IX Partial Differential Equations, Optimal Design and Numerics

Nodal Control on Networks of Vibrating Strings
via Duality Method and Constructive Method

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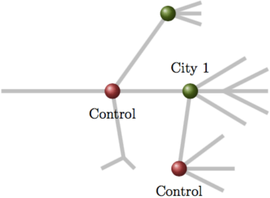
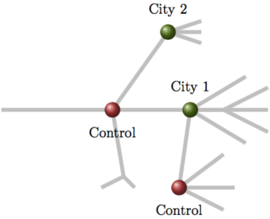
Motivation from Application



- The coupling of gas pipes.
- State function [isothermal Euler equations]
  - $\rho(t,x)$: the density of the gas,
  - $q(t,x)$: the flux in the pipe.
- Nodal Controls $u(t)$: Pressure increases at the compressor stations.

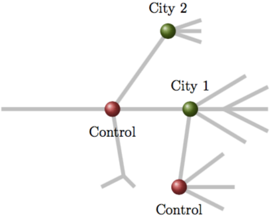
Motivation from Application



- Q: Can we find controls to satisfy the given demand of the cities?
  Aim: T*he boundary traces of state* to *exactly* fit any given profile as function of time on a node *after a suitable time* $t = T$ by means of *boundary controls*. [= Exact boundary controllability of nodal profile]

Origins of Nodal Control Problems

Motivation from Application



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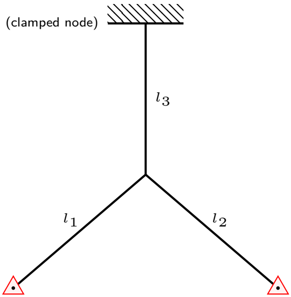
- Answer: Yes! (in local sense, and at least after a waiting time T^*). [M.Gugat 2010, 2014, T.Li 2010]

Nodal Control Problem on Network of Vibrating Strings

■ **Toy Model.** Vibrations of a non-trivial network formed by three elastic strings.



- $y^i = y^i(t,x) : [0, l_i] \to \mathbb{R}, i = 1, 2, 3$, the transversal displacements of the strings, governed by **linear/nonlinear** wave equation.

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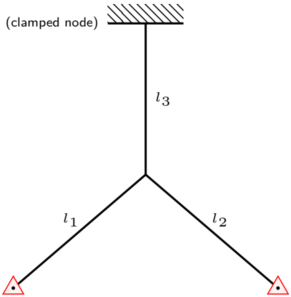

- $y^i = y^i(t,x) : [0, l_i] \to \mathbb{R}, i = 1, 2, 3$, the transversal displacements of the strings, governed by **linear/nonlinear** wave equation. (Working space? Wellposedness?)

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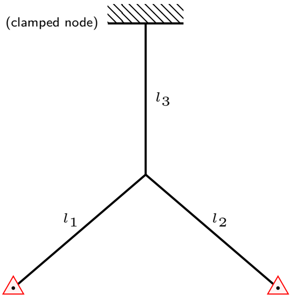

- $y^i = y^i(t,x) : [0, l_i] \to \mathbb{R}, i = 1, 2, 3$, the transversal displacements of the strings, governed by **linear/nonlinear** wave equation. (Working space? Wellposedness?)
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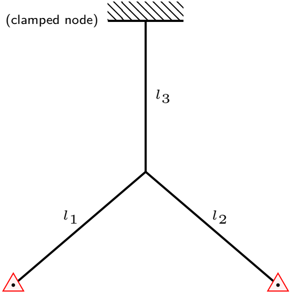

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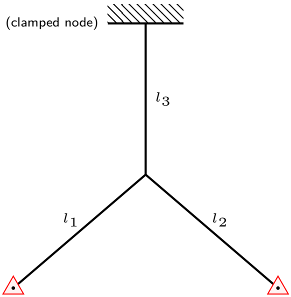

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- Boundary controls, such as Neumann controls $y^i_x(t, l_i) = u^i(t), i = 1, 2$.
- Coupled at one of their extremes at $x = 0$.

