# $L^p$ STABILITY FOR THE LINEAR WAVE EQUATION WITH LOCALIZED DAMPING


Presented by:

**Meryem KAFNEMER (UMA, ENSTA Paris)**

Under the supervision of:

**Frédéric JEAN (UMA, ENSTA Paris),**

**Yacine CHITOUR (LS2, CentraleSupélec)**


ECOLE DOCTORALE DE MATHEMATIQUES HADAMARD

CentraleSupélec | université PARIS-SACLAY

# Existing work about the stability of the damped wave equation with internal damping:

- Case p=2
  - ***“A new method to obtain decay rate estimates for dissipative systems with localized damping”,*** *Patrick Martinez (1999)*
  - ***“Exponential stability for the wave equation with weak nonmonotone damping”,*** *Patrick Martinez, Judith Vancostenoble, (2000)*

For extended references:
“On Some Recent Advances on Stabilization for Hyperbolic Equations”, Fatiha Alabau (2012)

- Case p≠2
  - ***"$L^p$-estimates of solutions to some nonlinear wave equations in one space dimension”,*** *Alain Haraux (2009)*
  - ***“$L^p$-asymptotic stability analysis of a 1D wave equation with a nonlinear damping”,*** *Yacine Chitour, Swann Marx, and Christophe Prieur (2019)*

For extended references:
“$L^p$-asymptotic stability of 1D damped wave equations with localized and linear damping”, Meryem Kafnemer and al. (2022)

# HILBERTIAN FRAMEWORK

LINEAR PROBLEM

# Case $p = 2$ :

Control

$$\begin{cases} z_{tt} - \Delta z + \overbrace{a(x)u}^{\text{Control}} = 0 & \text{in } \Omega \times \mathbb{R}, \\ z = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ z(0,\cdot) = z^0 \ , \ z_t(0,\cdot) = z^1 & \text{in } \Omega. \end{cases}$$

$\Omega \subset \mathbb{R}^N$ , $C^2$ bounded

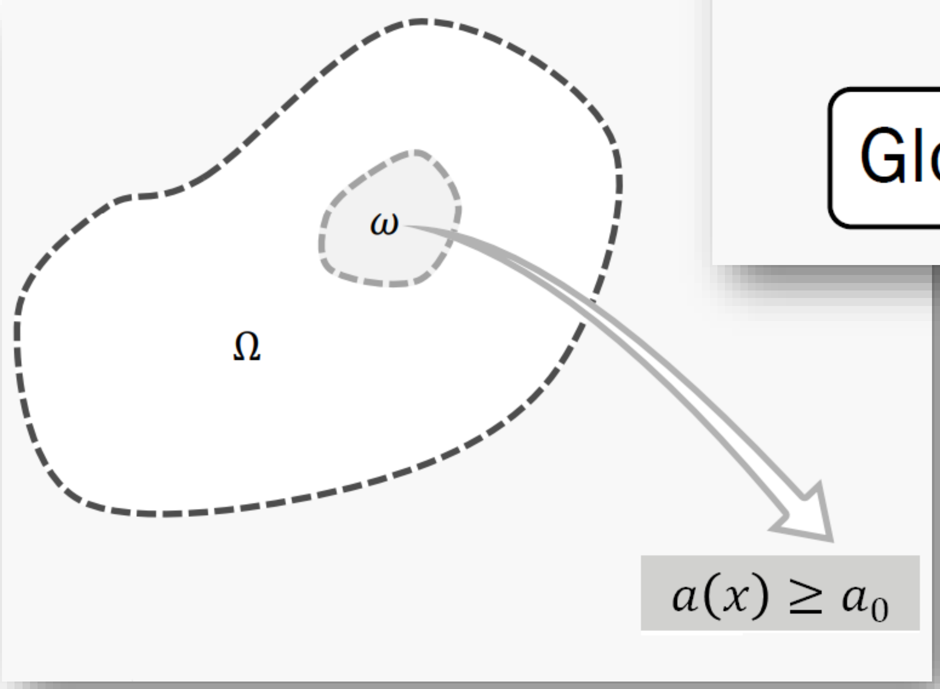
$(z_0, z_1) \in H_0^1(\Omega) \times L^2(\Omega)$ ,

