in (0,1). \end{cases} \tag{1}$$

Where $\alpha \in (0,1)$, the initial condition $u_0 \in L^2(0,1)$ and U denotes the control.

Previous Results

Theorem [L. Gagnon, P, Lissy & S. Marx (2021). SIAM J. Control Optim.]

There exists a discrete set $\mathscr{S} \subset \mathbb{R}$ such that for any $\lambda > 0$ with $\lambda \notin \mathscr{S}$, there exists $C(\lambda) > 0$ and a feedback law $U(t) = K(u(t))$, where $K \in L^2(0,1)'$, such that for any $u_0 \in L^2(0,1)$, there exists a unique solution u of (18) that verifies moreover : for any $t > 0$,

$$||u(t,\cdot)||_{L^2(0,1)} \leq C(\lambda)||u_0||_{L^2(0,1)}e^{-\lambda t}$$

M. Gueye (2014) Exact boundary controllability of 1-D parabolic and hyperbolic degenerate equations., *SIAM J. Control Optim* 52(4), 2037–2054.

L. Gagnon, P. Lissy & S. Marx (2021) A Fredholm transformation for the rapid stabilization of a degenerate parabolic equation, *SIAM J. Control Optim* 59(5), 3828–3929.

Goal

We want to prove a boundary rapid stabilization result for a degenerate parabolic equation, in the case where the control is localized at $x = 0$.

The stabilization problem is described by the following system :

$$\begin{cases} \partial_t u & = (x^{\alpha} \partial_x u)_x, & (t,x) \in \mathbb{R}^+ \times (0,1), \\ u(t,0) & = U(t),\ u(t,1) = 0, & t \in \mathbb{R}^+, \\ u(0,x) & = u_0(x), & x \in (0,1). \end{cases} \tag{2}$$

Where $\alpha \in (0,1)$, the initial condition $u_0 \in L^2(0,1)$ and U denotes the control.

Theorem 1 [Lissy, Pierre, Moreno, C. (submitted, 2022.)]

There exists a discrete set $\mathscr{S} \subset \mathbb{R}$ such that for any $\lambda > 0$ with $\lambda \notin \mathscr{S}$, there exists $C(\lambda) > 0$ and a feedback law $U(t) = K(u(t))$, where $K \in L^2(0,1)'$, such that for any $u_0 \in L^2(0,1)$, there exists a unique solution u of (18) that verifies, for any $t \geq 0$:

$$||u(t,\cdot)||_{L^2(0,1)} \leq C(\lambda)||u_0||_{L^2(0,1)}e^{-\lambda t}$$

Degenerate operator

We define the spaces

$$H^1_\alpha(0,1) := \{f \in L^2(0,1) \,|\, x^{\frac{\alpha}{2}} f_x \in L^2(0,1)\},$$

and

$$H^1_{\alpha,0}(0,1) := \{f \in H^1_\alpha(0,1) \,|\, f(0) = f(1) = 0\},$$

The unbounded operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ is given by :

$$\begin{cases} Au := (x^\alpha u_x)_x, \\ D(A) := \{u \in H^1_{\alpha,0}(0,1) \,|\, x^\alpha u_x \in H^1(0,1)\}. \end{cases}$$

There exists a Hilbert basis $\{\phi_n\}_{n\in\mathbb{N}^*}$ of $L^2(0,1)$ and an increasing sequence $(\lambda_n)_{n\in\mathbb{N}^*}$ of real positive numbers such that $\lambda_n \to +\infty$ and

$$-A\phi_n = \lambda_n \phi_n. \tag{3}$$

We have

$$\lambda_n := (\kappa j_{\nu,n})^2, \quad n \in \mathbb{N}^*, \tag{4}$$

and

$$\phi_n(x) = \frac{(2\kappa)^{1/2}}{J'_\nu(j_{\nu,n})} x^{(1-\alpha)/2} J_\nu(j_{\nu,n} x^\kappa), \quad x \in (0,1), \quad n \in \mathbb{N}^*. \tag{5}$$

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Where ν and κ are two parameters given by

$$\nu := \frac{1-\alpha}{2-\alpha} \text{ and } \kappa := \frac{2-\alpha}{2}. \tag{6}$$

