Friedrich-Alexander-Universität
Naturwissenschaftliche Fakultät

Probabilistic turnpike properties for optimal control problems

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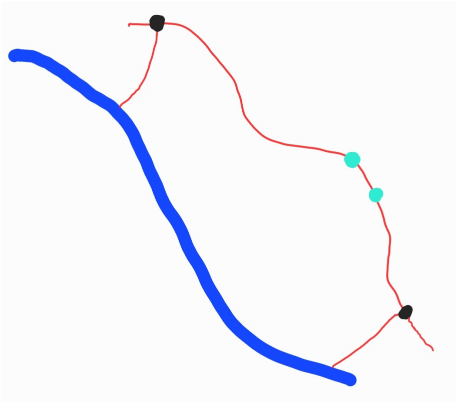
Department of Data Science

What is the turnpike property?

There is a fastest route between any two points;
and if the origin and destination
are close together and far from the turnpike,
the best route may not touch the turnpike.

But if origin and destination are far enough apart,
it will always pay to get on to the turnpike
and cover distance at the best rate of travel,
even if this means adding a little mileage at either end.
(LW MCKENZIE (1986))



If all the time-derivatives are set to zero
and initial conditions and terminal conditions are canceled,
this yields a **static optimal control problem**.

Turnpike results state relations between the **static** optimal state/control and the **dynamic** optimal states/controls.

Typically for large time intervals, close to its middle the dynamic optimal states/controls are **close** to the static optimal state/control. .

Transregio 154

Projects C03 and C05

This research was funded by the DFG through the **SFB Transregio 154**
Mathematical modelling, simulation and optimization using the example of gas networks.

C03:
Nodal control and the turnpike phenomenon

- Rüdiger Schultz
  Universität Duisburg–Essen (UDE)
- Michael Schuster (FAU)
- Martin Gugat

C05:
Observer-based data assimilation for time dependent flows on gas networks

- Jan Giesselmann (TUD)
- Teresa Kunkel (TUD)
- Martin Gugat

History of the DFG (from dfg.de)

- The history of the DFG extends back over almost 100 years.
- Its predecessor was the *Notgemeinschaft der Deutschen Wissenschaft*.
- From the very beginning, its work has focused decisively on funding and supporting excellent research.
- The *Notgemeinschaft der Deutschen Wissenschaft* (association for mutual assistance founded in an emergency) was founded in 1920 *'in order to avert a complete collapse of German science and scholarship'*.

Notgemeinschaft committee 1924 with Fritz Haber (front left) and Max Planck (centre)
(Copyright aus Zierold, Forschungsförderung in drei Epochen, 1968)

Control of linear time-discrete systems with random perturbations

Given some initial state $l_0 \in \mathbb{R}^n$, consider the evolution of state l:

$$l_t = A l_{t-1} + B x_t + \xi_t \quad (t = 1, \ldots, T)$$

Here, l_t (state), x_t (control), ξ_t (random vector) $\in \mathbb{R}^n$, A symmetric and positive definite, B regular.

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- We fix a *desired region* $F \subseteq \mathbb{R}^n$ (closed convex) and a *desired expected terminal state* $l^{(\delta)} \in F$
  E.g., filling level constraints $l_t \in [l_*, l^*] =: F$

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- As the state is random, we impose the following constraints:

$$\underbrace{\mathbb{P}(l_t \in F \quad \forall t = 1, \ldots, T)}_{\varphi(x)} \geq p \quad \text{probabilistic constraint}$$

$$\mathbb{E}l_T = l^{(\delta)} \quad \text{expected-value constraint}$$

Optimal control problem

For some weight $\gamma > 0$ we define an objective (control cost + tracking term for states):

$$J(x) := \sum_{t=0}^{T} \|\mathbb{E}l_t - l^{(\delta)}\|^2 + \gamma \sum_{t=1}^{T} \|Bx_t - Bx^{(\delta)}\|^2,$$

where $x^{(\delta)}$ is the uniquely defined 'equlibrium-control' keeping the state as desired in expectation, hence

$$l^{(\delta)} = Al^{(\delta)} + Bx^{(\delta)} + E.$$

The optimal control problem with a probabilistic constraint is

$$(P(l^0)) \qquad \min_x \{J(x) \mid \varphi(x) \geq p,\ l_0 = l^0,\ \mathbb{E}l_T = l^{(\delta)}\}$$

Standing assumption: Existence of a *'Slater point'* x_S : $\quad \varphi(x_S) > p,\ \mathbb{E}l_T = l^{(\delta)}$.

