Universidad Autónoma de Madrid

# Positivity of infinite dimensional linear systems

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IX Partial differential equations, optimal design and numerics, Benasque

August 26, 2022

European Commission

European Research Council

Established by the European Commission

Fundación Deusto Fundazioa

DyCon DYNAMIC CONTROL

# Outline

# Motivation and Preliminary

# Motivation

System of transport equations with memory on a network of scattered ramification nodes

Let us consider the following system of transport equations on a network:

$$(\Sigma_{\text{TN}}) \begin{cases} \dfrac{\partial}{\partial t} z_j(t,x,v) = v \dfrac{\partial}{\partial x} z_j(t,x,v) + q_j(x,v).z_j(t,x,v), & t \geq 0,\ (x,v) \in \Omega_j, \\ z_j(0,x,v) = f_j(x,v) \geq 0, \quad z_j(\theta,x,v) = \varphi_j(\theta,x,v) \geq 0, \qquad \theta \in [-r,0] & (x,v) \in \Omega_j,\ (\text{IC}) \\ \iota_{ij}^{out} z_j(t,1,\cdot) = \mathrm{w}_{ij} \sum_{k \in \mathcal{M}} \iota_{ik}^{in} [\mathbb{J}_k(z_k)(t,0,\cdot) + \mathbb{L}_k(z_k(t+\cdot,\cdot,\cdot))] + \sum_{l \in \mathcal{N}_c} \mathrm{b}_{il} u_l(t,.), & t \geq 0,\ (\text{BC}) \end{cases}$$

for $i \in \{1,\dots,N\} := \mathcal{N}$, $j \in \{1,\dots,M\} := \mathcal{M}$ and $l \in \{1,\dots,n\} := \mathcal{N}_c$ with $M \geq N \geq n$, where we set $\Omega_j := [0,l_j] \times [v_{\min}, v_{\max}]$, $l_j > 0$. Here, the vertex delay operators $\mathbb{L}_j$ are given by

$$\mathbb{L}_j(\varphi_j)(x,v) = \int_0^{l_j} \int_{v_{\min}}^{v_{\max}} \int_{-r}^{0} d[\eta_k](\theta) \varphi_k(\theta,x,v) dv dx, \qquad (x,v) \in \Omega_j,\ \varphi_j \in W^{1,p}([-r,0]; L^p(\Omega_j)).$$

where the first integral is in the Lebesgue-Stieltjes sense. Moreover, the scattering operators $\mathbb{J}_j$ are given by

$$(\mathbb{J}f)_j(x,v) = \int_{v_{\min}}^{v_{\max}} \ell_j(x,v,v') f_j(x,v') dv', \qquad (x,v) \in \Omega_j,\ f_j \in L^p(\Omega_j),$$

where $\ell_j \in L^\infty(\Omega_j \times [v_{\min}, v_{\max}])$ for every $j \in \mathcal{M}$.

# Motivation

size-structured population model with delayed birth process

We consider the following size-structured population model with delayed birth process:

$$(\Sigma_{\text{SPM}})\begin{cases} \frac{\partial z(t,s)}{\partial t}+\frac{\partial q(s)z(t,s)}{\partial s}=-\mu(s)z(t,s), & t\geq 0,\ s\in(0,s^*)\\ z(0,s)=f(s)\geq 0, \quad z(\theta,s)=\varphi(\theta,s)\geq 0, & \theta\in[-r,0],\ s\in(0,s^*)\\ g(0)z(t,0)=\int_0^{s^*}\int_{-r}^{0}\beta(s)z(t+\theta,s)d\eta(\theta)ds+bu(t), & t\geq 0. \end{cases}$$

Here $z(t,s)$ represents the population density of certain species of size $s\in(0,s^*)$ at time $t\geq 0$, where $s^*>0$ is the maximum size of of any individual in the population. The function $q(s)$ is the growth rate of size over time and the size dependent functions $\beta$ and $\mu$ denote the fertility and mortality, respectively. By $b$, we denotes the boundary control operator, $u$ define the control functions and $(f,\varphi)$ are the initial distributions of our target population.

# Motivation

Can be transformed as the following perturbed boundary value control system

$$(\mathrm{AG\Gamma})\begin{cases}\dot{z}(t)=\mathcal{A}_m z(t), & t>0,\\ z(0)=x\geq 0, & \\ Gz(t)=\Gamma z(t)+Ku(t), & t>0,\end{cases}$$

where the state variable $z(.)$ takes values in a Banach lattice $X$ and the control function $u(\cdot)$ is given in the Banach space $L^p([0,+\infty);U)$. Here,

- The maximal (differential) operator $A_m : D(A_m) \subset X \longrightarrow X$ is closed and densely defined;
- $K$ is a bounded linear operator from $U$ into $\partial X$ (both are Banach lattices),
- $G,\Gamma : D(A_m) \longrightarrow \partial X$ are linear continuous trace operators.

## Motivation

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- $G, \Gamma : D(A_m) \longrightarrow \partial X$ are linear continuous trace operators.

### Goal

To $(\mathrm{AG}\Gamma)$ we associate the following operator

$$\mathfrak{A}=A_m, \qquad D(\mathfrak{A})=\{x\in D(A_m): Gx=\Gamma x\}.$$

- Positivity and Well-posedness of $(\Sigma)$.

# Previous works

- J. L. Lions and E. Magens.
- W. Desch, I. Lasiecka, and W. Schappacher.
- D. Salamon.
- G. Weiss.
- R. Schnaubelt.
- O. J. Staffans.
- A. Chen and K. Morris.
- H. Zwart and B. Jacob.
- S.Hadd, R. Manzo, and A.Rhandi.

