Asymptotic analysis of partially and locally dissipated hyperbolic systems[1]

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BENASQUE IX Partial differential equations, optimal design and numerics
26 August 2022

[1] Joint works with Raphaël Danchin, Nicola De Nitti and Enrique Zuazua

Presentation Outline

Introduction

We consider *n*-component quasilinear hyperbolic systems of the form:

$$\frac{\partial U}{\partial t} + \sum_{j=1}^{d} A^j(U)\frac{\partial U}{\partial x_j} = \frac{BU}{\varepsilon}, \quad \text{where} \quad x \in \mathbb{R}^d, \varepsilon > 0 \quad \text{and} \quad U = \begin{pmatrix} u_1 \\ .. \\ u_n \end{pmatrix}$$

where the matrices A^j and B are symmetric $\rightarrow$ hyperbolicity of the system.

It models physical phenomena with finite speed of propagation or conservations laws, such as:

- The compressible Euler equation, Maxwell's equation, Einstein's equation, MHD equations, Yang-Mills equation, etc.
- Numerous applications: non-viscous fluid mechanics, kinetic theory, astrophysics, road traffic modelling, blood vessel circulation, etc.

Questions: When does the solution exists globally in time? And what happens when $\varepsilon \rightarrow 0$?

$\rightarrow$ There is a surprisingly strong connection between these questions and problems related to control theory and Villani's hypocoercivity theory.

Role of the dissipation term

For that issue, the 0-th order term BU, that acts as a **dissipation** term, plays an essential role.

- In the case $B \equiv 0$, local-in-time strong solutions may develop singularities (shock waves) in finite time (A. Majda, D. Serre).
- If BU acts directly on each component i.e. $B > 0$

  $\rightarrow$ global existence + convergence to the equilibrium exponentially fast. (T-T. Li)
- In practice, a more reasonable assumption is that the dissipation acts only on certain components of the system: $B + {}^T B \geq 0$.
- This can be written as:

$$BU = \begin{pmatrix} 0_{\mathbb{R}^{n_1}} \\ DU \end{pmatrix} \text{ where } D > 0, DU \in \mathbb{R}^{n_2}, n_1, n_2 \in \mathbb{N} \text{ and } n_1 + n_2 = n.$$

- Main application: compressible Euler system with damping

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + u \cdot \nabla u + \nabla P(\rho) + \dfrac{u}{\varepsilon} = 0. \end{cases}$$

Global-in-time existence of solutions

Q: Since the dissipation is only present in some equations of the system, how can one ensure the global existence of solutions?
As a toy-model, let us look at the damped p-system

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + v = 0. \end{cases}$$

For this simple system, performing standard energy estimates leads to:

$$\frac{d}{dt}\|(u,v)\| + \|v\|_{L^2}^2 \leq 0 \rightarrow \text{ no time-decay information on } u.$$

Idea: consider the following perturbed functional

$$\mathcal{L}^2 = \|(u, v, \partial_x u, \partial_x v)\|_{L^2}^2 + \int_{\mathbb{R}} v \partial_x u,$$

which allows to recover dissipation properties on all the components.

Indeed, after basic computations, one obtains

$$\frac{d}{dt}\mathcal{L}^2 + \|v\|_{L^2}^2 + \|(\partial_x u, \partial_x v)\|_{L^2}^2 \leq 0.$$

And since $\mathcal{L}^2 \sim \|(u, v, \partial_x u, \partial_x v)\|_{L^2}^2$, one can derive time-decay estimates (**depending on the frequencies!**)

SK condition, Kalman rank condition and Hypocoercivity

For the general system $\partial_t U + \sum_j A^j \partial x_j U + BU = 0$, the previous idea also holds under the following condition:

Definition (Shizuta-Kawashima '80s)

$$\forall \xi \in \mathbb{R}^d, \quad \ker L \cap \{\text{eigenvectors of } \sum_j A^j \xi_j\} = \{0\}. \tag{SK}$$

Actually such condition is **equivalent to the Kalman rank condition for the couple** $(\sum_j A^j \xi_j, B)$.

