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Time integration of Maxwell equations on a heterogeneous medium

IX Partial differential equations, optimal design and numerics, Benasque, August 21–September 2, 2022

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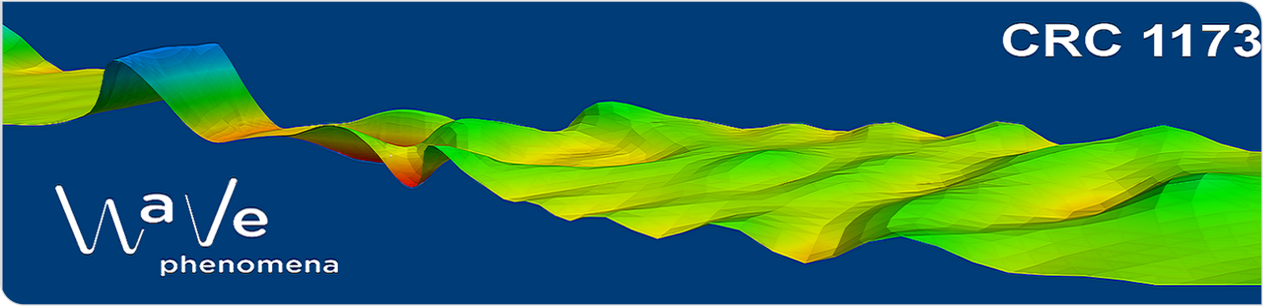
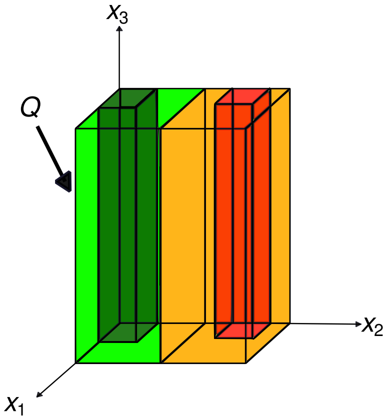
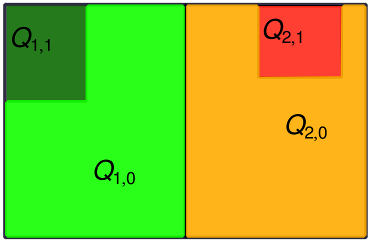


Agenda

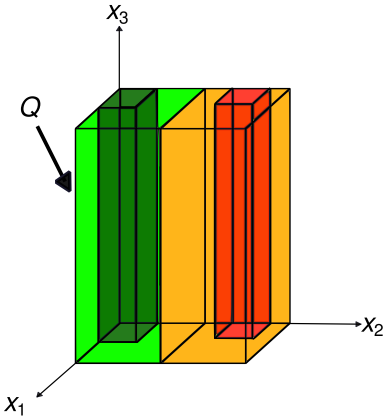
1. Model problem
2. Regularity theory
3. Construction and error analysis of a dimension splitting scheme
4. Numerical experiments

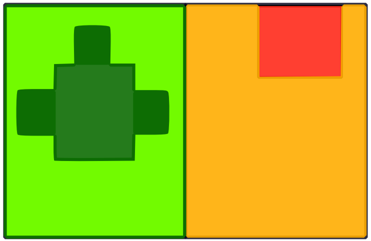
Model problem



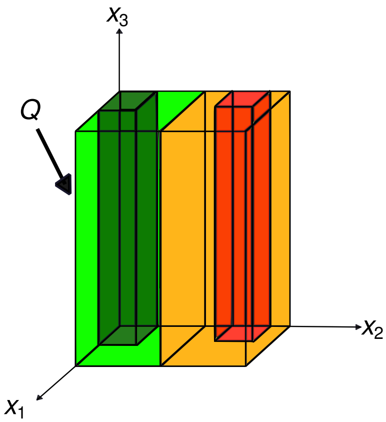


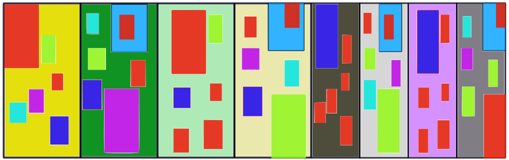
Model problem





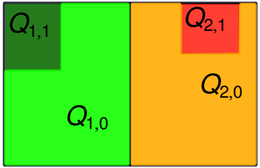
Model problem





Notation & assumptions

- Put $f^{(i,j)} := f|_{Q_{i,j}}$ for $f \in L^2(Q)$
- Set of all interfaces $\mathscr{F}_{\text{int}}$
- Denote by $[\![\cdot]\!]_{\mathcal{F}}$ the jump at $\mathcal{F}$, and by $\nu_{\mathcal{F}}$ the normal vector of $\mathcal{F}$
- Parameters $\varepsilon^{(i,j)}, \mu > 0$ are constant
  - **Note:** No monotonicity / jump restriction on ε