The derivative is given by

$$\phi_n'(y) = \frac{(2\kappa)^{1/2}}{J_\nu'(j_{\nu,n})}\left[\frac{1-\alpha}{2}y^{-\frac{1+\alpha}{2}}J_\nu(j_{\nu,n}y^\kappa) + y^{\frac{1-\alpha}{2}}\kappa j_{\nu,n}J_\nu'(j_{\nu,n}y^\kappa)y^{-\frac{\alpha}{2}}\right], \tag{7}$$

and $\lambda > 0$ was chosen in such a way that

$$\lambda_n - \lambda \neq 0,\ \lambda_n - \lambda \neq \lambda_k,\ \lambda_n - \lambda \neq \kappa^2 y_{\nu,k}^2 \text{ for any } k, n \in \mathbb{N}^*, \tag{8}$$

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and $\lambda > 0$ was chosen in such a way that

$$\lambda_n - \lambda \neq 0, \;\; \lambda_n - \lambda \neq \lambda_k, \;\; \lambda_n - \lambda \neq \kappa^2 y_{\nu,k}^2 \text{ for any } k, n \in \mathbb{N}^*, \tag{8}$$

For a real number ν, the Bessel functions are solutions of the following differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\nu^2)y=0,\ \ x\in(0,\infty),$$

Bessel functions of the first kind :

$$J_\nu(x)=\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}, \tag{9}$$

where $\Gamma(\cdot)$ is the gamma function.

Bessel functions of the first kind

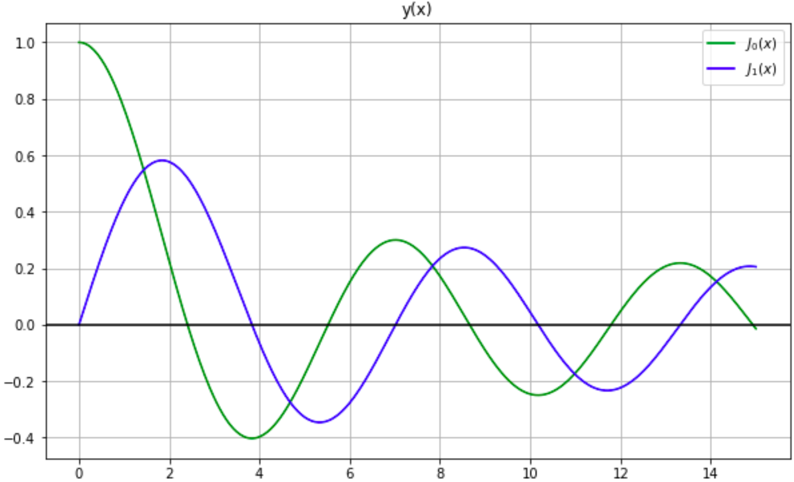

Figure 1: The first two Bessel functions

Bessel functions of the second kind

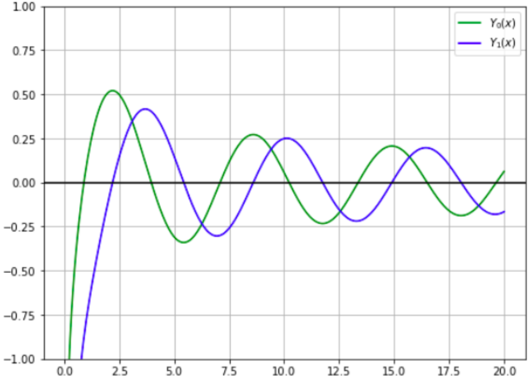

Figure 2: The first two Bessel functions

$$Y_\nu(x) := \frac{(\cos \nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}, \quad \nu \in (0,1). \tag{10}$$

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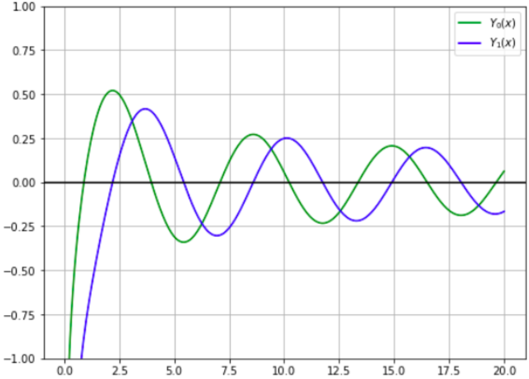

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Properties

Orthogonality property

$$\int_0^1 x^{1-\alpha} J_\nu(j_{\nu,n}x^\kappa) J_\nu(j_{\nu,m}x^\kappa) dx = \frac{\delta_{nm}}{2\kappa} J'_\nu(j_{\nu,n})^2. \tag{11}$$