Inspired by this fact and by the hypercoercivity theory, Beauchard and Zuazua constructed the following Lyapunov functional to recover decay estimates:

$$\mathcal{L} \triangleq \|U\|_{L^2}^2 + \int_{\mathbb{R}^d} \min(\xi, \xi^{-1})\mathcal{I} \quad \text{where} \quad \mathcal{I} \triangleq \Im \sum_{k=1}^{n-1} \varepsilon_k \big(LA_\omega^{k-1}\widehat{U} \cdot LA_\omega^k \widehat{U}\big).$$

Again, one obtains

$$\frac{d}{dt}\mathcal{L} + \kappa \min(1, |\xi|^2)\mathcal{L} \leq 0$$

- With this estimate at hand, one deduces the **global existence of small H^s solutions in the full space and**

$$\|U^h(t)\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)} \leq Ce^{-\lambda t}\|U_0\|_{L^2(\mathbb{R}^d,\mathbb{R}^n)},$$
$$\|U^\ell(t)\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^n)} \leq Ct^{-\frac{d}{2}}\|U_0\|_{L^1(\mathbb{R}^d,\mathbb{R}^n)}$$

  where U^h and U^ℓ correspond, respectively, to the high and low frequencies of the solution.
- Moreover, the method developed by Beauchard and Zuazua allows to treat situation when the (SK) condition is not satisfied.
- **However these hypercoercivity techniques do not give the full story of the behavior of the solution in the low frequency-regime and new considerations needs to be made to be able to study the limit $\varepsilon \to 0$.**

"New" observations

- Back to the damped p-system:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u + \dfrac{v}{\varepsilon} = 0. \end{cases} \tag{1}$$

A spectral analysis of the matrix

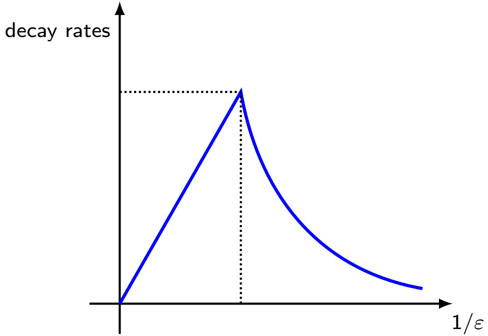
$$\begin{pmatrix} 0 & i\xi \\ i\xi & \dfrac{1}{\varepsilon} \end{pmatrix}$$

shows that:

- In low frequencies ($|\xi| \ll \dfrac{1}{\varepsilon}$), there are two **real** eigenvalues $\dfrac{1}{\varepsilon}$ and $\varepsilon\xi^2$.
- In high frequencies ($|\xi| \gg \dfrac{1}{\varepsilon}$), two complex conjugate eigenvalues coexist, whose real parts are asymptotically equal to $\dfrac{1}{2\varepsilon}$.
- The threshold between low and high frequencies is at $\dfrac{1}{\varepsilon}$
- $\rightarrow$ **The behavior of solution depend on the relation between ξ and ε.**

Insights from the spectral analysis

- There exists a damped mode in the low frequencies regime associated to the eigenvalue $\frac{1}{\varepsilon}$ $\rightarrow$ **Crucial uniform estimates**
- The asymptotic behaviour of the solution when $\varepsilon \to 0$ is not so intuitive.
  - Naively, we expect that as the damping coefficient becomes larger the dissipation becomes more dominant.
  - However, the so-called *overdamping* effect occurs: **the decay rate behavior is related to** $(\varepsilon, 1/\varepsilon)$**.**