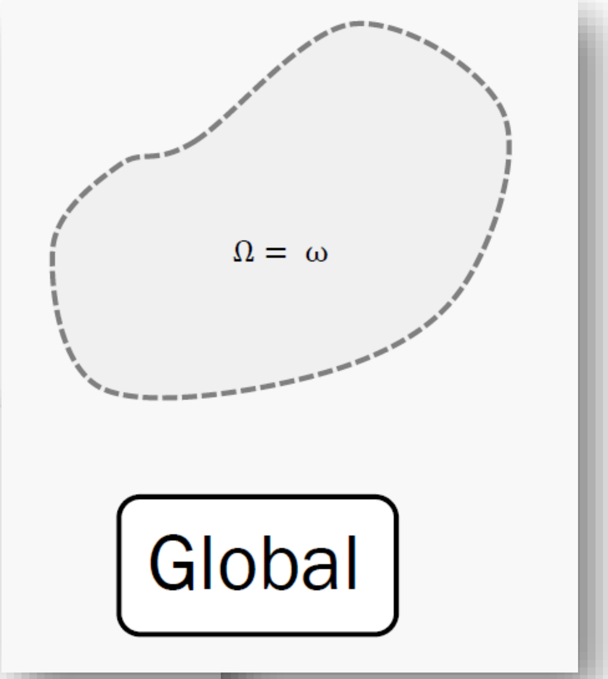
$a : \overline{\Omega} \to \mathbb{R}$ continuous function, satisfies:

$$a(x) \geq a_0 > 0 \ \ on\ \omega \subset \Omega.$$

In our case, $u$ is a linear damping given by:

$$\boldsymbol{u = z_t}$$

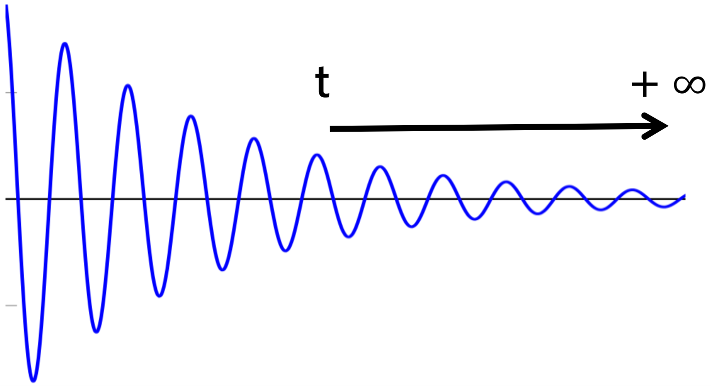


# Case $p = 2$ :

Stabilizing feedback

$$(P)\quad \begin{cases} z_{tt} - \Delta z + \overbrace{a(x) z_t}^{\text{Stabilizing feedback}} = 0 & \text{in } \Omega \times \mathbb{R}, \\ z = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ z(0,\cdot) = z^0 \ , \ z_t(0,\cdot) = z^1 & \text{in } \Omega. \end{cases}$$




$$E(t) = \frac{1}{2}\int_\Omega (z_t(t,\cdot)^2 + |\nabla z(t,\cdot)|^2) dx,$$

## Different types of stability

Strong stabilization: $E(t) \longrightarrow 0$ when $t \longrightarrow \infty$

Polynomial stability: $\exists C, \gamma > 0,\ E(t) \leq CE(0)t^{-\gamma}$

Exponential stability: $\exists C, \gamma > 0,\ E(t) \leq CE(0)e^{-\gamma t}$

To obtain exponential stability results, geometrical conditions are imposed on the damping domain $\omega$