Maxwell equations

- Study linear isotropic Maxwell equations on cuboid Q

$$\begin{aligned} \partial_t(\varepsilon \mathbf{E}) &= \operatorname{curl} \mathbf{H} - \mathbf{J}, & \partial_t(\mu \mathbf{H}) &= -\operatorname{curl} \mathbf{E} && \text{on } Q, \\ \mathbf{E} \times \nu &= 0, & \mu \mathbf{H} \cdot \nu &= 0 && \text{on } \partial Q \end{aligned}$$

- **Question:** How accurate are time discrete approximations from dimension splitting schemes?
- **Intermediate step:** Study regularity of Maxwell system
  - **Literature:** Bonnet-BenDhia/Hazard/Lohrengel 1999; Costabel/Dauge/Nicaise 1999; Bonito/Guermond/Luddens 2013; Ciarlet 2016, 2020; Ciarlet/Lefèvre/Lohrengel/Nicaise 2010.

Analytical framework

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- Define Maxwell operator

$$M\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{\varepsilon}\operatorname{curl}\\ -\frac{1}{\mu}\operatorname{curl} & 0\end{pmatrix}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix},$$

$$\mathscr{D}(M) := H_0(\operatorname{curl}, Q) \times H(\operatorname{curl}, Q),$$

$$H_0(\operatorname{curl}, Q) = \{\varphi \in L^2(Q)^3 \mid \operatorname{curl}\varphi \in L^2(Q)^3,\ \varphi \times \nu = 0 \text{ on } \partial Q\}$$

- Construct **state space**

$$X_2 := \{(\mathbf{E}, \mathbf{H}) \in \mathscr{D}(M^2) \mid \operatorname{div}(\mathbf{E}^{(i,j)}) \in H^1_{00}(Q_{i,j}),\ \operatorname{div}(\mathbf{E}) = 0 \text{ on } \Gamma_3,\ [\![\varepsilon\mathbf{E}\cdot\nu_{\mathcal{F}}]\!]_{\mathcal{F}} = 0,$$
$$\operatorname{div}(\mu\mathbf{H}) = 0 \text{ on } Q,\ \mu\mathbf{H}\cdot\nu = 0 \text{ on } \partial Q\}$$

- On X_2: Interpret Maxwell equations as Cauchy problem

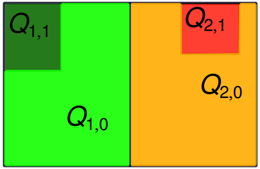
$$\text{(CP)} \qquad \partial_t\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = M\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} - \begin{pmatrix}\frac{1}{\varepsilon}\mathbf{J}\\ 0\end{pmatrix}, \qquad t \geq 0, \qquad \begin{pmatrix}\mathbf{E}(0)\\ \mathbf{H}(0)\end{pmatrix} = \begin{pmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{pmatrix}$$

Behavior of ε

- Let $\kappa_0 \in (2/3, 1]$ with

$$\max_{i=1,2} \frac{(\varepsilon^{(i,1)} - \varepsilon^{(i,0)})^2}{\varepsilon^{(i,1)} \varepsilon^{(i,0)}} = -\frac{4 \sin^2(\kappa_0 \pi)}{\sin(\frac{\kappa_0}{2}\pi) \sin(\frac{3\kappa_0}{2}\pi)}$$

- Note: κ_0 increases if jumps become smaller



Regularity statement

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- Recall

$$(CP) \quad \partial_t(\mathbf{E},\mathbf{H}) = M(\mathbf{E},\mathbf{H}) - (\tfrac{1}{\varepsilon}\mathbf{J},0), \quad t \geq 0, \quad (\mathbf{E}(0),\mathbf{H}(0)) = (\mathbf{E}_0,\mathbf{H}_0),$$

$$X_2 := \{(\mathbf{E},\mathbf{H}) \in \mathscr{D}(M^2) \mid \operatorname{div}(\mathbf{E}^{(i,j)}) \in H^1_{00}(Q_{i,j}),\ \operatorname{div}(\mathbf{E}) = 0 \text{ on } \Gamma_3,\ [\![\varepsilon\mathbf{E}\cdot\nu_{\mathcal{F}}]\!]_{\mathcal{F}} = 0,$$
$$\operatorname{div}(\mu\mathbf{H}) = 0 \text{ on } Q,\ \mu\mathbf{H}\cdot\nu = 0 \text{ on } \partial Q\}$$