Low frequencies in a simple case

Our idea is to follow exactly what the spectral analysis tells us. This can be done using:

$$\|f\|^h_{\dot{\mathbb{B}}^s_{2,1}} \triangleq \sum_{j\geq \frac{1}{\varepsilon}} 2^{js}\|\dot{\Delta}_j f\|_{L^2} \quad \text{and} \quad \|f\|^\ell_{\dot{\mathbb{B}}^{s'}_{p,1}} \triangleq \sum_{j\leq \frac{1}{\varepsilon}} 2^{js'}\|\dot{\Delta}_j f\|_{L^p}.$$

Back (again) to the damped p-system:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \partial_x u + \dfrac{v}{\varepsilon} = 0, \end{cases}$$

Defining the damped mode $w = v + \varepsilon\partial_x u$, the system can be rewritten

$$\begin{cases} \partial_t u - \varepsilon\partial^2_{xx} u = -\partial_x w \\ \partial_t w + \dfrac{w}{\varepsilon} = -\varepsilon\partial^2_{xx} v. \end{cases}$$

$\rightarrow$ **One directly get the uniform behaviour observed in the spectral analysis, not just heat effect as depicted in the previous references.**
$\rightarrow$ It is possible to study the two equations in L^p spaces in a decoupled way as the source terms can be absorbed in the low-frequency regime:

$$\|\partial_x f\|^\ell_{B^s_{p,1}} \leq \frac{1}{\varepsilon}\|f\|^\ell_{B^s_{p,1}}$$

To sum-up

- The hypocoercivity approach does not give the full story of the low-frequency behavior.
- New low-frequency analysis + high frequencies computation à la Beauchard et Zuazua, leads to uniform global existence result for general hyperbolic system satisfying the (SK) condition.
- And from these uniform estimates with a threshold depending on ε we can justify the relaxation limit when $\varepsilon \to 0$ with an explicit convergence rate.
- Introducing the diffusive rescaling:

$$(\widetilde{\rho}^{\varepsilon}, \widetilde{v}^{\varepsilon})(t,x) \triangleq (\rho, \frac{v}{\varepsilon})(\frac{t}{\varepsilon}, x).$$

The Euler system rewrites:

$$\begin{cases} \partial_t \rho^{\varepsilon} + \operatorname{div}(\rho^{\varepsilon} v^{\varepsilon}) = 0, \\ \varepsilon^2 \partial_t(\rho^{\varepsilon} v^{\varepsilon}) + \varepsilon^2 \operatorname{div}(\rho^{\varepsilon} v^{\varepsilon} \otimes v^{\varepsilon}) + \nabla P(\rho^{\varepsilon}) + \rho^{\varepsilon} v^{\varepsilon} = 0. \end{cases}$$

Relaxation result

Theorem (Danchin, C-B, Math. Ann. 2022)

Let $d \geq 1$, $p \in [2,4]$ and $\varepsilon > 0$.

- *Let $\bar{\rho}$ be a strictly positive constant and $(\rho - \bar{\rho}, v)$ be the solution of the compressible Euler system with damping (constructed with the previous arguments)*
- *Let $\mathcal{N} \in \mathcal{C}_b(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1}) \cap L^1(\mathbb{R}^+; \dot{\mathbb{B}}^{\frac{d}{p}+2}_{p,1})$ be the unique solution associated to the Cauchy problem:*

$$\begin{cases} \partial_t \mathcal{N} - \Delta P(\mathcal{N}) = 0 \\ \mathcal{N}(0,x) = \mathcal{N}_0 \in \dot{\mathbb{B}}^{\frac{d}{p}}_{p,1} \end{cases}$$

If we assume that

$$\|\widetilde{\rho}_0^\varepsilon - \mathcal{N}_0\|_{\mathbb{B}^{\frac{d}{p}-1}_{p,1}} \leq C\varepsilon,$$

then

$$\|\widetilde{\rho}^\varepsilon - \mathcal{N}\|_{L^\infty(\mathbb{R}_+;\dot{\mathbb{B}}^{\frac{d}{p}-1}_{p,1})} + \|\widetilde{\rho}^\varepsilon - \mathcal{N}\|_{L^1(\mathbb{R}_+;\dot{\mathbb{B}}^{\frac{d}{p}+1}_{p,1})} + \left\| \frac{\nabla P(\widetilde{\rho}^\varepsilon)}{\widetilde{\rho}^\varepsilon} + \widetilde{v}^\varepsilon \right\|_{L^1(\mathbb{R}^+;\dot{\mathbb{B}}^{\frac{d}{p}}_{p,1})} \leq C\varepsilon.$$