Can we find boundary controls such that for any given initial data, after a suitable time $t = \overline{T}$ the boundary traces of solution exactly fit $y^3(t, l_3) = \bar{y}_3(t), t \geq \overline{T}$?

How? Two Method

- Duality Method
  - > Duality between controllability and observability.
  - > Working space: Hilbert space.
  - > Exact controllability problem (Final data at T): J.-L. Lions[1], E. Zuazua[2], J. M.Coron[3].
  - > Main difficulty: observability inequalities.
  - > Main result: Observability and Controllability of nodal profile. [Linear case (Y. Wang 2022), can be extended to semi-linear case]

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[1] J. -L. Lions, Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués, Vol. I, Masson, (1988).

[2] E. Zuazua, Exact controllability for semilinear wave equations in one space dimension, Annales de II. H. P., section C, tome 10, no 1 (1993), p. 109-129.

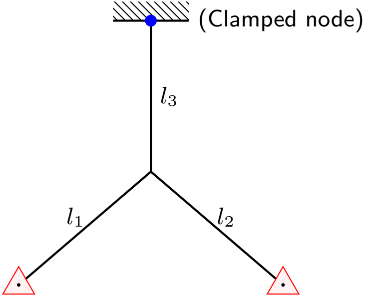
[3] J. M. Coron, Control and Nonlinearity, Mathematical Surveys and Monographs Vol.136, American Mathematical Society, 2007.

How? Two Method (ctd.)

- Duality Method
- Constructive Method
  - **Nonlinear case.** Lack of duality. (Implicit duality[4]. Exact Observability Problem for quasilinear hyperbolic problem.)
  - Working space: $C^k(k \geq 1)$ space. [Existence and uniqueness of **semi-global classical solution**] (small and smooth solution).
  - Design the desired control by explicit constructive method with modular structure. (Control is not unique!)
  - Main result: local controllability in nonlinear case, and global result in linear case. (Y.Wang 2018, 2021, 2022)
  - Can be extend to many **1d hyperbolic systems** (such as gas networks, water networks, etc. joint work with **G. Leugering, C.Rodriguez, G. Vergara-Hermosilla**). See my lecture in next week.

[4] Tatsien Li. Controllability and Observability for Quasilinear Hyperbolic Systems. 2010.

Three string Network with two Neumann controls



$$\begin{cases} y^i_{tt} - y^i_{xx} = 0, \qquad (t,x) \in (0,T) \times (0,l_i), \quad i = 1,2,3, \\ x = 0 : y^1(t,0) = y^2(t,0) = y^3(t,0), \qquad t \in (0,T), \\ y^0_x(t,0) + y^1_x(t,0) + y^2_x(t,0) = 0, \qquad t \in (0,T), \\ x = l_3 : y^3(t,l_3) = 0, \qquad t \in (0,T), \\ x = l_i : y^i_x(t,l_i) = u^i(t), \qquad t \in (0,T), \quad i = 1,2, \end{cases} \tag{1}$$

Working space

We introduce the working spaces

$$V = \left\{ \psi \in \prod_{i=1}^{3} H^1(0, l_i) \middle| \psi^1(0) = \psi^2(0) = \psi^3(0), \psi^3(l_3) = 0 \right\},$$
$$H = \prod_{i=1}^{3} L^2(0, l_i) \tag{2}$$

with the inner product

$$(\vec{\phi}, \vec{\psi})_V = \sum_{i=1}^{3} \frac{1}{2} \int_0^{l_i} \phi_x^i \cdot \psi_x^i \mathrm{d}x,$$
$$(\vec{\phi}, \vec{\psi})_H = \sum_{i=1}^{3} \frac{1}{2} \int_0^{l_i} \phi^i \cdot \psi^i \mathrm{d}x. \tag{3}$$

In the following, $\vec{y} = (y^1, y^2, y^3)$, and $\vec{y}_0, \vec{y}_1$ denotes the corresponding initial displacement and velocity, respectively.

