Existence of controls insensitizing the rotational of the solution of Navier-Stokes system having a vanishing component

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IX Partial Differential Equations, Optimal Design and Numerics

Outline

- Control problem: Navier Stokes system
- State of the art
- Method of the proof
- Some comments, perspectives

Insensitizing control for the Navier-Stokes system

- Ω bounded connected regular open subset of $\mathbb{R}^N$, $(N = 2, 3)$.
- $\partial\Omega$ is regular enough, $T > 0$.
- $\omega \subset \Omega$ (control set), $Q := \Omega \times (0, T)$, $\Sigma := \partial\Omega \times (0, T)$.

We consider the Navier-Stokes system with incomplete data:

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = f + v\mathbb{1}_\omega, \ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 + \tau \hat{y}_0 & \text{in } \Omega. \end{cases}$$

where τ is a small constant and $\|\hat{y}_0\|_{L^2(\Omega)^N} = 1$. Unknown.

Insensitizing control problem: To find control v in $L^2(\omega \times (0, T))^N$ such that the functional (Sentinel)

$$J_\tau(y) := \frac{1}{2} \iint_{\mathcal{O} \times (0,T)} |\nabla \times y|^2 \chi \mathrm{d}x \, \mathrm{d}t, \quad \mathcal{O} \subset \Omega \text{ (Observation set)},$$

$$\chi \in C_c^\infty(\mathcal{O}) : 0 \leq \chi \leq 1, \chi \equiv 1 \text{ in } \mathcal{O}_0 \Subset \mathcal{O}.$$

is not affected by the uncertainty of the initial data, that is,

$$\left.\frac{\partial J_\tau(y)}{\partial \tau}\right|_{\tau=0} = 0, \quad \forall \hat{y}_0 \in L^2(\Omega)^N \text{ s.t. } \|\hat{y}_0\|_{L^2(\Omega)^N} = 1.$$

A cascade system

The previous condition is equivalent to the following null controllability problem:
To find a control v such that $z(0) = 0$, where

$$\left\{\begin{array}{ll} w_t - \Delta w + (w\cdot\nabla)w + \nabla p_0 = f + v\mathbb{1}_\omega, \ \nabla\cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + \nabla p_1 = \nabla\times((\nabla\times w)\chi), \quad \nabla\cdot z = 0 & \text{in } Q, \end{array}\right.$$

with boundary and initial conditions:

$$\left\{\begin{array}{ll} w = z = 0 & \text{on } \Sigma, \\ w(0) = y^0, \ z(T) = 0 & \text{in } \Omega. \end{array}\right.$$

We are interested in controls of the form:

1. $v = (v_1, v_2, 0)$ or $v = (v_1, 0, v_3)$ or $v = (0, v_2, v_3)$ instead of $v = (v_1, v_2, v_3)$.

Previous results

First results (using N scalar controls)

1. Heat equation:
   -[Bodart,Fabre - 1995], [De Teresa - 2000], [Bodart, González-Burgos, Pérez-García - 2002], [Guerrero - 2007].
2. Stokes system:
   -[Guerrero, 2007], uses as observation functionals the L^2-norm and the L^2-norm of the rotational of Stokes solution.
3. Navier-Stokes equation:
   -[Gueye - 2013] uses all components of the control function.

Previous results

Reduced number of scalar controls.

1. Navier Stokes system:
   -[Carreño, Gueye - 2014]: They reduce one scalar control.
2. Boussineq system:
   -[Carreño, Guerrero, Gueye - 2015]: they work in the system in 3D and reduce two scalar controls.
   -[Carreño - 2017]: where reduce one scalar control with no control on temperature equation.

Principal result: Case $v = (v_1, v_2, 0)$

$$\begin{cases} w_t - \Delta w + (w \cdot \nabla) w + \nabla p_0 = f + (v_1, v_2, 0) \mathbb{1}_{\omega}, & \nabla \cdot w = 0 \quad \text{in } Q, \\ -z_t - \Delta z + (z \cdot \nabla^t) w - (w \cdot \nabla) z + \nabla p_1 = \nabla \times ((\nabla \times w) \chi), & \nabla \cdot z = 0 \quad \text{in } Q, \\ w = z = 0 & \text{on } \Sigma, \\ w(0) = y^0, \ z(T) = 0 & \text{in } \Omega. \end{cases}$$

Theorem[1]. Let $y^0 = 0$, and $\mathcal{O} \cap \omega \neq \emptyset$. There exists $\delta > 0$ such that if $\|e^{K/t^{14}} f\|_{L^2(Q)^3} < \delta$, there exists control v in $L^2(\omega \times (0,T))^3$ of the form $v = (v_1, v_2, 0)$ such that $z(0) = 0$.

Observation: We can choose $v = (v_1, 0, v_3)$ or $v = (0, v_2, v_3)$.

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Method of the proof

1. Linearization around zero.
2. Null controllability of the linearized system (Main part of the proof).
   Main tool: Carleman estimate for the adjoint system with source terms.
3. Inverse mapping theorem for the nonlinear system.