# Multipliers method:

$$\iint M \times \; ( z_{tt} - \Delta z + a(x) z_t ) = 0$$

⇩

Energy Estimate

$$\int_S^T E(t)dt \le CE(S),$$

⇩

$$E(t) \le E(0) e^{-\frac{t-C}{C}} \quad \forall\, t \ge 0.$$

Exponential stability

Using the multiplier $z_t$ :

$$\iint z_t \times \; ( z_{tt} - \Delta z + a(x) z_t ) = 0$$

⇩

$$E'(t) = -\int a(x) z_t^2 \, dx$$

# NON-HILBERTIAN FRAMEWORK

LINEAR PROBLEM

# Case $1 \leq p \leq \infty$ :

The study can be done in one dimension only

$$\begin{cases} z_{tt} - z_{xx} + a(x) z_t = 0 & \text{for } (t,x) \in \mathbb{R}_+ \times (0,1), \\ z(t,0) = z(t,1) = 0 & t \geq 0, \\ z(0,\cdot) = z_0 \ , \ z_t(0,\cdot) = z_1, & \end{cases} \tag{1}$$

$$X_p := W_0^{1,p}(0,1) \times L^p(0,1),$$
$$Y_p := \left(W^{2,p}(0,1) \cap W_0^{1,p}(0,1)\right) \times W_0^{1,p}(0,1),$$

$$\|(u,v)\|_{X_p} := \left(\frac{1}{p}\int_0^1 (|u' + v|^p + |u' - v|^p)\, dx\right)^{\frac{1}{p}},$$

$$\|(u,v)\|_{Y_p} := \left(\frac{1}{p}\int_0^1 (|u'' + v'|^p + |u'' - v'|^p)\, dx\right)^{\frac{1}{p}}.$$

$$E_p(t) = \frac{1}{p}\int_0^1 (|z_x + z_t|^p + |z_x - z_t|^p)\, dx$$

Haraux (2009)

Haraux (2009), $E_p$ is non-increasing with an explicit expression of $E_p'$ along strong solutions.

# Geometrical conditions:

$$\begin{cases} z_{tt} - z_{xx} + a(x) z_t = 0 & \text{for } (t,x) \in \mathbb{R}_+ \times (0,1), \\ z(t,0) = z(t,1) = 0 & t \geq 0, \\ z(0,\cdot) = z_0 \ , \ z_t(0,\cdot) = z_1, & \end{cases} \tag{1}$$

$a : \overline{\Omega} \to \mathbb{R} \quad a \geq 0 \;\; on \; [0,1] \;$ and $\; \exists \; a_0 > 0, \; a \geq a_0 \;\; on \;\; \omega$


Geometrical conditions: unnecessary


$(\mathbf{H_1})$ $a : [0,1] \to \mathbb{R}$ is a non-negative continuous function such that

$$\exists \; a_0 > 0, \; a \geq a_0 \;\; \text{on} \;\; \omega = ]c,d[ \subset [0,1],$$

# Well-posedness

**Theorem (Well-posedness)** *Let* $p \in [1, \infty)$. *For any initial data* $(z_0, z_1) \in X_p$ *(resp.* $Y_p$*), there exists a unique weak (resp. strong) solution* $z$ *such that*

$$z \in L^\infty(\mathbb{R}_+, W_0^{1,p}(0,1)) \cap W^{1,\infty}(\mathbb{R}_+, L^p(0,1)),$$

$$(\text{resp.} \qquad z \in L^\infty(\mathbb{R}_+, W^{2,p}(0,1) \cap W_0^{1,p}(0,1)) \cap W^{1,\infty}(\mathbb{R}_+, W_0^{1,p}(0,1)).)$$