Lemma 1

For all $(a, b) \in \mathbb{R}$ such that $a \neq b$,

$$\begin{aligned} \int_0^1 x J_\nu(ax) Y_\nu(bx) dx = {} & \frac{1}{a^2 - b^2} \left[b J_\nu(a) Y'_\nu(b) - a J'_\nu(a) Y_\nu(b) \right] \\ & - \lim_{z \to 0^+} \frac{z}{a^2 - b^2} \left[b J_\nu(az) Y'_\nu(bz) - a J'_\nu(az) Y_\nu(bz) \right]. \end{aligned} \tag{12}$$

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Lemma 2

We have

$$\phi_n'(1) = (2\kappa)^{1/2}\kappa j_{\nu,n}, \tag{13}$$

and

$$\lim_{y\mapsto 0^+} y^\alpha \phi_n'(y) = \frac{(1-\alpha)(2\kappa)^{\frac{1}{2}} j_{\nu,n}^{\nu}}{2^\nu J_\nu'(j_{\nu,n})\Gamma(1+\nu)}. \tag{14}$$

There exist two constants $C_1 > 0$ and $C_2 > 0$ such that, for any $n \in \mathbb{N}^*$, we have

$$\frac{C_1}{n^{\frac{3}{2}}} \leqslant \left| J_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}\right)\right| \leqslant \frac{C_2}{n^{\frac{3}{2}}}. \tag{15}$$

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Interpolation spaces

We define, for $s \in [0,1]$ the spaces

$$D(A^s) = \left\{ f = \sum_{n=1}^{\infty} a_n \phi_n \in L^2(0,1) \,\middle|\, \sum_{n=1}^{\infty} \lambda_n^{2s} a_n^2 < \infty \right\} \tag{16}$$

and

$$D(A^s)' = \left\{ f = \sum_{n=1}^{\infty} a_n \phi_n \in \mathscr{D}'(0,1) \,\middle|\, \sum_{n=1}^{\infty} \frac{a_n^2}{\lambda_n^{2s}} < \infty \right\}. \tag{17}$$

The degenerate parabolic equation :

$$\begin{cases} \partial_t u & = (x^{\alpha}\partial_x u)_x, & (t,x) \in \mathbb{R}^+ \times (0,1), \\ u(t,0) & = U(t), \; u(t,1) = 0, & t \in \mathbb{R}^+, \\ u(0,x) & = u_0(x), & x \in (0,1). \end{cases} \tag{18}$$

Where $\alpha \in (0,1)$, the initial contion $u_0 \in L^2(0,1)$ and U denotes the control.

The target system is described by the following equation :

$$\begin{cases} \partial_t v & = (x^\alpha \partial_x v)_x - \lambda v, \quad (t,x) \in \mathbb{R}^+ \times (0,1), \\ v(t,0) & = v(t,1) = 0, \quad t \in \mathbb{R}^+, \\ v(0,x) & = v_0(x), \quad x \in (0,1). \end{cases}, \tag{19}$$

Let $V(t) := \int_0^1 v(t,x)^2 dx$, for any $t \geqslant 0$

$$V(t) \leqslant V(0)e^{-2\lambda t} \implies ||v||^2_{L^2(0,1)} \leq e^{-2\lambda t}||v_0||^2_{L^2(0,1)} \tag{20}$$

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$$V(t) \leqslant V(0) e^{-2\lambda t} \implies ||v||^2_{L^2(0,1)} \leq e^{-2\lambda t} ||v_0||^2_{L^2(0,1)} \tag{20}$$

Backstepping transformation

We introduce the linear operator given by

$$Tf : x \longmapsto f(x) - \int_0^1 k(x,y) f(y) dy, \tag{21}$$

Then

$$v = Tu : x \longmapsto u(t,x) - \int_0^1 k(x,y) u(t,y) dy, \tag{22}$$

verifies the target system.

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verifies the target system.