# Preliminary

Let $E$ be a real vector space and $\leq$ be a partial order on this space. Then $E$ is said to be an ordered vector space if it satisfies the following properties:

- If $f, g \in E$ and $f \leq g$, then $f + h \leq g + h$ for all $h \in E$.
- If $f, g \in E$ and $f \leq g$, then $\alpha f \leq \alpha g$ for all $\alpha \geq 0$.

If, in addition, $E$ is lattice with respect to the partial ordered, that is, $\sup\{f, g\}$ and $\inf\{f, g\}$ exist for all $f, g \in E$, then $E$ is said to be a *vector lattice*. For $f \in E$, the *positive part* of $f$ is defined by $f_+ := \sup\{f, 0\}$, the *negative part* of $f$ by $f_- := \sup\{-f, 0\}$ and the absolute value of $f$ by $|f| := \sup\{f, -f\}$, where 0 is the zero element of $E$. The set of all positive elements of $E$, denoted by $E_+$, is a convex cone with vertex 0. In particular, it generates a canonical ordering $\leq$ on $E$ which is given by: $f \leq g$ if and only if $g - f \in E_+$. For each $f, g \in E$ with $f \leq g$, the set $[f, g] = \{h \in E : \ f \leq h \leq g\}$ is called an *order interval*. A norm complete vector lattice $E$ such that its norm satisfies the following property

$$|f| \leq |g| \quad \Rightarrow \quad \|f\| \leq \|g\|$$

for $f, g \in X$, is called *Banach lattice*. If $E$ is a Banach lattice, its topological dual $E'$, endowed with the dual norm and the dual order, is also a Banach lattice.

# Preliminary

A Banach lattice $E$ is said to be *AL-space* whenever its norm is additive on the positive cone. A Banach lattice $E$ is said to be a *KB-space* (Kantorovič-Banach space) whenever every increasing norm bounded sequence of $E_+$ is norm convergent. Important examples of *KB*-spaces are reflexive Banach lattices and *AL*-spaces. A norm of a Banach lattice is *order continuous* if for each generalized sequence $(x_\alpha)$ such that $x_\alpha \downarrow 0$ in $E$, the sequence $(x_\alpha)$ converges to 0 for the norm $\|\cdot\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence $(x_\alpha)$ is decreasing (in symbol $\downarrow$), its infimum exists and $\inf(x_\alpha) = 0$. Note that each *KB-space* has an order continuous norm, but a Banach lattice with an order continuous norm is not necessary a *KB-space*.
A vector lattice $E$ is called *order* (or *Dedekind*) *complete* if every nonempty bounded above subset has a supremum or equivalently if any nonnegative generalized sequence $(x_\alpha)$ has a supremum in $E$, that is, whenever $0 \leq x_\alpha \uparrow \leq x$ implies the existence of $\sup\{x_\alpha\}$. Here, the notation $x_\alpha \uparrow \leq x$ means that $x_\alpha$ is increasing (in symbol $\uparrow$) and $x_\alpha \leq x$ for all $\alpha$. As an example, every Banach lattice with an order continuous norm is order complete.

# Preliminary

Now, if we denote by $\mathcal{L}(E,F)$ the Banach algebra of all linear bounded operators from an order Banach space $E$ to an order Banach space $F$, then an operator $P \in \mathcal{L}(E,F)$ is positive if $PE_+ \subset F_+$.
An everywhere defined positive operator from a Banach lattice to a normed vector lattice is bounded. The set of all positive operators from a Banach lattice $E$ to another Banach lattice $F$, denoted by $\mathcal{L}_+(E,F)$, is a convex cone in the space $\mathcal{L}(E,F)$. This cone, however, in general does not generate $\mathcal{L}(E,F)$. This fact motivates the following definition.

## Definition

Let $E,F$ be ordered Banach spaces. An operator $P \in \mathcal{L}(E,F)$ is said to be regular, if there exist $P_1, P_2 \in \mathcal{L}(E,F)$ positive such that $P = P_1 - P_2$.

The vector space of all regular operators $P \in \mathcal{L}(E,F)$ is denoted by $\mathcal{L}^r(E,F)$. Moreover, if $E,F$ are Banach lattices such that $F$ is *order complete*, then $\mathcal{L}^r(E,F)$ under the norm

$$\|P\|_r := \||P|\|_{\mathcal{L}(E,F)}, \qquad (\text{called the r-}\textit{norm})$$

is an ordered complete Banach lattice, where $|P| \in \mathcal{L}(E,F)$ denotes the so-called modulus operator defined by $|P| := \sup\{P, -P\}$


- C.D. Aliprantis and O. Burkinshaw, Positive operators, Pure and Applied Mathematics, 119. Academic Press. Inc., Orlando FL. (1985).
- H.H. Schaefer, Banach lattices and positive operators. Springer-Verlag, Berlin-Heidelberg (1974).

# Preliminary

Now, let $X$ be a real Banach lattice and $(A, D(A))$ the generator of a $C_0$-semigroup $\mathbb{T} := (T(t))_{t\geq 0}$ on $X$. The type of $\mathbb{T}$ is defined as $\omega_0 := \inf\{t^{-1}\log\|T(t)\| : t > 0\}$. We denote by $\rho(A)$ the resolvent set of $A$, i.e., the set of all $\mu \in \mathbb{C}$ such that $\mu - A$ is invertible, and by $R(\mu, A) := (\mu - A)^{-1}$ the resolvent operator of $A$. The complement of $\rho(A)$, is called the spectrum and is denoted by $\sigma(A) := \mathbb{C}\backslash\rho(A)$. The so-called spectral radius of the operator $A$ is defined by $r(A) := \sup\{|\mu| : \mu \in \sigma(A)\}$. The spectral bound $s(A)$ of $A$ is defined by $s(A) := \sup\{\Re e\mu : \mu \in \sigma(A)\}$.