Proposition

Problem $(P(l_0))$ has a solution.

Proof: $\varphi(x) = \mathbb{P}(\max_{t=1,\dots,T} \underbrace{\operatorname{dist}(l_t(x,\xi), F)}_{\text{continuous}} \leq 0)$ is upper semicontinuous, hence the feasible set of $(P(l_0))$ is closed and nonempty (contains x_S). J is coercive.

Turnpike result for small probabilities

For small probability levels p, the probabilistic constraint is likely to be **nonactive** at a solution of

$$(P(l^0)) \qquad \min_x \{J(x)\ | \varphi(x) \geq p,\ l_0 = l^0,\ \mathbb{E} l_T = l^{(\delta)}\}$$

(e.g., completely redundant for $p = 0$).

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Theorem (Turnpike for inactive probabilistic constraint)

Assume that $\varphi(x^*) > p$ for a solution of $(P(l_0))$.
Then, such solution is unique and there exists a number $z_\gamma \in (0, 1)$ that is ***independent*** *of l_0 and T such that*

$$\|\mathbb{E}(l_t) - l^{(\delta)}\|^2 \leq z_\gamma^t \, \|l_0 - l^{(\delta)}\|^2 \text{ for all } t \in \{1, ..., T\}.$$

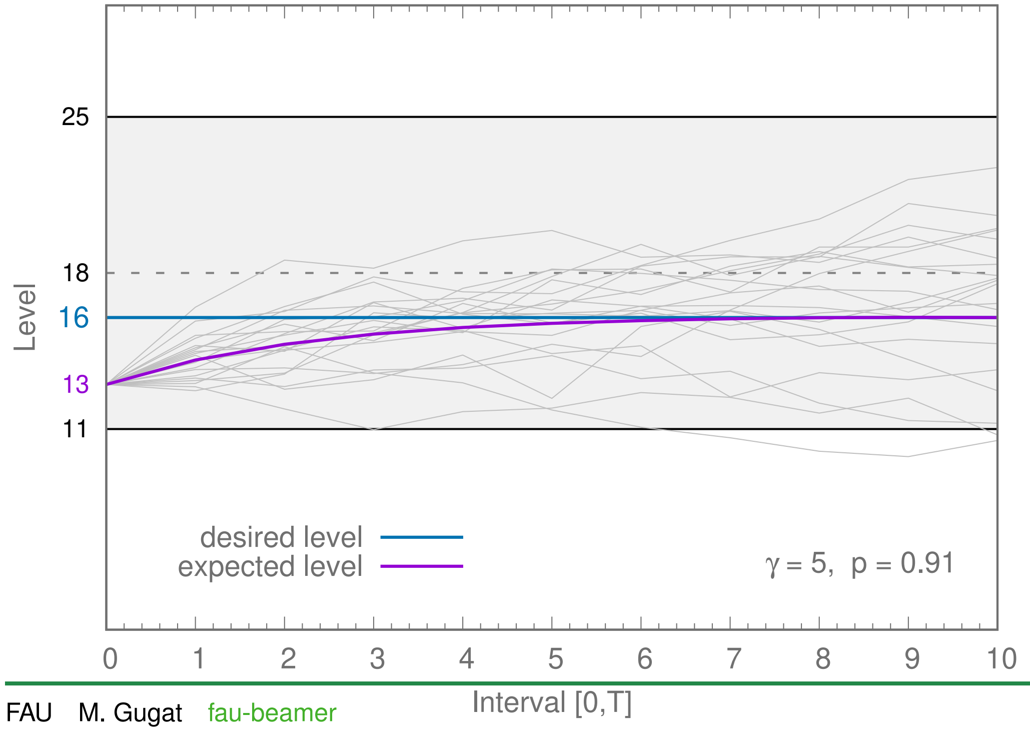
One may explicitly choose: $z_\gamma = \max_{k \in \{1,\ldots,n\}} \; \min_{z \in \mathbb{C}: p_k(z)=0} |z|^2,$