Kernel PDE

The kernel k has to satisfy the following system :

$$\begin{cases} -(y^\alpha k_y(x,y))_y + (x^\alpha k_x(x,y))_x - \lambda k(x,y) & = -\lambda \delta_{x=y}, & (x,y) \in (0,1)^2, \\ \lim\limits_{y \to 0^+} y^\alpha k_y(x,y) & = 0, & (x,y) \in (0,1)^2, \\ k(x,0) & = 0, & x \in (0,1), \\ k(x,1) & = 0, & x \in (0,1), \\ k(1,y) & = 0, & y \in (0,1). \end{cases} \tag{23}$$

We can decompose the kernel k formally as

$$k(x,y) = \sum_{n \in \mathbb{N}^*} \psi_n(x) \phi_n(y). \tag{24}$$

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We can decompose the kernel k formally as

$$k(x,y) = \sum_{n\in\mathbb{N}^*} \psi_n(x)\phi_n(y). \tag{24}$$

We obtain for $n \in \mathbb{N}^*$ the system

$$\begin{cases} -\lambda_n \psi_n(x) + (x^\alpha \partial_x \psi_n(x))_x - \lambda \psi_n(x) & = -\lambda \phi_n(x), \quad x \in (0,1), \\ \psi_n(1) & = 0, \\ \lim_{y \to 0^+} y^\alpha \sum_{n \in \mathbb{N}^*} \psi_n(x) \phi_n'(y) & = 0, \quad x \in (0,1). \end{cases} \tag{25}$$

Using the change of unknowns $\psi_n = \phi_n - \xi_n$, we get the system

$$\begin{cases} -(\lambda_n - \lambda)\xi_n(x) - (x^\alpha \partial_x \xi_n(x))_x & = 0, & x \in (0,1), \\ \xi_n(1) & = 0, \\ \lim_{y \to 0^+} y^\alpha \sum_{n \in \mathbb{N}^*} \xi_k(x) \phi_k'(y) & = \lim_{y \to 0^+} y^\alpha \sum_{n \in \mathbb{N}^*} \phi_k(x) \phi_k'(y), & x \in (0,1). \end{cases} \tag{26}$$

Kernel

We obtain for $n \in \mathbb{N}^*$ the system

$$\begin{cases} -\lambda_n \psi_n(x) + (x^\alpha \partial_x \psi_n(x))_x - \lambda \psi_n(x) & = -\lambda \phi_n(x), \quad x \in (0,1), \\ \psi_n(1) & = 0, \\ \lim\limits_{y \to 0^+} y^\alpha \sum\limits_{n \in \mathbb{N}^*} \psi_n(x) \phi_n'(y) & = 0, \qquad\qquad x \in (0,1). \end{cases} \tag{25}$$

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$$\begin{cases} -(\lambda_n - \lambda)\xi_n(x) - (x^\alpha \partial_x \xi_n(x))_x & = 0, & x \in (0,1), \\ \xi_n(1) & = 0, & \\ \lim\limits_{y \to 0^+} y^\alpha \sum\limits_{n \in \mathbb{N}^*} \xi_k(x) \phi_k'(y) & = \lim\limits_{y \to 0^+} y^\alpha \sum\limits_{n \in \mathbb{N}^*} \phi_k(x) \phi_k'(y), & x \in (0,1). \end{cases} \tag{26}$$

Sturm-Liouville

The Sturm-Liouville problem is described by the following equation

$$\begin{cases} -(x^{\alpha} y'(x))' & = \mu y(x), \quad x \in (0,1),\ \mu \in \mathbb{R}, \\ y(0) & = 0, \\ y(1) & = 0. \end{cases} \tag{27}$$

The problem (27) is a particular case of the Bessel equation

$$x^2 y'' + axy' + (bx^l + c)y = 0, \ \ x \in (0, \infty)$$

where a, b, and c are real numbers and $l \neq 0$. Then, we infer

$$y(x,\mu) = x^{\frac{1}{2}(1-\alpha)} Z_{\nu}\left(\frac{\sqrt{\mu} x^{\kappa}}{\kappa}\right), \ x \in (0, \infty),$$

where Z_{ν} is any Bessel function general solution.

We obtain that the solution of system (26) can be written under the form

$$\xi_n(x) = A_n x^{\frac{1}{2}(1-\alpha)} J_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa} x^\kappa\right) + B_n x^{\frac{1}{2}(1-\alpha)} Y_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa} x^\kappa\right),$$

where An, Bn are real numbers and $\xi_n(0) = 1$, then

$$B_n = -\frac{A_n J_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}\right)}{Y_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}\right)}.$$

We consider

$$\widetilde{\xi}_n(x) = \frac{\sqrt{2\kappa}x^{\frac{1-\alpha}{2}}}{J'(j_{\nu,n})} \left(J_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}x^\kappa\right) - \frac{J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)} Y_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}x^\kappa\right)\right), \tag{28}$$

and ψ_n under the form

$$\psi_n = \phi_n - c_n\tilde{\xi}_n, \tag{29}$$

with $c_n \in \mathbb{R}$ to be determined.