### Definition

A linear operator $(A, D(A))$ on a Banach lattice $X$ is called resolvent positive if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A)$ and $R(\mu, A) \geq 0$ for each $\mu > \omega$.

Moreover, a $C_0$-semigroups on a Banach lattice is positive if and only if the corresponding generator $(A, D(A))$ is resolvent positive. On the other hand, the extrapolation space associated with $X$ and $A$, denoted by $X_{-1}$, is the completion of $X$ with respect to the norm $\|x\|_{-1} := \|R(\mu, A)x\|$ for $x \in X$ and some $\mu \in \rho(A)$. Note that the choice of $\mu$ is not important, since by the resolvent equation different choices lead to equivalent norms on $X_{-1}$.

### Semigroup Frorum, (Bátkai et. all, 2018)

$$X_+ = X_{+,-1} \cap X.$$

The unique extension of $\mathbb{T}$ on $X_{-1}$ is a $C_0$-semigroup which we denote by $\mathbb{T}_{-1} := (T_{-1}(t))_{t\geq 0}$ and whose generator is denoted by $A_{-1}$. Note that $\mathbb{T}_{-1}$ is positive whenever $\mathbb{T}$ is.

# Boundary control systems

# Boundary control system

Consider the following boundary control system

$$(\mathrm{CS})\begin{cases}\dot{z}(t)=A_m z(t), & z(0)=x, \quad t>0,\\ Gz(t)=v(t), & t>0,\end{cases}$$

for $x\in X_+$ and $v\in L^p_+(\mathbb{R}_+;\partial X)$, where we recall that $X,\partial X$ are Banach lattices.

**Main Assumptions:**

H1. $A:=(A_m)_{|D(A)}$ with $D(A)=\ker G$, generates a strongly continuous positive semigroup $\mathbb{T}:=(T(t))_{t\geq 0}$ on $X$;

H2. $\mathrm{Range}\,(G)=\partial X$ and $\Gamma$ is positive.

For any $\mu \in \rho(A)$, we have

$$D(A_m) = D(A) \oplus \ker(\mu - A_m)$$

and the following inverse

$$D_\mu := \left(G_{|\ker(\mu - A_m)}\right)^{-1} \in \mathcal{L}(\partial X, X)$$

exists. Now we define the control operator

$$B := (\mu - A_{-1})D_\mu \in \mathcal{L}(\partial X, X_{-1}), \qquad \mu \in \rho(A).$$


- G. Greiner, Perturbing the boundary conditions of a generator, Houston J. Math., 13 (1987), 213-229.

# Boundary control system

The system (CS) is reformulated as the following distributed one

$$\dot{z}(t) = A_{-1}z(t) + Bv(t), \quad z(0) = x, \quad t \geq 0.$$

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$$\dot{z}(t) = A_{-1}z(t) + Bv(t), \quad z(0) = x, \quad t \geq 0.$$

- An integral solution of the above equation is given by: $z(t) = T(t)x + \Phi_t v \in X_{-1}$, for any $t \geq 0$, $x \in X$ and $v \in L^p([0,+\infty), \partial X)$, where

$$\Phi_t v := \int_0^t T_{-1}(t-s)Bv(s)ds \qquad (*).$$

- An operator $B \in \mathcal{L}(\partial X, X_{-1})$ is called $L^p$-admissible control operator for $A$ if, for some $\tau > 0$, the operator $\mathrm{Range}\,\Phi_\tau \subset X$. In particular, $z(\cdot) \in C(\mathbb{R}_+, X)$ for all $x \in X$ and $v \in L^p([0,+\infty), \partial X)$.
- Let $\mathfrak{B}_p(\partial X, X, \mathbb{T})$ denote the vector space of all $L^p$-admissible control operators $B$, which is a Banach space with the norm

$$\|B\|_{\mathfrak{B}_p} := \sup_{\|v\|_{L^p([0,\tau];\partial X)} \leq 1} \left\| \int_0^\tau T_{-1}(\tau - s)Bv(s)ds \right\|,$$

where $\tau > 0$ is fixed.

# Boundary control system

We have the following characterization of the well-posedness of equation (CS).

**Proposition (Yassine, 2022)**

Let $X, \partial X$ be Banach lattices, let $(A, D(A))$ generates a strongly continuous semigroup $\mathbb{T}$ on $X$ and $B \in \mathcal{L}(\partial X, X_{-1})$. Then for every $x \in X$ and $v \in L^p_{loc}(\mathbb{R}_+; \partial X)$, the system (CS) has a unique solution $z(\cdot) \in C(\mathbb{R}_+; X)$ if one of the following equivalent assertions holds.

*(i)* for any $x \in X_+$ and $v \in L^p_+(\mathbb{R}_+; \partial X)$, the solution $z(\cdot)$ of (CS) remains in $X_+$.

*(ii)* $\mathbb{T}$ is a positive and for some $\tau > 0$, $\Phi_\tau v \in X_+$ for all $v \in L^p_+(\mathbb{R}_+; \partial X)$.

# Boundary control system

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(*i*) for any $x \in X_+$ and $v \in L^p_+(\mathbb{R}_+; \partial X)$, the solution $z(\cdot)$ of (CS) remains in $X_+$.