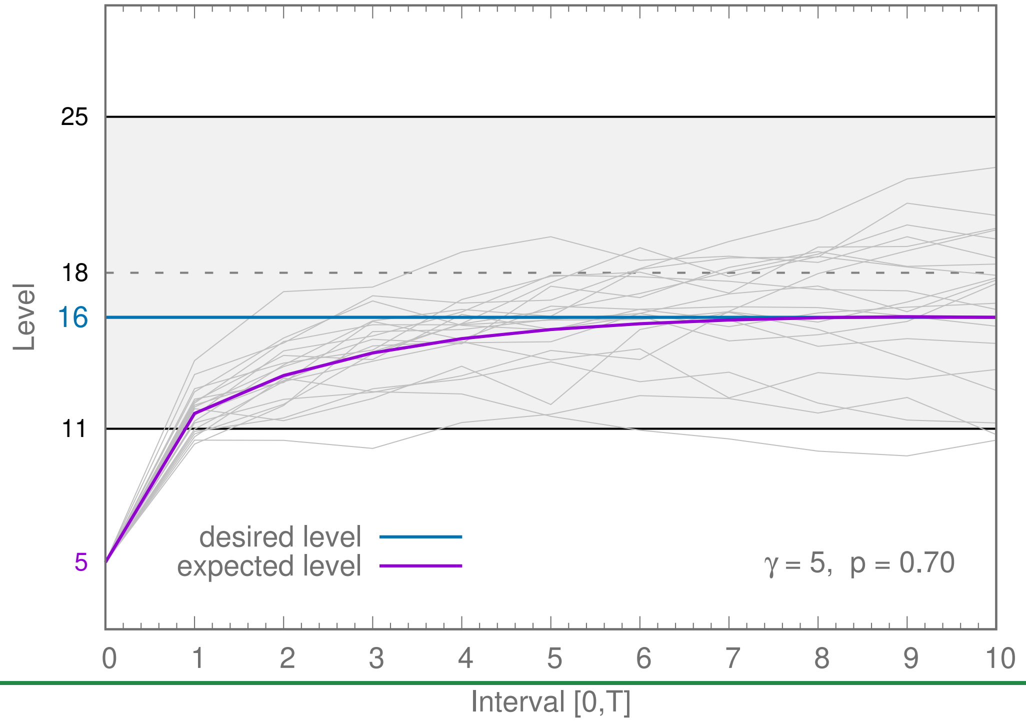
where - with the eigenvalues λ_k of the matrix A , the polynomials p_k are defined as

$$p_k(\omega) = \omega^2 - \left[\frac{1}{\lambda_k}\left(1 + \frac{1}{\gamma}\right) + \lambda_k\right] \omega + 1.$$