The proof is based on D'Alembert formula and fixed point theory

# Exponential stability:

**Theorem 4.1** *(Exponential stability) Fix $p \in (1, \infty)$ and suppose that Hypothesis* $(\mathbf{H_1})$ *is satisfied. Then the $C^0$-semigroup $(S_p(t))_{t\geq 0}$ is exponentially stable.*

$$E_p(t) = \frac{1}{p}\int_0^1 \left(|z_x + z_t|^p + |z_x - z_t|^p\right) dx$$

$$\forall\ 0 \leq S \leq T\ ,\ \int_S^T E_p(t)\, dt \leq C\, C_p E_p(S),$$

Energy estimate

$$\iint M \times \ ( z_{tt} - \Delta z + a(x) z_t) = 0$$

# Exponential stability:

**Theorem 4.1** *(Exponential stability)* *Fix* $p \in (1, \infty)$ *and suppose that Hypothesis* $(\mathbf{H_1})$ *is satisfied. Then the* $C^0$*-semigroup* $(S_p(t))_{t \geq 0}$ *is exponentially stable.*

$$E_p(t) = \frac{1}{p} \int_0^1 \left( |z_x + z_t|^p + |z_x - z_t|^p \right) dx$$

Riemann coordinates:

$$\rho(t,x) = z_x(t,x) + z_t(t,x),$$
$$\xi(t,x) = z_x(t,x) - z_t(t,x).$$

$$\begin{cases} \rho_t - \rho_x = -\frac{1}{2} a(x)(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0,1), \\ \xi_t + \xi_x = \frac{1}{2} a(x)(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0,1), \\ \rho(t,0) - \xi(t,0) = \rho(t,1) - \xi(t,1) = 0 & \forall t \in \mathbb{R}_+, \\ \rho_0 := \rho(0,.) = z_0' + z_1 \ , \ \xi_0 := \xi(0,.) = z_0' - z_1, & \end{cases}$$

$$E_p(t) = \frac{1}{p} \int_0^1 (|\rho|^p + |\xi|^p) dx.$$

# Exponential stability:

$$\begin{cases} \rho_t - \rho_x = -\frac{1}{2}a(x)(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0,1), \\ \xi_t + \xi_x = \frac{1}{2}a(x)(\rho - \xi) & \text{in } \mathbb{R}_+ \times (0,1), \\ \rho(t,0) - \xi(t,0) = \rho(t,1) - \xi(t,1) = 0 & \forall t \in \mathbb{R}_+, \\ \rho_0 := \rho(0,.) = z_0' + z_1 \ , \ \xi_0 := \xi(0,.) = z_0' - z_1, & \end{cases}$$

$$\begin{cases} \iint M(\rho) \times \left( \rho_t - \rho_x + \frac{1}{2}a(x)(\rho - \xi) \right) = 0 \\ \iint M(\xi) \times \left( \xi_t + \xi_x - \frac{1}{2}a(x)(\rho - \xi) \right) = 0 \end{cases}$$

$$\Downarrow$$

$$\forall\ 0 \leq S \leq T\ , \ \int_S^T E_p(t)\,dt \leq C\, C_p E_p(S),$$

# Case p≥2: Generalizing the multipliers

- $x\,\psi\, sign(\rho)|\rho|^{p-1}$ ; $x\,\psi\, sign(\xi)|\xi|^{p-1}$
- $\phi\,(p-1)|\rho|^{p-2}z$ ; $\phi\,(p-1)|\xi|^{p-2}z$
- $v$ where, $v$ is the solution of

$$\begin{cases} v_{xx} = \beta sign(z)|z|^{p-1} & x \in (0,1), \\ v(0) = v(1) = 0, \end{cases}$$

$$\begin{cases} 0 \leq \psi \leq 1, \\ \psi = 0 \text{ on } Q_0, \\ \psi = 1 \text{ on } (0,1) \setminus Q_1, \end{cases} \qquad \begin{cases} 0 \leq \phi \leq 1, \\ \phi = 1 \text{ on } Q_1, \\ \phi = 0 \text{ on } (0,1) \setminus Q_2, \end{cases} \qquad \begin{cases} 0 \leq \beta \leq 1, \\ \beta = 1 \text{ on } Q_2 \cap (0,1), \\ \beta = 0 \text{ on } \mathbb{R} \setminus \omega. \end{cases}$$

## Case $1 < p < 2$ :

- Problem in the second multiplier:

$$(p-1)\phi|\rho|^{p-2}z\ ;\ (p-1)\phi|\xi|^{p-2}z$$

- Solution ?