(*ii*) $\mathbb{T}$ is a positive and for some $\tau > 0$, $\Phi_\tau v \in X_+$ for all $v \in L^p_+(\mathbb{R}_+; \partial X)$.

For $\alpha > s(A)$, let $v \in L^p_\alpha(\mathbb{R}_+; \partial X)$, the space of all functions of the form $v(t) = e^{\alpha t} u(t)$, where $u \in L^p(\mathbb{R}_+; \partial X)$. Then, $u$ and $z$ from (CS) have Laplace transforms related by

$$\hat{z}(\mu) = R(\mu, A)x + \widehat{(\Phi.u)}(\mu), \quad \text{with} \quad \widehat{(\Phi.u)}(\mu) = R(\mu, A_{-1})B\hat{v}(\mu),$$

for all $\mathfrak{Re}\mu > \alpha$, where $\hat{v}$ denote the Laplace transform of $v$.

**Lemma**

Let $X, \partial X$ be Banach lattices and let $\mathbb{T}$ be a positive $C_0$-semigroup on $X$. Then, we have $B$ is positive iff $D_\mu := R(\mu, A_{-1})B$ is positive for all $\mu > s(A)$, iff $\Phi_t$ is positive for all $t \geq 0$.

# Boundary control system

### Observation

- One could relax the definition of $L^p$-admissible positive control operators to $\Phi_\tau L^p_+(\mathbb{R}_+;\partial X) \subset X_+$, for some $\tau \geq 0$.
- It is not difficult to see that $\mathfrak{B}_{p,+}(\partial X, X, \mathbb{T})$ is the positive convex cone in $\mathfrak{B}_p(\partial X, X, \mathbb{T})$.
- Therefore it generates the order: for $B, B' \in \mathfrak{B}_p(\partial X, X, \mathbb{T})$, we have $B' \leq B$ whenever $B - B' \in \mathfrak{B}_{p,+}(\partial X, X, \mathbb{T})$.

### Lemma (Yassine, 2022)

Let $X, \partial X$ be Banach lattices and $\mathbb{T}$ be a positive $C_0$-semigroup on $X$, and let $\Phi_\tau$ be given by (*). If $B \in \mathcal{L}^r(\partial X, X_{-1})$ is $L^p$-admissible, then

$$\Phi_\tau \in \mathcal{L}(L^p(\mathbb{R}_+;\partial X), X) \cap \mathcal{L}^r(L^p(\mathbb{R}_+;\partial X), X_{-1})$$

for all $\tau \geq 0$. Moreover, if there exist $B_1, B_2 \in \mathfrak{B}_{p,+}(\partial X, X, \mathbb{T})$ such that $B = B_1 - B_2$, then $\Phi_\tau \in \mathcal{L}^r(L^p(\mathbb{R}_+;\partial X), X)$ for all $\tau \geq 0$.

# Boundary control system

Result 1 (Yassine, 2022)

Let $X, \partial X$ be Banach lattices and $\mathbb{T}$ be a positive $C_0$-semigroup on $X$. Define the vector space

$$\mathfrak{B}^r_p(\partial X, X, \mathbb{T}) := \{B \in \mathcal{L}^r(\partial X, X_{-1}) : \ \Phi_\tau \in \mathcal{L}^r(L^p(\mathbb{R}_+;\partial X), X), \ \ \forall \tau \geq 0\},$$

the space of all $L^p$-admissible regular control operators. Then, the following vector space equality holds:

$$\mathfrak{B}^r_p(\partial X, X, \mathbb{T}) = \mathfrak{B}_{p,+}(\partial X, X, \mathbb{T}) - \mathfrak{B}_{p,+}(\partial X, X, \mathbb{T}).$$

Moreover if $X$, in addition, is order complete, then $\mathfrak{B}^r_p(\partial X, X, \mathbb{T})$ is an order complete Banach lattice under the norm

$$\|B\|_{\mathfrak{B}^r_p} = \sup\left\{ \||\Phi_\tau|v\| : \ v \geq 0, \ \|v\|_{L^p([0,\tau];\partial X)} \leq 1 \right\}.$$

where $\tau > 0$ is fixed.

# Boundary control system

Result 1

### Remark

It is to be that the extrapolation space $X_{-1}$ is not, in general, a Banach lattice (more precisely, a vector lattice). A striking consequence of the above observation is that $\mathcal{L}^r(\partial X, X_{-1})$ cannot be expected to be a Banach lattice. On the other hand, we have the following vector subspace inclusions:

$$\begin{array}{ccc} \mathcal{L}^r(\partial X, X) & \subseteq & \mathcal{L}(\partial X, X) \\ \cap & & \cap \\ \mathfrak{B}_p^r(\partial X, X, \mathbb{T}) & \subset & \mathfrak{B}_p(\partial X, X, \mathbb{T}) \\ \cap & & \cap \\ \mathcal{L}^r(\partial X, X_{-1}) & \subseteq & \mathcal{L}(\partial X, X_{-1}). \end{array}$$


- A. Bátkai, B. Jacob, J. Voigt, and J. Wintermayr, Perturbation of positive semigroups on *AM*-spaces. Semigroup Forum. **96** (2018) 33-347.

# Boundary control system

## Sketch of proof

Step 1: We show the existence of $|B|$ in the canonical order of $\mathfrak{B}_p^r(\partial X, X, \mathbb{T})$.