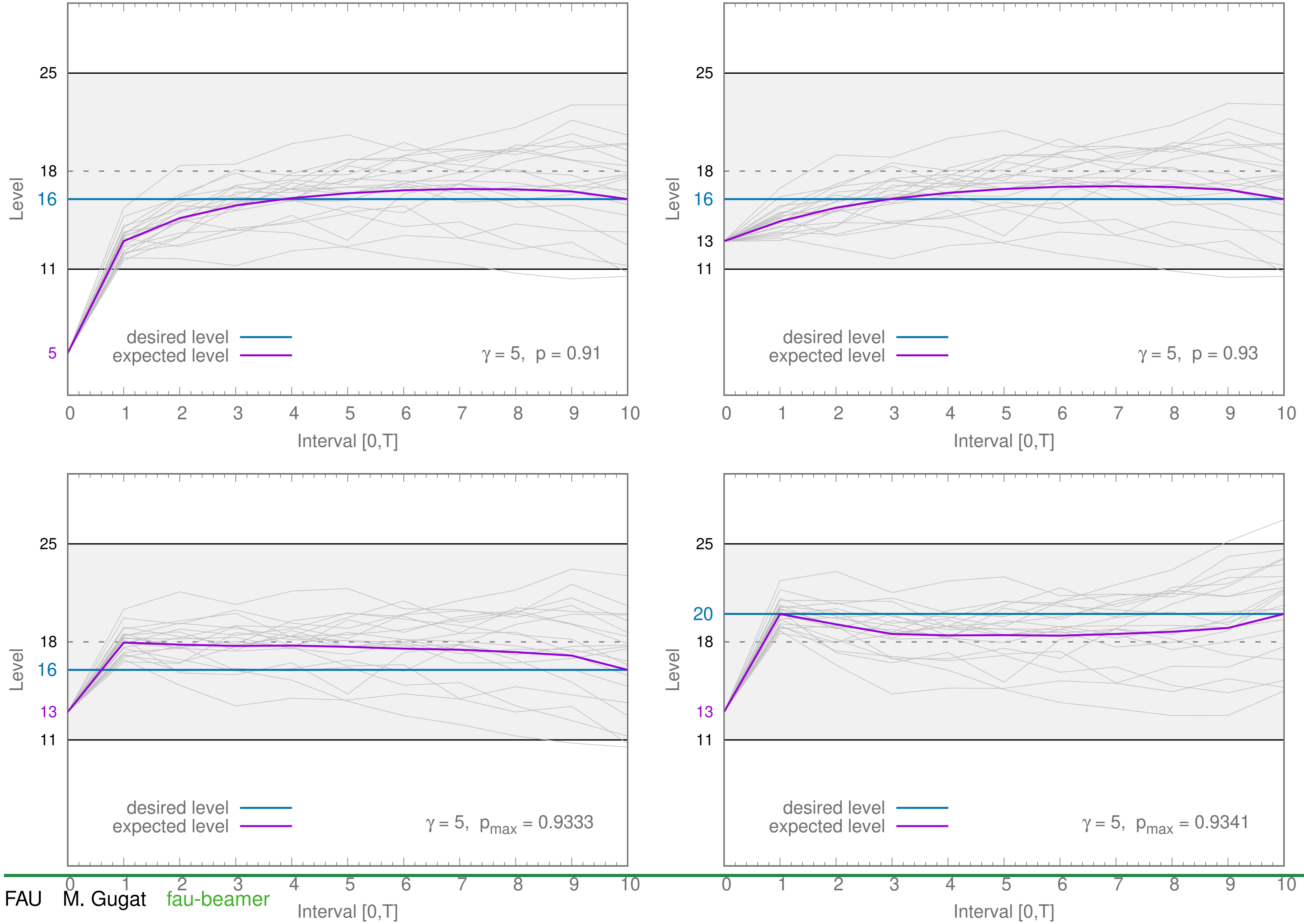
Numerical Illustration (small probabilities)

Desired region:	F	$=$	$[11, 25]$
Initial level:	l_0	$=$	$5,\ 13$
Desired level:	$l^{(\delta)}$	$=$	16
Time horizon:	T	$=$	10
Control cost factor:	γ	$=$	5
Distribution of inflow t:	ξ_t	$\sim_{i.i.d.}$	$\mathcal{N}(E, 1);\ E = -1$





Numerical Illustration (large probabilities up to $p^{\max}$)



Penalizing the probabilistic constraint

For high probability levels, the probabilistic constraint is likely to be binding and cannot be omitted from our problem

$$(P(l^0)) \qquad \min_x \{J(x) \mid \varphi(x) \geq p,\ l_0 = l^0,\ \mathbb{E} l_T = l^{(\delta)}\}.$$

It is advantageous to equivalently reformulate $(P(l^0))$ as problem where the probabilistic constraint is removed and placed as a penalty term. For $\lambda \in [0, 1]$, consider the problem

$$(Q(l^0, \lambda)) \qquad \min_x \{\lambda\, J(x) - (1 - \lambda) \ln(\varphi(x)) \mid l_0 = l^0, \mathbb{E} l_T = l^{(\delta)}\}$$

Lemma

If ξ has a log-concave density (e.g., Gaussian and may others), then for each $\lambda \in [0, 1]$, $(Q(l^0, \lambda))$ is a convex optimization problem.

As above, we use the representation $\varphi(x) = \mathbb{P}(\max_{t=1,\ldots,T} \operatorname{dist}(l_t(x, \xi), F) \leq 0)$.

$l(x, \xi)$ *is affine linear and* F *is convex and closed. Hence,*

$$(x, \xi) \mapsto \max_{t=1,\ldots,T} \operatorname{dist}(l_t(x, \xi), F) \leq 0$$

is convex.
Assumption on density of ξ implies that ξ has a log-concave distribution (Prékopa's Theorem).
Both properties yield that φ is a log-concave function (once more Prékopa).