Theorem (Wellposedness)

For any given initial data $(\vec{y}_0, \vec{y}_1) \in V \times H$ and boundary functions $u^1, u^2 \in L^2(0,T)$, the corresponding compatibility conditions are satisfied at the nodes $(0, l_1)$ and $(0, l_2)$, then system admits a unique solution [5]:

$$\vec{y} \in C([0,T];V) \bigcap C^1([0,T];H). \tag{4}$$

Furthermore, we can get a Neumann trace hidden regularity result at the end $x = l_3$ with homogenous Dirichlet boundary condition that: the map

$$\{\vec{y}_0, \vec{y}_1\} \mapsto y^3_x\big|_{x=l_3} : V \times H \to L^2\big(\{x = l_3\} \times (0,T)\big) \tag{5}$$

is continuous.

[5] J.E. Lagnese, G. Leugering, Schmidt, Modeling, Analysis and Control of Dynamic Elastic Multi-Link Structures, 1994

Duality Method in Nodal Control Problem

Nodal Control Problem: Let $T > \overline{T} > 0$. For given desired profile function $\bar{y}_3(t) \in L^2(\overline{T}, T)$ to find boundary controls $u^1, u^2 \in L^2(0, T)$ so that

$$\mathcal{F}(u^1, u^2) = \bar{y}_3(t).$$

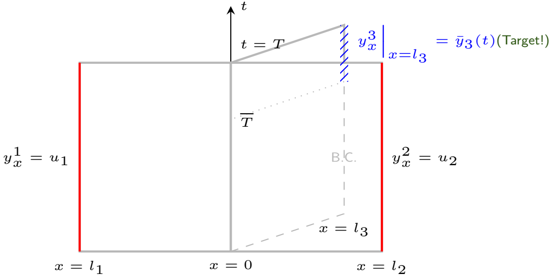


Figure: Nodal Control Problem: y-system

Control Operator: Define a mapping to describe the control problem
For given $T > \overline{T} > 0$, define a mapping

$$\boxed{\begin{aligned} \mathcal{F}:\ U &\longrightarrow L^2(\overline{T}, T) \\ (u^1, u^2) &\mapsto y_x^3(t, l_3) \end{aligned}} \tag{6}$$

where $U = L^2(0,T) \times L^2(0,T)$, $y_x^3(t,l_3)$ is the Neumann trace at the end $x = l_3$ of the corresponding solution $\vec{y} \in C^0([0,T]; V) \cap C^1([0,T]; H)$ to the system (1) with null initial data $(\vec{y}_0, \vec{y}_1)(x) = (0,0)$ and boundary controls u^1, u^2 at $x = l_1$ and $x = l_2$. Then

Definition [Boundary Controllability of Nodal Profile]

The system (1) is controllable of nodal profile at charged node $x = l_3$ in time $t \in (\overline{T}, T)$ if the control operator $\mathcal{F}$ is onto.

Main Result

Our main controllability result is

Theorem (Leugering-Li-Wang 2022)

Let

$$\overline{T} = l_3 + \max\{l_1, l_2\} \tag{7}$$

and $T > \overline{T}$. Then the system (1) is controllable of nodal profile at $x = l_3$ in time $t \in (\overline{T}, T)$.

Proof via Duality Method

Framework:

- Controllability of nodal profile $\Longleftrightarrow \mathcal{F}$ is onto

[7] Rudin, Functional Analysis, 1973, P97;
[7] Coron, Control and Nonlinearity, P35

Proof via Duality Method

Framework:

- Controllability of nodal profile $\Longleftrightarrow$ $\mathcal{F}$ is onto
- the surjectivity of a certain linear map $\mathcal{F}$ from a Hilbert space H_1 to another Hilbert space H_2 $\Longleftrightarrow$ the existence of $c > 0$ such that

$$\|\mathcal{F}^*(x_2)\|_{H_1} \geq c\|x_2\|_{H_2}, \quad \forall x_2 \in H_2, \tag{8}$$

where $\mathcal{F}^* : H_2 \to H_1$ is the adjoint of $\mathcal{F}$. This inequality (8) is called the observability inequality. [6][7]

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- So, we need to

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- and then prove (8).