Localized damping

Damping active outside of a ball

We consider the one-dimensional linear hyperbolic system

$$\begin{cases} \partial_t U + A\partial_x U = -BU\mathbf{1}_\omega, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ U(0,x) = U_0(x), & x \in \mathbb{R}, \end{cases}$$

where $U = (u_1, u_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and

$$\omega := \mathbb{R} \setminus B_R(0) = \{x \in \mathbb{R} : \|x\| \geq R\} \quad \text{for a fixed } R > 0.$$

We assume:

- $B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ with $D > 0$
- The matrix A is a *strictly hyperbolic matrix*, i.e. A has n real distinct eigenvalues

$$\lambda_1 < \lambda_p < 0 < \lambda_{p+1} < \lambda_n.$$

- The couple (A, B) satisfies the (SK) condition.

In other words: **we are in the same situation as before but the damping is only effective in ω (the complementary of a ball).**

Goal: Quantify the decay as in the classical case.

Difficulties:

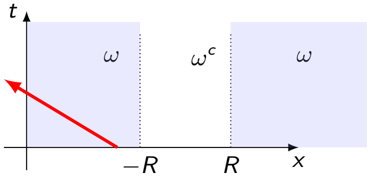
- Application of the Fourier transform to the term $1_\omega U$.
- Using a perturbed functional approached is not suitable to solve such problems, as it is known for the damped wave equations. equations.

Idea:

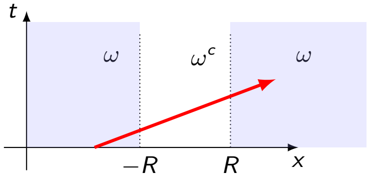
- The characteristic lines of the system spend only a finite time in the undamped region.
- When a characteristic is outside the undamped region, the solution decays as in the classical analysis.

This motivates us to develop a method involving only the consideration of the characteristics curves and a semigroup-wise decomposition.

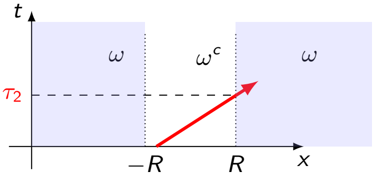
Propagation of characteristics and their location with respect to the region $\omega = \mathbb{R} \setminus B_R$ where the damping is active.



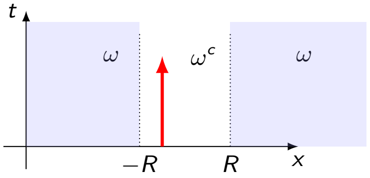
(a) **Case 1:** The initial support is in the damped region and the characteristics are going away from the un-damped region.



(b) **Case 2:** The initial support is in the damped region and the characteristics cross the un-damped region



(c) **Case 3:** The initial support is in the un-damped region



(d) **Case 4:** There is one zero eigenvalue. $\rightarrow$ Standing wave

Reformulation of the system

As A is symmetric with n real distinct eigenvalues, there exists a matrix $P \in O(n, \mathbb{R})$ such that

$$P^{-1}AP = \Lambda \quad \text{where} \quad \Lambda = \text{diag}(\lambda_1, ..., \lambda_n).$$

Setting $V = P^{-1}U$, the system can be reformulated into

$$\begin{cases} \partial_t V + \Lambda \partial_x V = P^{-1}BPV\mathbf{1}_\omega(x), & (t,x) \in (0,\infty) \times \mathbb{R}, \\ V(0,x) = V_0(x), & x \in \mathbb{R}, \end{cases} \tag{2}$$