Theorem

Let $\kappa > 1 - \kappa_0$. Let $T > 0$, $(\mathbf{E}_0,\mathbf{H}_0) \in X_2$, and $(\frac{1}{\varepsilon}\mathbf{J},0) \in C([0,T],X_2)$.
Then (CP) has a unique solution

$$(\mathbf{E},\mathbf{H}) \in C^1([0,T],PH^{1-\kappa}(Q)^6) \cap C([0,T],PH^{2-\kappa}(Q)^6).$$

- Tools: Regularity theory for elliptic transmission problems, interpolation theory, semigroup theory[1]

[1] Costabel, Dauge, Nicaise 1999; Z. 2022a; Z. 2022b

Construction of a dimension splitting scheme

- Set $t_n := \tau n$
- **1. Step:** Split Maxwell operator

$$M\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon}(\partial_2\mathbf{H}_3 - \partial_3\mathbf{H}_2) \\ \frac{1}{\varepsilon}(\partial_3\mathbf{H}_1 - \partial_1\mathbf{H}_3) \\ \frac{1}{\varepsilon}(\partial_1\mathbf{H}_2 - \partial_2\mathbf{H}_1) \\ \frac{1}{\mu}(\partial_3\mathbf{E}_2 - \partial_2\mathbf{E}_3) \\ \frac{1}{\mu}(\partial_1\mathbf{E}_3 - \partial_3\mathbf{E}_1) \\ \frac{1}{\mu}(\partial_2\mathbf{E}_1 - \partial_1\mathbf{E}_2) \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon}\partial_2\mathbf{H}_3 \\ \frac{1}{\varepsilon}\partial_3\mathbf{H}_1 \\ \frac{1}{\varepsilon}\partial_1\mathbf{H}_2 \\ \frac{1}{\mu}\partial_3\mathbf{E}_2 \\ \frac{1}{\mu}\partial_1\mathbf{E}_3 \\ \frac{1}{\mu}\partial_2\mathbf{E}_1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{\varepsilon}\partial_3\mathbf{H}_2 \\ -\frac{1}{\varepsilon}\partial_1\mathbf{H}_3 \\ -\frac{1}{\varepsilon}\partial_2\mathbf{H}_1 \\ -\frac{1}{\mu}\partial_2\mathbf{E}_3 \\ -\frac{1}{\mu}\partial_3\mathbf{E}_1 \\ -\frac{1}{\mu}\partial_1\mathbf{E}_2 \end{pmatrix} =: A\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} + B\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

Construction of a dimension splitting scheme

- **2. Step:** Put $w_j^0 := (\mathbf{E}_0, \mathbf{H}_0)$. Approximate $(\mathbf{E}(t_{n+1}), \mathbf{H}(t_{n+1})) \approx w^{n+1}$ with[2]

$$
\begin{aligned}
(I - \tfrac{\tau}{2}A)w^{n+1/2} &= (I + \tfrac{\tau}{2}B)w^n, \\
(I - \tfrac{\tau}{2}B)w^{n+1} &= (I + \tfrac{\tau}{2}A)\Big(w^{n+1/2} - \tfrac{\tau}{2\varepsilon}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1}), 0)\Big) \\
\rightsquigarrow w^{n+1} &= \mathscr{T}_{n+1}(w^n) := (I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)\Big[(I - \tfrac{\tau}{2}A)^{-1}(I + \tfrac{\tau}{2}B)w^n - \tfrac{\tau}{2\varepsilon}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1}), 0)\Big]
\end{aligned}
$$

$\rightsquigarrow$ **a**lternating **d**irection **i**mplicit (**ADI**) scheme

- **Advantages:**
  - scheme unconditionally stable
  - implicit steps simplify to 1D elliptic problems

[2] Zheng, Chen, Zhang 2000; Namiki 2000; Eilinghoff, Schnaubelt 2018.