# Case $1 < p < 2$ :

$$\phi\ (p-1)|\rho|^{p-2}z\ ;\ \phi\ (p-1)|\xi|^{p-2}z$$

⇓

$$\phi\ (p-1)(|\rho|+1)^{p-2}z\ ;\ \phi\ (p-1)(|\xi|+1)^{p-2}z$$

$$E_p(t) = \frac{1}{p}\int_0^1 (|\rho|^p + |\xi|^p)dx.$$

⟹

$$\mathcal{E}_p(t) = \int_0^1 (G(\rho) + G(\xi))\ dx.$$

$$G(y) = \frac{1}{p}[(|y|+1)^p - 1] - |y|.$$

# Case $1 < p < 2$ :

**Proposition 4.3** *Fix $p \in (1,2)$ and suppose that Hypothesis $(\mathbf{H_1})$ is satisfied. Then there exist positive constants $C$ and $C_p$ such that, for every $(z_0, z_1) \in Y_p$ verifying*

$$E_p(0) \leq 1, \tag{105}$$

*we have the following energy estimate:*

$$\forall\ 0 \leq S \leq T,\ \int_S^T \mathcal{E}_p(t)\,dt \leq C\,C_p \mathcal{E}_p(S). \tag{106}$$

$$\mathcal{E}_p(t) \leq \mathcal{E}_p(0)e^{1-\gamma_p t} \leq e^{1-\gamma_p t},$$

$$\|S_p(t_p)\|_{X_p} = \sup_{E_p(0)\leq 1} E_p(t)^{\frac{1}{p}} \leq \left(\frac{1}{2}\right)^{\frac{1}{p}} < 1,$$

$$E_p(t) \leq \tfrac{1}{2} \text{ if } t \geq t_p$$

⇓

Exponential stability

# Case of a global constant damping

$$a(x) \equiv 2\alpha, \quad \forall\, x \in (0,1),$$

**Proposition 5.1** *For $p = 1$ or $p = \infty$, the semi-group $(S(t))_{t\geq 0}$ is exponentially stable for a global constant damping if $\alpha \in (0,2)$.*

Idea of proof: $z(t,x) = e^{-\alpha t} v(t,x), \quad x \in (0,1), \quad t \geq 0,$

$$\begin{cases} v_{tt} - v_{xx} = \alpha^2 v & \text{in } \mathbb{R}_+ \times (0,1), \\ v(t,0) = v(t,1) = 0 & t \geq 0, \\ v(0,\cdot) = z_0 \ , \ v_t(0,\cdot) = z_1 + \alpha z_0. & \end{cases}$$

$$E_p(t) \leq e^{-\alpha p t}\left(2^{p-1} + \frac{\alpha^p}{p^2}\right) V_p(t).$$

$$V_p(t) \leq V_p(0) e^{\alpha^2 p K_p^{\frac{1}{p}}\, t}.$$

$\Longrightarrow$

$$E_p(t)^{\frac{1}{p}} \leq (2+\alpha)^2 e^{M_\alpha t} E_p(0)^{\frac{1}{p}},$$

$$M_\alpha := -\alpha + \alpha^2 K_p^{\frac{1}{p}} = -\alpha\left(1 - \frac{\alpha}{2p^{\frac{1}{p}}}\right).$$

# Conclusion:

- Generalizing the multiplier method in the $L^p$ framework.
- Exponential stability for $1 < p < \infty$.
- Conjecture for $p = 1, p = \infty$.

# THANK YOU FOR YOUR ATTENTION ☺