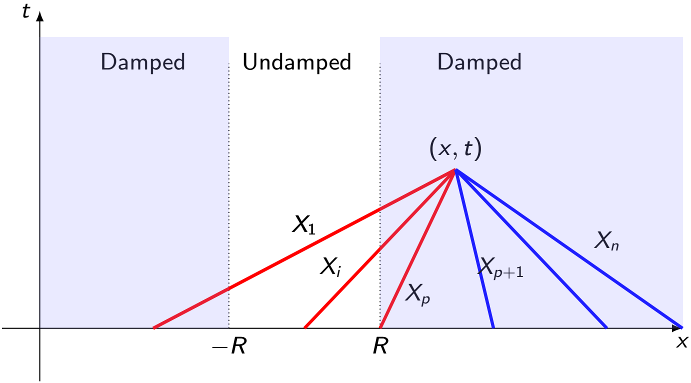
Decomposing $V = (v_1, \dots, v_n)$, (2) is equivalent to the following system of coupled transport equations:

$$\begin{cases} \partial_t v_1 + \lambda_1 \partial_x v_1 & = \sum_{j=1}^n b_{1,j} v_j \, \mathbf{1}_\omega(x) \\ & \vdots \\ \partial_t v_n + \lambda_n \partial_x v_n & = \sum_{j=1}^n b_{n,j} v_j \, \mathbf{1}_\omega(x) \end{cases}$$

For all $1 \leq i \leq n$, the characteristic lines X_i of each equations passing through the point $(x_0, t_0) \in \mathbb{R} \times [0, T]$ are given by

$$X_i\,(t, x_0, t_0) := \lambda_i(t - t_0) + x_0, \quad t \in [0, T].$$

Figure: Characteristics passing through a point $(x, t) \in \mathbb{R} \times \mathbb{R}_+$.



1. The total time spend by all the characteristics in the undamped region is *finite*.
2. Whenever *one* of the characteristic is in the undamped region, then the solution does not, in general, undergo any decay.

Main Theorem

Theorem (De Nitti-Zuazua-CB '22)

Assume that the matrix A is symmetric, strictly hyperbolic and does not admit the eigenvalue 0 and that the couple (A, B) satisfies the (SK) condition.
Let $U_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.
Then, there exists a constant $C > 0$ and a finite time $\bar{\tau} > 0$ such that for $t \geq \bar{\tau}$, the solution satisfies

$$\|U^h(\cdot, t)\|_{L^2(\mathbb{R})} \leq Ce^{-\gamma(t-\bar{\tau})}\|U_0\|_{L^2(\mathbb{R})},$$
$$\|U^\ell(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq C(t-\bar{\tau})^{-1/2}\|U_0\|_{L^1(\mathbb{R})}$$

where

$$\bar{\tau} = \max\left(\sum_{i=1}^{p} \frac{2R}{|\lambda_i|}, \sum_{i=p+1}^{n} \frac{2R}{|\lambda_i|}\right).$$

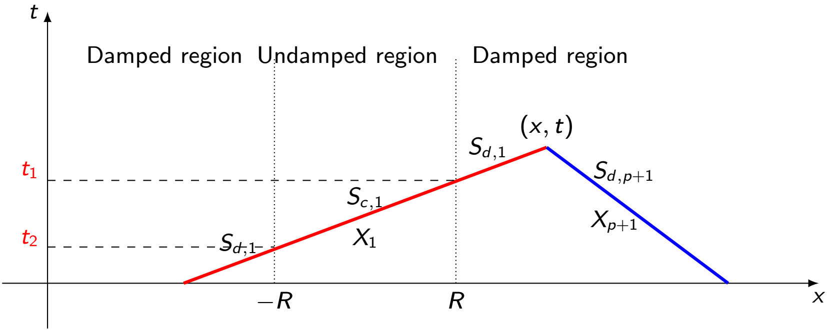
The decay estimates are delayed by the time each characteristic spend in the undamped region

Idea of proof

For every $(x, t) \in \mathbb{R}^2$, there exists suitable times t_1, t_2 such that

$$v_i(x,t) = S_{d,i}(t)S_{c,i}(t_1)S_{d,i}(t_2)v_{i,0}(x),$$

- S_d: **dissipative semigroup** associated to the equation with $\omega = \mathbb{R}$;
- S_c: **conservative semigroup** associated to the equation with $\omega = \emptyset$.