Linearized system

The linearized system around zero with source terms:

$$\begin{cases} w_t - \Delta w + \nabla p_0 = f^w + v \mathbb{1}_\omega, & \nabla \cdot w = 0 \quad \text{in } Q, \\ -z_t - \Delta z + \nabla p_1 = f^z + \nabla \times ((\nabla \times w)\chi), & \nabla \cdot z = 0 \quad \text{in } Q, \end{cases}$$

with

$$\begin{cases} w = z = 0 & \text{on } \Sigma, \\ w(0) = 0, \ z(T) = 0 & \text{in } \Omega. \end{cases}$$

We want to prove $z(0) = 0$ with controls of the form

$$v = (v_1, v_2, v_3) \quad \text{and} \quad \begin{cases} v = (v_1, v_2, 0) \text{ or} \\ v = (v_1, 0, v_3) \text{ or} \\ v = (0, v_2, v_3) \end{cases}$$

We prove an observability inequality for the adjoint system

Adjoint system and observability inequality

Dual variables: $\varphi \leftrightarrow w$, $\psi \leftrightarrow z$

$$\left\{ \begin{array}{lll} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g^\varphi + \nabla \times ((\nabla \times \psi)\chi), & \nabla \cdot \varphi = 0 & \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\pi_\psi = g^\psi, & \nabla \cdot \psi = 0 & \text{in } Q, \end{array} \right.$$

with

$$\left\{ \begin{array}{ll} \varphi = \psi = 0 & \text{on } \Sigma, \\ \varphi(T) = 0, \ \psi(0) = \psi^0 & \text{in } \Omega. \end{array} \right.$$

For general control $v = (v_1, v_2, v_3)$:

$$\iint_Q \rho_1(t)\Big(|\varphi|^2 + |\psi|^2\Big) \le C \left\| \rho_2(t)(g^\varphi, g^\psi) \right\|_X^2 + C \iint_{\omega\times(0,T)} \rho_3(t)\Big(|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2\Big)$$

for some $C > 0$ where $\rho_i(t) \sim \exp(-C_i/t^{14}(T-t)^{14})$ and X is certain space.

Observability inequality

<u>Our Case:</u>

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g^\varphi + \nabla \times ((\nabla \times \psi)\chi), & \nabla \cdot \varphi = 0 \quad \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\pi_\psi = g^\psi, & \nabla \cdot \psi = 0 \quad \text{in } Q. \end{cases}$$

with

$$\begin{cases} \varphi = \psi = 0 & \text{on } \Sigma, \\ \varphi(T) = 0, \ \psi(0) = \psi^0 & \text{in } \Omega. \end{cases}$$

- For controls $v = (v_1, v_2, 0)$: only local terms φ_1 and φ_2.

$$... \leq ... + C \iint_{\omega \times (0,T)} \rho_3(t) \Big(|\varphi_1|^2 + |\varphi_2|^2 \Big).$$

Proof Sketch. Case: $v = (v_1, v_2, v_3)$

Observation functional

$$J(y) := \iint_{\mathcal{O}\times(0,T)} |y|^2 \mathrm{d}x\,\mathrm{d}t$$

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g^\varphi + \psi \mathbb{1}_{\mathcal{O}}, & \nabla\cdot\varphi = 0 \quad \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\pi_\psi = g^\psi, & \nabla\cdot\psi = 0 \quad \text{in } Q. \end{cases}$$

- Carleman for $\varphi = (\varphi_1, \varphi_2, \varphi_3)$.
- Carleman for $\psi = (\psi_1, \psi_2, \psi_3)$ (with local term like $\nabla \times \psi$).
- Estimate the local term $\nabla \times \psi$, in terms of φ using:

$$\nabla \times \psi = -(\nabla \times \varphi)_t - \Delta(\nabla \times \varphi) - (\nabla \times g^\varphi) \text{ in } \omega \cap \mathcal{O}.$$

- Combine the Carleman of φ and ψ.

Proof Sketch. Case: $v = (v_1, v_2, 0)$

Observation functional

$$J(y) := \iint_{\mathcal{O}\times(0,T)} |\nabla \times y|^2 \chi \mathrm{d}x\, \mathrm{d}t$$

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g^\varphi + \nabla \times ((\nabla \times \psi)\chi), & \nabla \cdot \varphi = 0 \quad \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\pi_\psi = g^\psi, & \nabla \cdot \psi = 0 \quad \text{in } Q. \end{cases}$$

- Carleman for φ_1 and φ_2.
- Carleman for ψ_1 and ψ_2 (with local terms like $\Delta^2\psi_1$ and $\Delta^2\psi_2$).
- Estimate the local terms $\Delta^2\psi_j$ in terms of φ_j $(j = 1, 2)$, using:

$$\Delta^2\psi_j = (\Delta\varphi_j)_t + \Delta^2\varphi_j + \Delta g_1^\varphi - \partial_1\nabla \cdot g^\varphi \text{ in } \omega \cap \mathcal{O}_0.$$

- Combine the Carleman of ψ and φ_j, $j = 1, 2$.

Perspectives and open problems

- The method reduces the quantify of vanishing components to one (some cases to two).
- Possible extension problem: Insensitizing control for the Boussinesq system:

$$\begin{cases} y_t - \Delta y + (y,\nabla)y + \nabla p = f + v\mathbb{1}_\omega + \theta e_N, \ \nabla\cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + y\cdot\nabla\theta = f_0 + v_0\mathbb{1}_\omega & \text{in } Q, \\ y = 0, \ \theta = 0 & \text{on } \Sigma, \\ y|_{t=0} = y^0 + \tau\hat{y}^0, \ \theta|_{t=0} = \theta^0 + \tau\hat{\theta}^0 & \text{in } \Omega. \end{cases}$$

Here,

$$e_N = \begin{cases} (0,1) & \text{if } N = 2, \\ (0,0,1) & \text{if } N = 3, \end{cases}$$

Insensitize the functional

$$J_\tau(y) := \frac{1}{2}\iint_{\mathcal{O}_1\times(0,T)} |\nabla\times y|^2 \mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\iint_{\mathcal{O}_2\times(0,T)} |\nabla\theta|^2 \mathrm{d}x\,\mathrm{d}t,$$

with $\mathcal{O}_1, \mathcal{O}_2 \in \Omega$ such that $\mathcal{O}_1\cap\mathcal{O}_2\cap\omega \neq \emptyset$.

Thanks for your attention