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Penalized equivalent

Consider our original and penalized problems:

$$(P(l^0)) \qquad \min_x \{ J(x) \mid \varphi(x) \geq p,\ l_0 = l^0,\ \mathbb{E} l_T = l^{(\delta)} \}.$$

$$(Q(l^0, \lambda)) \qquad \min_x \{ \lambda\, J(x) - (1 - \lambda) \ln \left(\varphi(x) \right) \mid l_0 = l^0, \mathbb{E} l_T = l^{(\delta)} \}$$

Theorem

Let ξ have a log-concave density and assume that for all $\lambda \in (0, 1]$, the solution $\tilde{x}$ of problem $(Q(l^0, \lambda))$ satisfies $\varphi(\tilde{x}) > 0$. Then, for all $\lambda \in (0,\ 1]$, problem $Q(l^0,\ \lambda)$ has a unique solution and there exists a number $\lambda^* \in (0, 1]$ such that the solution of $Q(l^0,\ \lambda^*)$ is equal to the solution of $P(l^0)$.

Proof by comparison of optimality conditions for convex optimization problems.

Both assumptions of the theorem are automatically fulfilled if ξ is Gaussian and $\mathrm{int}\, F \neq \emptyset$.

Candidate for turnpike solution

Consider our original and the equivalent penalized problems:

$$\min_x \{J(x) \mid \varphi(x) \geq p,\ l_0 = l^0,\ \mathbb{E} l_T = l^{(\delta)}\} \quad (P(l^0)).$$

$$\min_x \{\lambda^*\, J(x) - (1 - \lambda^*) \ln(\varphi(x)) \mid l_0 = l^0, \mathbb{E} l_T = l^{(\delta)}\} \quad (Q(l^0, \lambda^*))$$

Given the penalized form of the problem, a turnpike candidate would result from $(Q(l^0, \lambda^*))$ by leaving the initial state l_0 as well as the expected terminal state $\mathbb{E} l_T$ free:

$$(\hat{Q}) \qquad \min_{x, l_0} \{\lambda^*\, J(x) - (1 - \lambda^*) \ln(\varphi(x))\}$$

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Theorem

Let $\mathbb{E}l_t$ and $\mathbb{E}\hat{l}_t$ be the expected states associated with the optimal solutions of the original problem $(Q(l^0, \lambda^*)) = (P(l^0))$ and the turnpike problem $(\hat{Q})$ In addition to the assumptions of the previous Theorem (e.g., ξ Gaussian), suppose that

$$\varkappa := \mathbb{P}(\hat{l}_T - \mathbb{E}\hat{l}_T + l^{(\delta)} \in F \mid \hat{l}_t \in F\ \forall t \in \{1, ..., T-1\}) > 0$$

(e.g., if ξ Gaussian). Then, one may estimate the difference between the expected states can be estimated by

$$\sum_{t=0}^{T} \|\mathbb{E}l_t - \mathbb{E}\hat{l}_t\|^2 \leq (C(R) + \log \varkappa^{-1})/\lambda^*,$$

where $C(R) = aR^2 + bR + c$ (with a, b, c independent of T), $R := \max_{t=0,...,T} \|\mathbb{E}\hat{l}_t\|$.

What happens when $T \to \infty$?

Previous estimate for fixed T : $\sum_{t=0}^{T} \|\mathbb{E}l_t - \mathbb{E}\hat{l}_t\|^2 \leq (C(R) + \log \varkappa^{-1})/\lambda^*$

Indexation by T : $\sum_{t=0}^{T} \|\mathbb{E}l_t^{(T)} - \mathbb{E}\hat{l}_t^{(T)}\|^2 \leq (C(R_{(T)}) + \log \varkappa_{(T)}^{-1})/\lambda_{(T)}^* \quad (T \in \mathbb{N})$

If the right-hand side could be uniformly bouded by some K, then one would have a turnpike result

$$T^{-1} \sum_{t=0}^{T} \|\mathbb{E}l_t^{(T)} - \mathbb{E}\hat{l}_t^{(T)}\|^2 \leq T^{-1} K \to_{T\to\infty} 0$$

Uniformly bounding from above $C(R_{(T)}),\ \varkappa_{(T)}^{-1},\ 1/\lambda_{(T)}^*$?

1. $\inf_{T\in\mathbb{N}} \varkappa_{(T)} = \inf_{T\in\mathbb{N}} \mathbb{P}(\hat{l}_T - \mathbb{E}\hat{l}_T + l^{(\delta)} \in F \mid \hat{l}_t \in F\ \forall t \in \{1, ..., T-1\}) > 0$
2. $\|\mathbb{E}\hat{l}_t^{(T)}\| \leq R^* < \infty \quad \forall T \in \{0, \ldots, T\}\ \forall T \in \mathbb{N}$
3. $1/\lambda_{(T)}^*$?