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Key: Observability Inequality

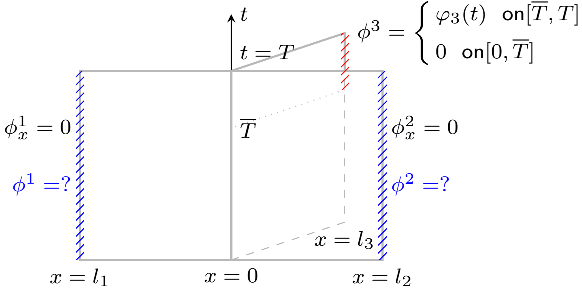


Figure: Dual Problem: ϕ -system

$$\mathcal{F}^*(-\varphi_3(t)) = (\phi^1(\cdot, l_1), \phi^2(t, l_2)) \quad \forall \varphi_3(t) \in H^1_0(\overline{T}, T),$$

Theorem

Let $\overline{T} > l_3 + \max\{l_1, l_2\}$ (sharp!) and $T > \overline{T}$. There exists a constant $c > 0$, such that the observability inequality of the adjoint problem holds:

$$\| \left(\phi^1(\cdot, l_1), \phi^2(\cdot, l_2)\right) \|_{L^2(0,T)\times L^2(0,T)} \geq c\|\varphi_3(\cdot)\|_{L^2(\overline{T},T)}, \forall \varphi_3 \in H^1_0(\overline{T}, T). \quad (9)$$

Key: Observability Inequality (ctd.)

From the densely defined of $\mathcal{F}^*$, here we only prove the observability inequality for $\varphi_3 \in H_0^1(\overline{T}, T)$.

Two methods:

- the explicit solution of the adjoint problem, side-wise D' Alembert Formula[8]. (Dáger, Zuazua)
- the multiplier method. (Alabau)

[8] René Dáger, Enrique Zuazua. Wave Progagation, Observation and Control in 1 d Flexible Multi-Structures

Networks of nonlinear vibrating strings

Consider the following coupled system of 1-D quasilinear wave equations $(i = 1, ..., n)$:

$$
(\mathbf{E}) \begin{cases} y^i_{tt} - (K^i(y^i, y^i_x))_x = F(\mathbf{y}, \mathbf{y}_x, \mathbf{y}_t), \quad x \in [0, L_i], t \in [0, T] \\ \sum K^i(y^i(t,0), y^i_x(t,0)) = 0, \ t \in [0, T] \\ y^j(t,0) = y^i(t,0), i \neq j, \\ y^i(t, L_i) = u^i(t), \ t \in [0, T] \\ (y^i, y^i_t)(0, x) = (\phi^i(x), \psi^i(x)), \quad x \in [0, L_i]. \end{cases}
$$

where

- $\mathbf{y} = (y^1, ..., y^n)^{\mathrm{T}}$ is an unknown vector function of (t, x),
- $K^i = K^i(y^i, y^i_x)$ are given C^2 functions of y^i and y^i_x,
- $\frac{\partial}{\partial y^i_x} K^i(y^i, y^i_x) > 0$,
- u^i can be considered as 0 (no control) or control function.

Setting of the Nodal Profile Problem

- Initial data: $C^2([0, L_i]; \mathbb{R}) \times C^1([0, L_i]; \mathbb{R})(i = 1, ..., n)$.
- Control space: $u^j \in C^2[0, T])$
- We ask the boundary trace of the solution to (E) satisfies:

$$y_x^i(t, L_i) = \bar{y}_i(t), \quad \overline{T} \leq t \leq T,$$

where $\bar{y}_i(t) \in C^1[\overline{T}, T]$ are given profile as functions of time and compatible to the I.C.