### Abstract control system

Let $X, \partial X$ be Banach lattices. An abstract control system on $X, \partial X$ is a pair $(\mathbb{T}, \Phi)$ such that

$$\Phi_{\tau+t} v = T(\tau)\Phi_t v_{|[0,t]} + \Phi_\tau v(\cdot + t), \tag{1}$$

for $\tau, t \geq 0$ and $v \in L_+^p([0, \tau+t], \partial X)$, where $\Phi := (\Phi_\tau)_{\tau \geq 0}$ is a family of bounded operators from $L^p(\mathbb{R}_+; \partial X)$ to $X$.

### Lemma (Yassine, 2022)

Let $X, \partial X$ be Banach lattices such that $X$ is order complete and let $\mathbb{T}$ be a positive $C_0$-semigroup on $X$. Assume that $(\mathbb{T}, \Phi)$ is an abstract control system on $X, \partial X$ such that $\Phi_\tau \in \mathcal{L}^r(L^p(\mathbb{R}_+; \partial X), X)$ for all $\tau \geq 0$. Then, for each $\tau \geq 0$, there exist unique $\Phi_\tau^+, \Phi_\tau^- \in \mathcal{L}_+(L^p(\mathbb{R}_+; \partial X), X)$ splitting $\Phi_\tau$ as $\Phi_\tau = \Phi_\tau^+ - \Phi_\tau^-$ and satisfying the composition property (1).

# Boundary control system

## Sketch of proof

Let $B \in \mathfrak{B}_p^r(\partial X, X, \mathbb{T})$ and let $\Phi_\tau \in \mathcal{L}^r(L^p(\mathbb{R}_+;\partial X), X)$ be the corresponding input map. Thus, for each $\tau \geq 0$, the modulus of $\Phi_\tau$ exists (in the canonical ordering of $\mathcal{L}(L^p(\mathbb{R}_+;\partial X), X)$) and is given by

$$|\Phi_\tau| = \Phi_\tau^+ + \Phi_\tau^-, \qquad \Phi_\tau = \Phi_\tau^+ - \Phi_\tau^-.$$

Step 2: We establish that $\mathfrak{B}_p^r(\partial X, X, \mathbb{T})$ is order complete. Indeed, let $(B_\alpha)$ be a set of nonnegative elements such that $0 \leq B_\alpha \uparrow \leq B$ holds in $\mathfrak{B}_p^r(\partial X, X, \mathbb{T})$. Then, for each $\alpha$, the operators $B_\alpha$ satisfy

$$\Phi_\tau^\alpha v = \int_0^\tau T_{-1}(\tau - s) B_\alpha v(s) ds,$$

for all $v \in L_+^p(\mathbb{R}_+, \partial X)$ and for some fixed $\tau \geq 0$.

Step 3: $\|\cdot\|_{\mathfrak{B}_p^r}$-completeness of $\mathfrak{B}_p^r(\partial X, X, \mathbb{T})$

# Boundary control system

### Result 2 (Yassine, 2022)

Let $X, \partial X$ be Banach lattices such that $X$ is a *KB-space* and let $\mathbb{T}$ be positive. Then

$$\mathfrak{B}_1^r(\partial X, X, \mathbb{T}) = \mathcal{L}^r(\partial X, X). \tag{2}$$

### Sketch of proof

Let $B \in \mathfrak{B}_{1,+}(\partial X, X, \mathbb{T})$ and $\tau > 0$. Let $v \in \partial X$ and consider the sequence of functions

$$v_n(s) := \begin{cases} 0, & \text{if } 0 \leq s \leq \tau - \frac{1}{n}, \\ n|v|, & \text{if } \tau - \frac{1}{n} < s \leq \tau. \end{cases}$$

### Consequence

Let $X, \partial X$ be Banach lattices with $X$ is an *AL-space* and let $\mathbb{T}$ be positive. Then the vector space equality (2) holds. Moreover, if there exists a positive contractive projection $\Pi : \partial X'' \to \partial X$, $((\partial X)''$ is the bidual of $\partial X$), then

$$\mathfrak{B}_1(\partial X, X, \mathbb{T}) = \mathcal{L}(\partial X, X).$$

# Boundary control system

**Result 3 (Yassine, 2022)**

Let $X, \partial X$ be Banach lattices, let $A$ be a densely defined resolvent positive operator such that

$$\|R(\mu_0, A)x\| \geq c\|x\|, \qquad x \in X_+, \qquad (**)$$

for some $\mu_0 > s(A)$ and constant $c > 0$. Let $B \in \mathcal{L}(\partial X, X_{-1})$ be a positive control operator. Then $B$ is a positive zero-class $L^p$-admissible control operators.

**Consequence**

In particular, if $B \in \mathcal{L}^r(\partial X, X_{-1})$, then $B$ is a zero-class $L^p$-admissible control operator.


- C. J. K. Batty and D. W. Robinson, Positive one-parameter semigroups on ordered Banach space. Acta Appl. Math. **2** (1984) 221-296.
- W. Arendt, Resolvent positive operators. Proc. London Math. Soc. **54** (1987) 321-349.