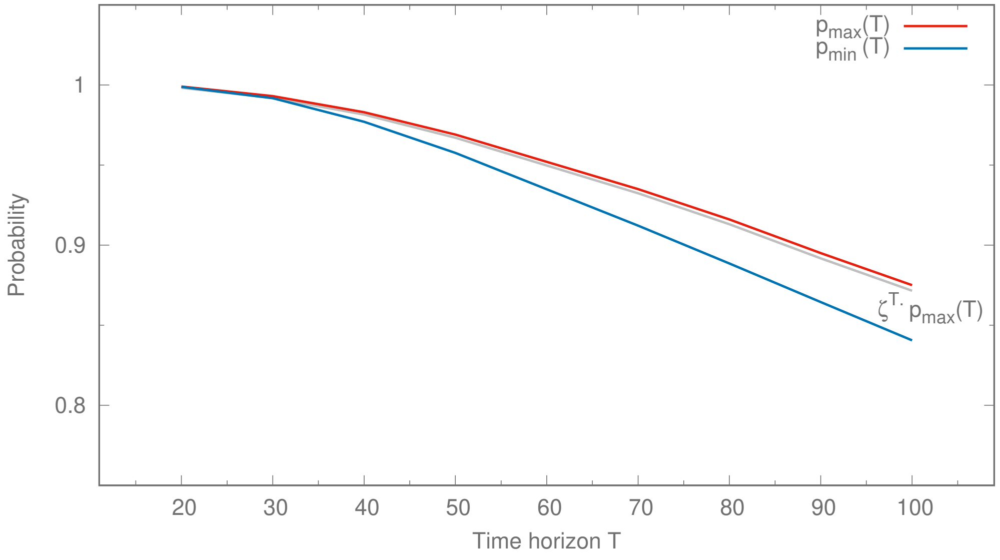
Boundedness away from zero of $1/\lambda^*_{(T)}$

Lemma

In the probabilistic constraint $\varphi(x) \geq p_{(T)}$ *let* $p_{(T)} = \zeta^T p^{\max}_{(T)}$ *for some* $\zeta \in (0,1)$ *and assume that*

$$\|\mathbb{E}l_t\| \leq \tilde{R} < \infty \quad \forall T \in \{0, \ldots, T\} \ \forall T \in \mathbb{N}.$$

Then, there exists some $C^* > 0$ *such that* $\lambda^*_{(T)} > C^*$ *for all* $T \in \mathbb{N}$.