Difficulty: t_1 and t_2 depend on the point (x, t). **But** $t_1 - t_2$ is unif bounded!

Other remarks/difficulties

Difficulties due to partial dissipation:

- It is only possible to obtain dissipation for the solution V if all the semigroups $S_{d,i}$ are active on a same time-interval i.e. the "full" semigroup $S_d = (S_{d,1}, \cdots, S_{d,p})$ needs to be active.
- For instance, the action of $S_{d,1}$ on the first component does not, in general, imply any time-decay properties for the component v_1.
- This means that if one of the conservative semigroups $S_{c,i}$ is active on a time-interval then the whole solution does not experience any decay on this time-interval;

Recalling that $\forall i \in [1, p]$

$$v_i(x,t) = S_{d,i}(t) S_{c,i}(t_1) S_{d,i}(t_2) v_{i,0}(x).$$

With the previous considerations one ends up studying:

$$\mathcal{I}(x,t) = \bigcup_{i=1}^{p} [t_{1,i}(x,t), t_{2,i}(x,t)] \tag{3}$$

which corresponds to the union of time-interval where the dissipation is not active. $\rightarrow |\mathcal{I}|$ quantifies the delay.

And, essentially, our theorem derives from

$$\sup_{x \geq R, t > 0} |\mathcal{I}(x,t)| \leq \sum_{i=1}^{p} \frac{2R}{|\lambda_i|} = \bar{\tau}. \tag{4}$$

And computations of the following type:

$$\begin{aligned} \|v_1(.,t)\|_{L^2(\mathbb{R})} &= \|S_{d,1}(t) S_{c,1}(t_1) S_{d,1}(t_2) v_{1,0}\|_{L^2(\mathbb{R})} \\ &\leq e^{-c(t-t_1)} \|S_{c,1}(t_1) S_{d,1}(t_2) v_{1,0}\|_{L^2} \\ &\leq e^{-c(t-t_1)} \|S_{d,1}(t_2) v_{1,0}\|_{L^2} \\ &\leq e^{-c(t-t_1)} e^{-c(t_2-0)} \|v_{1,0}\|_{L^2} \\ &\leq e^{-c(t-(t_1-t_2))} \|v_{1,0}\|_{L^2} \end{aligned}$$

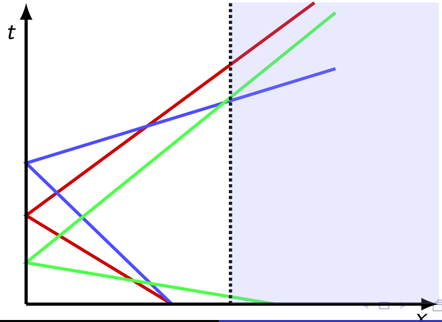
Conclusion

For general hyperbolic system verifying the (SK) condition:

- Exhibiting a damped mode in low frequencies and using a suitable framework leads to new results concerning relaxation limit of such systems.

For locally damped linear hyperbolic systems, our characteristic argument yields:

- Accurate quantification of the delay appearing in decay rates depending on the size of the undamped region and the eigenvalues of the system.
- Possible generalizations to the half-line and to ω^c being finite union of bounded stripes.



Thank you for your attention!

T. Crin-Barat, R. Danchin, Global existence for partially dissipative hyperbolic systems in the L^p framework, and relaxation limit, Mathematische Annalen (2022).

T. Crin-Barat, N. De Nitti, and E. Zuazua. On the decay of one-dimensional locally and partially dissipated hyperbolic systems. *Submitted*, 2022. ArXiv:2206.00555.