Existence and uniqueness of the kernel

Theorem 2 [Lissy, Pierre, Moreno, C. (submitted, 2022.)]

Assume that (8) holds. There exists a unique sequence $(c_n)_{n\in\mathbb{N}^*}$ such that

$$c_n - 1 \in l^2(\mathbb{N}^*) \tag{30}$$

and such that for any $n \in \mathbb{N}^*$, the corresponding ψ_n defined in (29) verifies (25).

Proposition 1 [Lissy, Pierre, Moreno, C. (submitted, 2022.)]

The family $\{(\lambda_n)^{\frac{\nu}{2}+\frac{1}{4}}\tilde{\xi}_n\}_{n\in\mathbb{N}^*}$ is a Riesz basis in $D(A^{\frac{\nu}{2}+\frac{1}{4}})'$.

We define the linear operators $B : \mathbb{R} \to D(A^{\nu/2+3/4})'$ as follows :

$$\langle Bz, \varphi \rangle_{D(A^{\nu/2+3/4})', D(A^{\nu/2+3/4})} = z \lim_{x \to 0^+} x^{\alpha} \varphi'(x), \; \varphi \in D(A^{\nu/2+3/4}). \tag{31}$$

and $K : L^2(0,1) \to \mathbb{R}$ given by

$$Kf : x \longmapsto \int_0^1 k(0,y) f(y) dy. \tag{32}$$

The transformation K belongs to $(L^2(0,1))'$.

Linear Operators

Setting $U = Ku$, the degenerate equation can be rewritten as

$$\begin{cases} \partial_t u = (A + BK)u, & t \in (0,T), \\ u(0,\cdot) = u_0, & x \in (0,1). \end{cases}$$

We define the spaces

$$H^1_{\alpha,R}(0,1) := \{f \in H^1_\alpha(0,1) \,|\, f(1) = 0\}$$

,

$$D(A)_R := \{f \in H^1_{\alpha,R}(0,1) \,|\, x^\alpha f_x \in H^1(0,1)\},$$

and

$$D(A + BK) = \{f \in D(A)_L \,|\, f(0) = Kf\}.$$

Let us remember that the operator $T : L^2(0,1) \to L^2(0,1)$ is given by

$$Tf : x \longmapsto f(x) - \int_0^1 k(x,y)f(y)dy. \tag{33}$$

The transformation T belongs to $\mathscr{L}(L^2(0,1))$.

Identities

1. $TB = B$ in $D(A^{\frac{\nu}{2}+\frac{3}{4}})'$.

2. $T(A + BK)f = (A - \lambda I)Tf$ in $L^2(0,1)$.

The transformation T is invertible from $L^2(0,1)$ to $L^2(0,1)$.

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Proof of Theorem 1.

Steps

- 1. The operator $A + BK$ is dissipative.
- 2. $A + BK$ is maximal.
- 3. Exponential stability.

Exponential stability

Using that $v = Tu$ and (20), we have, for all $t \geq 0$,

$$\begin{aligned} ||u(t,\cdot)||_{L^2(0,1)} =& ||T^{-1}Tu(t,\cdot)||_{L^2(0,1)} \\ &\leq ||T^{-1}||_{L^2(0,1)}||Tu(t,\cdot)||_{L^2(0,1)} \\ &\leq ||T^{-1}||_{L(L^2(0,1))}e^{-\lambda t}||Tu_0||_{L(L^2(0,1))} \\ &\leq ||T^{-1}||_{L(L^2(0,1))}||T||_{L^2(0,1)}e^{-\lambda t}||u_0||_{L^2(0,1)}. \end{aligned}$$

This concludes Step 3 and the proof of Theorem 1.

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&\leq ||T^{-1}||_{L(L^2(0,1))}e^{-\lambda t}||Tu_0||_{L(L^2(0,1))} \\
&\leq ||T^{-1}||_{L(L^2(0,1))}||T||_{L^2(0,1)}e^{-\lambda t}||u_0||_{L^2(0,1)}.
\end{aligned}
$$

This concludes Step 3 and the proof of Theorem 1.

Proof of Theorem 1.

Steps

- 1. The operator $A + BK$ is dissipative.
- 2. $A + BK$ is maximal.
- 3. Exponential stability.

Exponential stability

Using that $v = Tu$ and (20), we have, for all $t \geq 0$,

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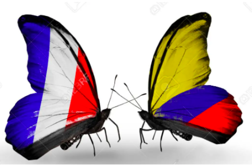

Thank you for your attention.