The solution $x^*_{(T)}$ of '$p^{\max}$'-problem

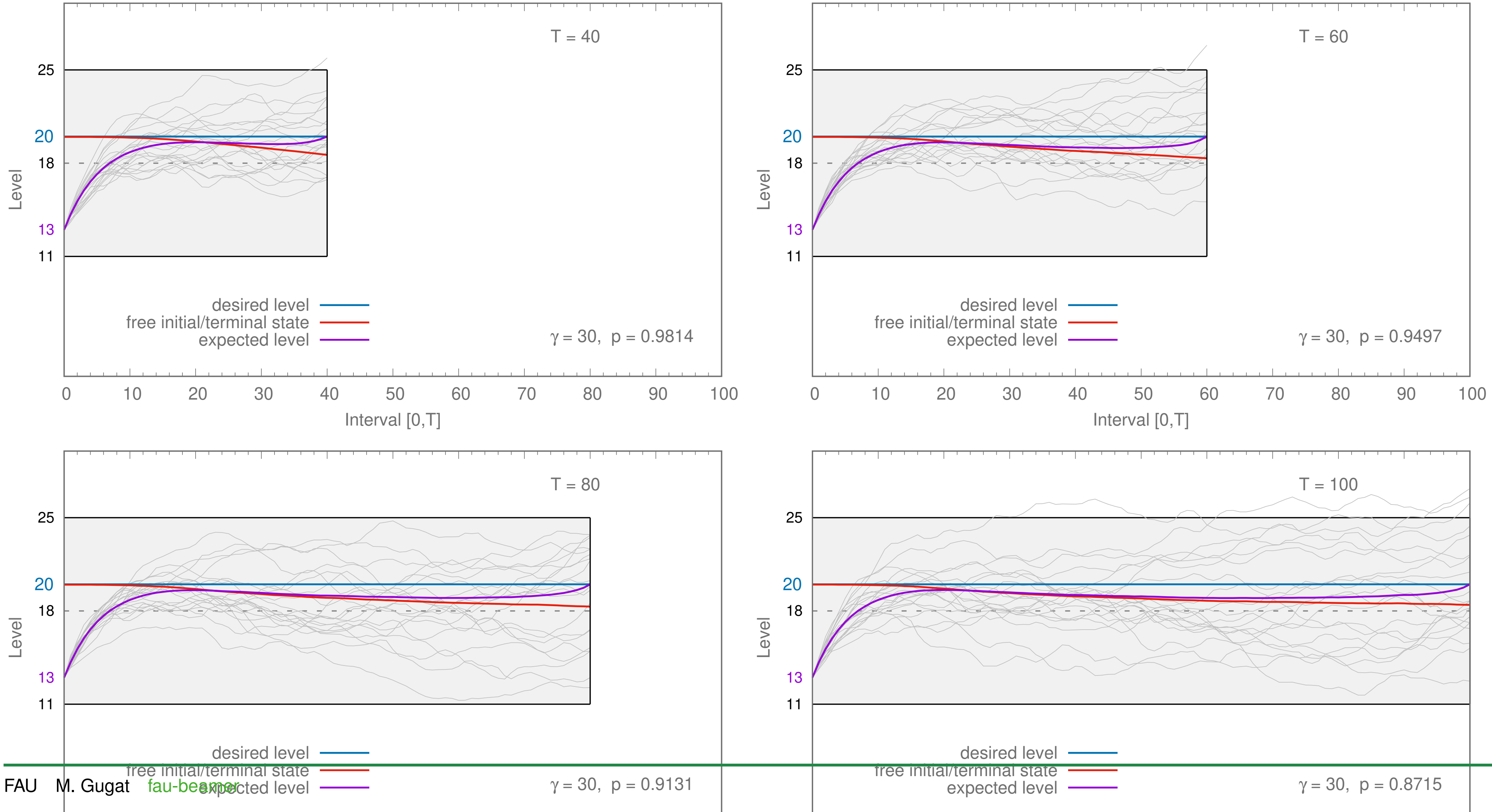
$$\max\{\varphi(x) \mid l_0 = l^0, \ \mathbb{E}l^{(T)}_T = l^{(\delta)}\}$$

is a *Slater point* of the given problem:

$$\varphi(x^*_{(T)}) = p^{\max} > p_{(T)}.$$

Numerical computations were based on the *spheric-radial decomposition* of Gaussian (more generally: elliptical) random distributions in order to compute values and gradients (w.r.t. control) of the probability function φ. See results by Wim v. Ackooij.

Numerical Illustration (large probabilities)



Open questions

We have established a turnpike result under the assumptions

1. $\inf_{T\in\mathbb{N}} \mathbb{P}(\hat{l}_T - \mathbb{E}\hat{l}_T + l^{(\delta)} \in F \mid \hat{l}_t \in F \ \forall t \in \{1, ..., T-1\}) > 0$
2. $\|\mathbb{E}\hat{l}_t^{(T)}\|, \|\mathbb{E}l_t\| \leq \hat{R} < \infty \quad \forall T \in \{0, \ldots, T\} \ \forall T \in \mathbb{N}$

While the meaning of 2. is clear, 1. is purely technical. How to guarantee both conditions in terms of the problem data?

1. can be probably guaranteed by a uniformity condition on the variances of ξ_t, e.g., $Var(\xi_t) \leq c$ for all $t \in \mathbb{N}$.

2. seems intuitively obvious, but it is not. Conditions found so far are unrealistic (e.g., independence of random states).

Conclusions

- The **uncertainty** of the data is **relevant** to obtain controls that work sufficiently well in the set of data that is expected.
- If information on the **probability distribution** of the data is available, it should be used in an optimal control model.
- We include the **probability** that state constraints are satisfied as a part of the **objective function**.
  In this way, the optimal controls are **robust** in the sense that the pressure bounds are satisfied with a high probability.
- An approach to the numerical solution that will be presented by *Michael Schuster* uses **a kernel density estimator** to obtain a differentiable approximation of the objective functional, similar as in [Sch+21].

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Thank you for your attention!

[Sch+21] M. Schuster, E. Strauch, M. Gugat, and J. Lang. "Probabilistic Constrained Optimization on Flow Networks". In: *Optimization and Engineering* (2021). DOI: 10.1007/s11081-021-09619-x.