Nodal Controllability

Theorem

In a neighbourhood of an equilibrium (around 0), the system (E) is locally exact boundary controllable of nodal profile by **only 1 control** when (controllability time, sharp)

$$T > \bar{T}.$$

- Remark 1: The wellposedness of IBVP: the **existence** and **uniqueness** of semi-global classical solution with small norm (Y.Wang, 2017).
- Remark 2: HUM method (J.Lions, 1980s) and duality method (E.Zuazua, 1990s) can not be applied on this case.
- Local controllability: (ϕ, ψ) and $\bar{y}_i$ are close to the equilibrium point (Y.Wang, Li, Leugering, 2019).

Nodal Controllability

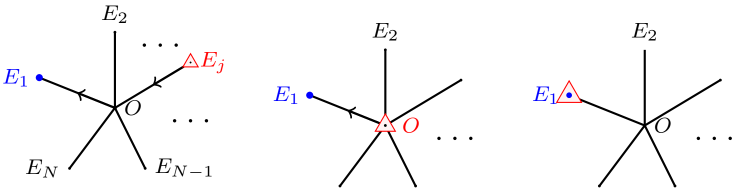
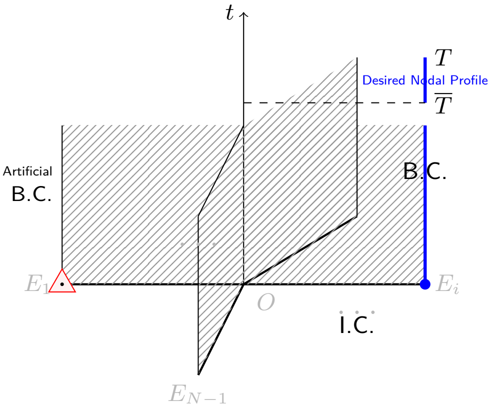


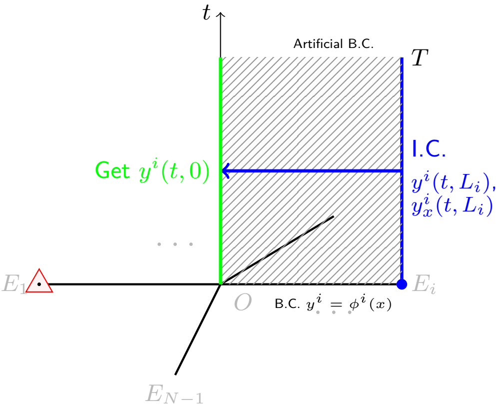
Fig.	Charged node	Controlled node	Controllability Time T
(a)	E_1	$E_j (j \neq 1)$	$\overline{T} > \frac{L_1}{\sqrt{K^1_{y^1_x}(0,0)}} + \frac{L_j}{\sqrt{K^j_{y^j_x}(0,0)}}$
(b)	E_1	O	$\overline{T} > \frac{L_1}{\sqrt{K^1_{y^1_x}(0,0)}}$
(c)	E_1	E_1(in-situ)	$\overline{T} > 0$

Constructive Method



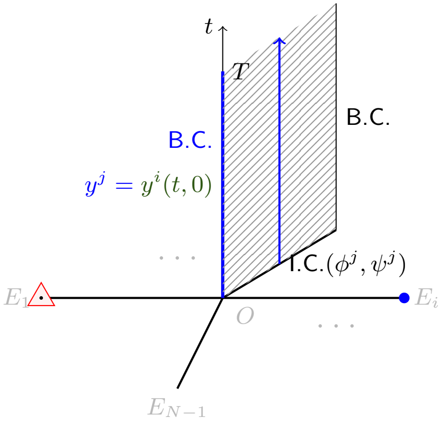
1. Forward problem on the whole network.
2. Connect the trace. (not unique)

Constructive Method



1. Forward problem on the whole network.
2. Connect the trace. (not unique)
3. Change the role of t and x. Sidewise problem from the charged node.