Error bound

- Notation:
  - Solution of (CP) with $w_0 = (\mathbf{E}_0, \mathbf{H}_0)$ is w
  - Set $\quad W_T := C([0,T], \mathscr{D}(M^3)) \cap W^{1,\infty}([0,T], X_2) \cap W^{2,1}([0,T], \mathscr{D}(A) \cap \mathscr{D}(B))$

Theorem

Let $\theta \in (0, 1/2)$. Let $T > 0$, $w^0 = (\mathbf{E}_0, \mathbf{H}_0) \in \mathscr{D}(M^3) \cap X_2$, $(\frac{1}{\varepsilon}\mathbf{J}, 0) \in W_T$.
There is $C = C(\varepsilon, \mu, \theta, Q) > 0$ with

$$\|w^n - w(t_n)\|_{L^2} \leq CT\tau^{3/2-\theta}\Big(\|w_0\|_{\mathscr{D}(M^3)\cap X_2} + \|(\tfrac{1}{\varepsilon}\mathbf{J}, 0)\|_{W_T}\Big)$$

for all $\tau \in (0, \tau_0)$, $n \in \mathbb{N}$ with $n\tau \leq T$.[3]

[3] Convergence with order 2 for standard ADI scheme with regular ε, μ: Hochbruck/Jahnke/Schnaubelt 2015; Eilinghoff/Schnaubelt 2017

Insights into the proof: estimate the local error

- For simplicity: $\mathbf{J} = 0$.
- **Important fact:** $X_2 \hookrightarrow \mathscr{D}(A) \cap \mathscr{D}(B) \cap PH^{2-\kappa}(Q)^6$
- Use the extrapolated operator A_{-1} of A on $L^2(Q)^6$
- Set

$$\Lambda_k(t) := \frac{1}{t^k(k-1)!} \int_0^t (t-s)^{k-1} e^{sM} ds, \qquad k \in \mathbb{N},\ t > 0$$

- Estimate the local error $\mathscr{T}_k(w(t_k)) - w(t_{k+1}) =: e_k(\tau)$

Insights into the proof: estimate the local error

- Arrive at the identity

$$e_k(\tau) = (I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A_{-1})^{-1}\Big[\underbrace{\big(-\tau^3 M^3 \Lambda_3(\tau) + \tfrac{\tau^3}{2}M^3\Lambda_2(\tau)\big)w(t_k)}_{\|\cdot\|_{L^2} \le C\tau^3 \|Mw(t_k)\|_{X_2}} - \tfrac{\tau^3}{4}A_{-1}BM\Lambda_1(\tau)w(t_k)\Big]$$

- Employ a functional calculus approach to show

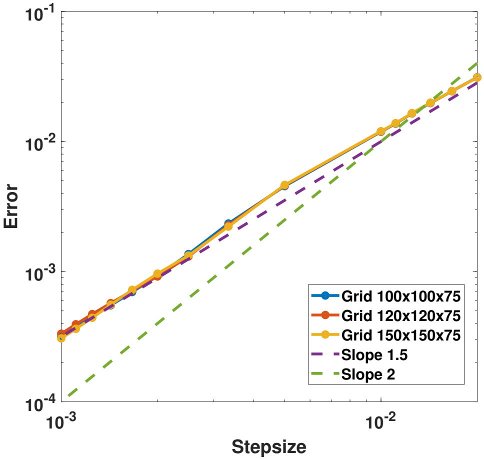
$$\|(I - \tfrac{\tau}{2}A_{-1})^{-1}A_{-1}BM\Lambda_1(\tau)w(t_k)\|_{L^2} \le C\tau^{-1/2-\theta}\sum_{l=0}^{1}\|M^l w(t_k)\|_{X_2}$$

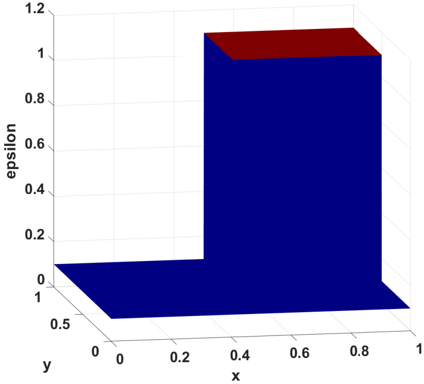
- Conclude the estimate

$$\|e_k(\tau)\|_{L^2} \le C\tau^{5/2-\theta}\sum_{l=0}^{1}\|M^l w(t_k)\|_{X_2}$$

Numerical experiments

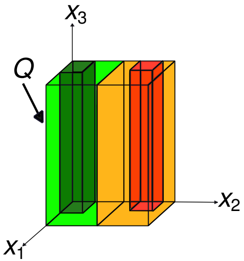
Joint work with Tobias Jahnke (KIT), see also Hochbruck/Jahnke/Schnaubelt 2015





Conclusion

- **Model:** heterogeneous cuboid with discontinous material parameters
- **Regularity:** solution of Maxwell equations belongs to $PH^{2-\kappa}$
- **Schemes:** split Maxwell operator along space dimensions
  - Integrate in time with Peaceman-Rachford method
- **Error analysis in L^2:** convergence with $\tau^{3/2-}$



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