# Boundary control system

## Perturbation result

### Desch-Schappacher perturbation, (Yassine, 2022)

Let $(A, D(A))$ be a densely defined resolvent positive operator such that the inverse estimate (**) holds, and let $B \in \mathcal{L}(X, X_{-1})$ be a positive operator. Then $(A_{-1} + B)_{|X}$ generates a positive $C_0$-semigroup on $X$ and

$$s((A_{-1} + B)_{|X}) = w_0((A_{-1} + B)_{|X})$$

# Observation systems

# Observation systems

Consider the observed linear system

$$(\mathrm{OS})\begin{cases}\dot{z}(t)=Az(t), & z(0)=x, \quad t>0,\\ y(t)=Mz(t), & \quad\quad\quad\quad t>0,\end{cases}$$

We say that the system $(\mathrm{OS})$ is well-posed if the output function $y(\cdot)$ can be extended to a function $y\in L^p_{loc}([0,+\infty),\partial X)$ such that

$$\|y(\cdot;x)\|_{L^p([0,\alpha],\partial X)}\leq\gamma\|x\|_X, \qquad (x\in X),$$

for any $\alpha>0$ and some constant $\gamma:=\gamma(\alpha)>0$.

# Observation systems

In order to overcome this obstacle we need the following class of operators $C := M_{|D(A)}$.

### Definition

An operator $C \in \mathcal{L}(D(A), \partial X)$ is called an $L^p$-admissible observation operator for $A$ if for some (hence for all) $\alpha > 0$, there exists a constant $\gamma := \gamma(\alpha) > 0$ such that

$$\int_0^\alpha \|CT(t)x\|^p dt \leq \gamma^p \|x\|^p, \qquad \forall x \in D(A). \tag{3}$$

If in addition $\lim_{\alpha \mapsto 0} \gamma(\alpha) = 0$, then $C$ is called a zero-class $L^p$-admissible observation operator for $A$.

# Observation systems

In order to overcome this obstacle we need the following class of operators $C := M_{|D(A)}$.

### Definition

An operator $C \in \mathcal{L}(D(A), \partial X)$ is called an $L^p$-admissible observation operator for $A$ if for some (hence for all) $\alpha > 0$, there exists a constant $\gamma := \gamma(\alpha) > 0$ such that

$$\int_0^\alpha \|CT(t)x\|^p dt \leq \gamma^p \|x\|^p, \qquad \forall x \in D(A). \tag{4}$$

If in addition $\lim_{\alpha \mapsto 0} \gamma(\alpha) = 0$, then $C$ is called a zero-class $L^p$-admissible observation operator for $A$.

In the following lemma we show that the admissibility of a positive observation operator is fully characterized on the positive part of $D(A)$.

### Lemma

Let $X, \partial X$ be Banach lattices, and let $\mathbb{T}$ be a positive $C_0$-semigroup on $X$ and $C \in \mathcal{L}(D(A), \partial X)$ be a positive operator. If for some $\alpha > 0$ the estimate (3) hold for any $0 \leq x \in D(A)$, then $C$ is a positive $L^p$-admissible observation operator for $A$.

# Observation systems

In view of the above lemma, the following result can be obtained by a slight modification of the proof of Lemma 2.1 in J. Voigt (1989).

**Lemma**

Let $X, \partial X$ be Banach lattices such that $\partial X$ is a real AL-space. Assume that $\mathbb{T}$ is a positive $C_0$-semigroup on $X$ and $C \in \mathcal{L}(D(A), \partial X)$ is positive. Then, $C$ is an $L^1$-admissible observation operator for $A$.


- J. Voigt, On resolvent positive operators and positive $C_0$-semigroups on *AL*-spaces. Semigroup Forum. **38** (1989) 263-266.

# Observation systems

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## Positive observation system

Let $X, \partial X$ be Banach lattices. An abstract positive observation system on $X, \partial X$ is a pair $(\mathbb{T}, \Psi)$ such that

$$(***) \qquad \begin{cases} \Psi_\tau(x+y) = \Psi_\tau x + \Psi_\tau y, & \\ \Psi_{\tau+t} x = \Psi_t x, & \text{on } [0,t], \\ \Psi_{\tau+t} x = [\Psi_\tau T(t)x](\cdot - t), & \text{on } [t, \tau+t], \end{cases}$$

for $\tau, t \geq 0$ and $x, y \in X_+$, where $\Psi := (\Psi_\tau)_{\tau \geq 0}$ is a family of bounded operators from $X_+$ to $L^p_+(\mathbb{R}_+; \partial X)$.

# Observation systems

### Observtion

It should be noted that for each operator $C \in \mathcal{L}(D(A), \partial X)$ we associate an operator $C_\Lambda$, called the *Yosida extension* of $C$ with respect to $A$, whose domain, denoted by $D(C_\Lambda)$, consists of all $x \in X$ for which $\lim_{\mu \mapsto \infty} C\mu R(\mu, A)x$ exists. If $C$ is positive and $A$ is a resolvent positive, then by the closedness of the positive cone $Y_+$ we have $C_\Lambda$ is also positive. Moreover, for an $L^p$-admissible positive observation operator $C$, we have $0 \leq T(t)x \in D(C_\Lambda)$ and

$$\Psi x = C_\Lambda T(.)x \geq 0,$$

for a.e $t \geq 0$ and all $x \in X_+$.

# Observation systems

### Definition

We denote by $\mathfrak{C}_p(X,\mathbb{T},\partial X)$ the vector space of all $L^p$-admissible observation operators for $A$. Let $\tau > 0$, then $\mathfrak{C}_p(X,\mathbb{T},\partial X)$ endowed with the operator norm

$$\|C\|_{\mathfrak{C}_p} := \sup_{\|x\|\leq 1} \|\Psi_\tau x\|_{L^p([0;+\infty),\partial X)}.$$

for any $x \in D(A)$, is a Banach space. Moreover, we shall denote by $\mathfrak{C}_{p,+}(X,\mathbb{T},\partial X)$ the set of all positive $L^p$-admissible observation operators, which is a positive cone in $\mathfrak{C}_p(X,\mathbb{T},\partial X)$. On the other hand, let

$$\mathfrak{C}_p^r(X,\mathbb{T},\partial X) := \{C \in \mathcal{L}^r(D(A),\partial X): \ \Psi_\tau \in \mathcal{L}^r(X, L^p([0,\infty);\partial X)), \ \forall \tau \geq 0\},$$

be the space of all regular $L^p$-admissible observation operators.