Constructive Method



1. Forward problem on the whole network.
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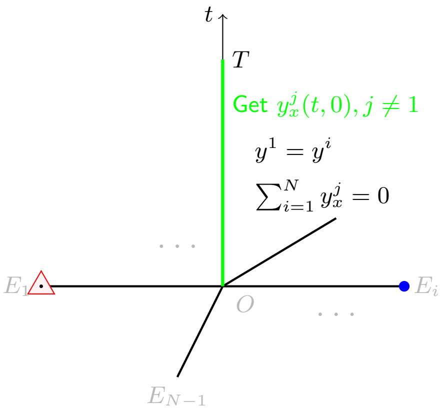
4. Solve forward problem on uncontrolled strings.

Constructive Method



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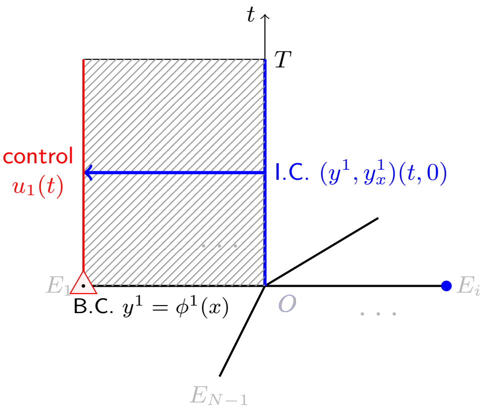

1. Forward problem on the whole network.
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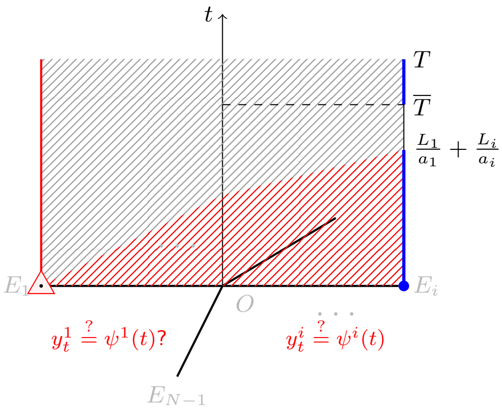
5. Solve sidewise problem from $x = 0$ to the controlled end
6. Determine the boundary control by the trace.

Constructive Method



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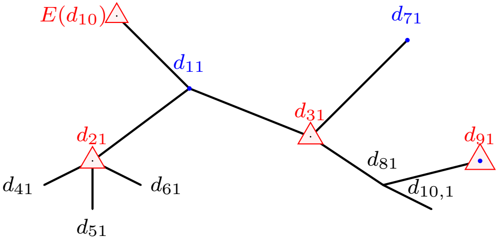
6. Determine the boundary control by the trace.
7. Verify I.C.; Uniqueness theory.



* Optimal controllability time T^*.
* Minimum number of controls.
* Placement of controls.
* Calculation of controls.

- Nodal Profile Control: Our aim is to fit (a part of) the boundary traces to a given profile after a suitably long time $t = T$ by means of boundary controls. **[Project: Control theory on planar or spacial string networks: controllability and partial nodal control for quasilinear hyperbolic systems. (Individual funding & NSFC-1121101.Joint work with T.Li.]**

Outline

- Duality Method
  - Linear case
  - **Duality between controllability and observability**
  - **Observability inequality**
  - **Observability and Controllability of nodal profile**
- Constructive Method
  - **Nonlinear case**
  - **semi-global classical solution**
  - **local controllability**
  - **Design** the desired control by explicit constructive method with modular structure.

Extension & Perspectives

- Extension to nonlinear mechanical systems: elastic multi-body structures, flexible structures, robotic systems.
- Extension to pipe-flow: gas network, water networks, flow-structure interaction. [EU Project: Conflex; DFG project TRR154]
- Damage problem: the sustainable control of mechanical structures, which suffer damage and possibly failure, especially in coupling points.[funded by DFG]
- Open Problem in Math: Nonlinear Analysis, Controllability Properties, Realization and Numerical Approximation.
- Large Scale Network, hybrid systems: A combined Model and Data Based approach. [funded ETI]

https://dcn.nat.fau.eu/yue-wang/