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be the space of all regular $L^p$-admissible observation operators.

## Result 1, (Yassine, 2022)

Let $X$ and $Y$ be Banach lattices such that $\partial X$ is order complete. Then the space $\mathfrak{C}_p^r(X,\mathbb{T},\partial X)$ endowed with norm

$$\|C\|_{\mathfrak{C}_p^r} := \sup_{\substack{\|x\|\leq 1 \\ x\geq 0}} \left\| |\Psi_\tau| x \right\|_{L^p([0;\infty),\partial X)}, \qquad \tau \geq 0,$$

is an order complete Banach lattice.

# Observation systems

## Sketch of the proof

Step 1: We show the existence of $|C|$ in the canonical order of $\mathfrak{C}_p^r(X,\mathbb{T},\partial X)$. Indeed, let $\tau \geq 0$ and $C \in \mathfrak{C}_p^r(X,\mathbb{T},Y)$. Then, there is $\Psi_\tau \in \mathcal{L}^r(X, L^p([0,\infty);\partial X))$ such that

$$(\Psi_\tau x)(t) = \begin{cases} CT(t)x, & t \in [0,\tau), \\ 0, & t \in [\tau,\infty), \end{cases}$$

for any $x \in D(A)$.

### Lemma

Let $X, \partial X$ be Banach lattices such that $\partial X$ is order complete and let $\mathbb{T}$ be a positive $C_0$-semigroup on $X$. Assume that $(\Psi,\mathbb{T})$ is an abstract observation system on $X,\partial X$ with $\Psi_\tau \in \mathcal{L}^r(X, L^p(\mathbb{R}_+;\partial X))$ for all $\tau \geq 0$. Then, for each $\tau \geq 0$, there exist unique $\Psi_\tau^+, \Psi_\tau^- \in \mathcal{L}_+(X, L^p(\mathbb{R}_+;\partial X))$ splitting $\Psi_\tau$ as $\Psi_\tau = \Psi_\tau^+ - \Psi_\tau^-$ and satisfying the properties (***).

# Observation systems

## Sketch of proof

Let $C \in \mathfrak{C}_p^r(X,\mathbb{T},\partial X)$ and let $\Psi_\tau \in \mathcal{L}^r(X, L^p(\mathbb{R}_+;\partial X))$ be the corresponding output map. Thus, for each $\tau \geq 0$, the modulus of $\Psi_\tau$ exists (in the canonical ordering of $\mathcal{L}(X, L^p(\mathbb{R}_+;\partial X))$) and is given by

$$|\Psi_\tau| = \Psi_\tau^+ + \Psi_\tau^-, \qquad \Psi_\tau = \Psi_\tau^+ - \Psi_\tau^-.$$

Step 2: We establish that $\mathfrak{C}_p^r(X,\mathbb{T},\partial X)$ is order complete. Indeed, let $(C_\alpha)$ be a set of nonnegative elements such that $0 \leq C_\alpha \uparrow\leq C$ holds in $\mathfrak{C}_p^r(X,\mathbb{T},\partial X)$. Then, for each $\alpha$, the operators $C_\alpha$ satisfies

$$\Psi_\tau^\alpha x = C_\alpha T(t)x$$

for all $0 \leq x \in D(A)$ and for some fixed $\tau \geq 0$.

Step 3: $\|\cdot\|_{\mathfrak{C}_p^r}$-completeness of $\mathfrak{C}_p^r(X,\mathbb{T},\partial X)$.

# Observation systems

## Observation

It is to be noted that if $\partial X$ is a real $AL$-space we have $\mathfrak{C}_1^r(X,\mathbb{T},\partial X) = \mathcal{L}^r(D(A),\partial X)$. However, in general, we only have the following vector subspace inclusion: $\mathfrak{C}_p^r(X,\mathbb{T},\partial X) \subset \mathfrak{C}_p(X,\mathbb{T},\partial X)$. Moreover, the result of the above proposition assert that for $C \in \mathfrak{C}_p^r(X,\mathbb{T},\partial X)$, then there exist unique $C_+, C_- \in \mathcal{L}(D(A),\partial X)$, both are positive $L^p$-admissible observation operators such that $C = C_+ - C_-$. Roughly speaking, $\mathfrak{C}_p^r(X,\mathbb{T},\partial X) = \mathfrak{C}_{p,+}(X,\mathbb{T},\partial X) - \mathfrak{C}_{p,+}(X,\mathbb{T},\partial X)$..

# Observation systems

## Observation

It is to be noted that if $\partial X$ is a real $AL$-space we have $\mathfrak{C}^r_1(X,\mathbb{T},\partial X) = \mathcal{L}^r(D(A),\partial X)$. However, in general, we only have the following vector subspace inclusion: $\mathfrak{C}^r_p(X,\mathbb{T},\partial X) \subset \mathfrak{C}_p(X,\mathbb{T},\partial X)$. Moreover, the result of the above proposition assert that for $C \in \mathfrak{C}^r_p(X,\mathbb{T},\partial X)$, then there exist unique $C_+, C_- \in \mathcal{L}(D(A),\partial X)$, both are positive $L^p$-admissible observation operators such that $C = C_+ - C_-$. Roughly speaking, $\mathfrak{C}^r_p(X,\mathbb{T},\partial X) = \mathfrak{C}_{p,+}(X,\mathbb{T},\partial X) - \mathfrak{C}_{p,+}(X,\mathbb{T},\partial X)$..

## Properties

Let $C_\Lambda$ be the *Yosida extension* of $C$ with respect to $A$. If $C \in \mathfrak{C}^r_p(X,\mathbb{T},\partial X)$ then the following properties hold:

(i) $C_\Lambda \in \mathfrak{C}^r_p(X,\mathbb{T},\partial X)$ and there exist $|C_\Lambda|, (C_\Lambda)_+, (C_\Lambda)_- \in \mathfrak{C}_{p,+}(X,\mathbb{T},\partial X)$ such as

$$|C_\Lambda| = (C_\Lambda)_+ + (C_\Lambda)_-, \qquad C_\Lambda = (C_\Lambda)_+ - (C_\Lambda)_-, \qquad \text{on} \qquad D((C_\Lambda)_+) \cap D((C_\Lambda)_-).$$

(ii) $(C_\Lambda)_+ = (C_+)_\Lambda$, $(C_\Lambda)_- = (C_-)_\Lambda$ and $|C_\Lambda| = |C|_\Lambda$.

## Proof.

The proof follows from the fact that $\sup$ is an uniformly continuous map from $X \times X$ into $\partial X$. □

# Observation systems

The following result provide a sufficient condition on admissibility of observation operators for positive semi-groups.

**Result 2, (Yassine, 2022)**

Let $X, \partial X$ be Banach lattices, and let $(A, D(A))$ be a densely defined resolvent positive operator such that

$$\|R(\mu_0, A)x\| \geq c\|x\|, \qquad x \in X_+,$$

for some $\mu_0 > s(A)$ and $c > 0$. Let $C \in \mathcal{L}(D(A), \partial X)$ be a positive operator. Then $C$ is a positive zero-class $L^p$-admissible observation operator for $A$. Moreover, if $C \in \mathcal{L}^r(D(A), \partial X)$ then $C$ is a zero-class $L^p$-admissible observation operator for $A$.

# Observation systems

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The last result at hand, one can deduce the following perturbation result of positive semigroups.

### Result 3, (Yassine, 2022)

Let $A$ be a densely defined resolvent positive operator on a Banach lattice $X$. Let $P : D(A) \subset X \longrightarrow X$ be a positive operator, and assume that

$$\|R(\mu_0, A)x\| \geq c\|x\|, \qquad x \in X_+,$$

for some $\mu_0 > s(A)$ and $c > 0$. Then $A + P$ generates a positive $C_0$-semigroup on $X$ and $s(A+P) = w_0(A+P)$.

Yassine, (2022)

Let $X, \partial X$ be Banach lattices and let assume that the following assumptions hold:

- (H1) $A = (A_m)_{|\ker G}$ is a resolvent positive operator such that

$$\|R(\mu_0, A)f\| \geq c\|f\|, \qquad \forall\, X \ni f \text{ psoitive},$$

- (H2) $G$ is surjective.

Furthermore, let assume that the operator $\Gamma$ is positive and the Dirichlet operator

$$D_\mu := \left(G_{|_{\ker(\mu - A_m)}}\right)^{-1} \geq 0, \qquad \forall \mu > s(A).$$

Then, the operator

$$\mathfrak{A} = A_m, \qquad D(\mathfrak{A}) = \{x \in D(A_m) : Gx = \Gamma x\}.$$

generates a positive $C_0$-semigroup $\mathcal{T} := (\mathcal{T}(t))_{t \geq 0}$ on $X$ and $\omega_0(\mathfrak{A}) = s(\mathfrak{A})$. Moreover, we have

$$\begin{aligned} R(\mu, \mathfrak{A}) &= R(\mu, A) + D_\mu (I_{\partial X} - \Gamma D_\mu)^{-1} \Gamma R(\mu, A) \\ &= R(\mu, A) + D_\mu \textstyle\sum_{n=0}^{\infty} (\Gamma D_\mu)^n \Gamma R(\mu, A) \geq R(\mu, A) \geq 0, \end{aligned}$$

for $\mu > s(A)$. Furthermore, there exists $\mu_0 > s(\mathfrak{A})$ and a constant $c' > 0$ such that

$$\|R(\mu_0, \mathfrak{A})f\| \geq c'\|f\|, \qquad \forall f \in X_+.$$

# Conclusion

$$(A,B,C)\begin{cases}\dot{z}(t)=Az(t)+Bu(t), & t\geq 0,\\ z(0)=x, & ,\\ y(t)=Cz(t), & t\geq 0.\end{cases}$$

Here $A$ generates a strongly continuous positive semigroup $\mathbb{T}$ on $X$, $B\in\mathcal{L}(U,X_{-1})$ is a (positive) control operator, $C\in\mathcal{L}(D(A),Y)$ is a (positive) observation operator, and $D\in\mathcal{L}(U,Y)$ is the (positive) feedthrough operator.

(1) We fully describe the structural properties of the spaces of $L^p$-admissible control/observation operators in the Banach lattice setting.

(2) We give new insight into the perturbation theory, namely, positive perturbations of positive semigroups.

# Perspective

- Controllability properties with positivity constraints of $(A, B, C)$ such as feedback stabilization, optimal control problems, the regulator problem, and $H^{\infty}$-control problem..

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THANK YOU FOR YOUR ATTENTION