IX Partial differential equations, optimal design and numerics
Benasque 2022, Aug 21 – Sep 02

Optimal design versus maximal Monge-Kantorovich metrics.

joined work with:

Karol Bolbotowski, Faculty of Civil Engineering, Warsaw University of Technology

Motivations

- Optimal design in mechanical engineering:
  - pre-stessed elastic membrane ($\sigma : \Omega \subset \mathbb{R}^2 \mapsto \mathcal{S}_+^{2\times 2}$)
  - optimal vault problem (parametrized surface $z = u(x, y)$)
  - Prager problem ($\sigma : \Omega \times (-h, h) \subset \mathbb{R}^3 \mapsto \mathcal{S}^{3\times 3}$)
- Mathematics:
  - GMT approach for lower dimensional structures
  - Connexion with Monge optimal transport
  - Optimal metrics and related geodesics

- Karol Bolbotowski: Optimal vault problem – form finding through 2D convex program, *Comp. and Math. with Appli., (2022)*

- K. Bolbotwski, GB: Optimal design versus maximal Monge-Kantorovich metrics, *Arch. Ration. Mech. Anal. 243 (2022), no. 3, 1449-1524.*

Plan of the talk

I- The classical optimal compliance problem (Euclidean metric)

II- From free material design (FMD) to optimal pre-stressed membrane Pb (OM)

III- Duality and PDE approach (smooth case)

IV- The geometric OT approach (via maximal monotone maps)

V- Two-point scheme and truss-like solutions (conjecture and numerics)

I- The classical optimal compliance problem

Scalar setting

- $\Omega \subset \mathbb{R}^d$ convex bounded design domain ($d = 2, 3$)
- $f \in \mathcal{M}(\Omega)$ a scalar measure (load or source term)
- $a : \Omega \to \mathbb{R}_+$ a conductivity (or stiffness) coefficient subject to $\int_\Omega a\,dx \le m$ (design variable).
- $u : \Omega \to \mathbb{R}$ solving the state equation

$$-\operatorname{div} q = f \quad , \quad q = a \nabla u \quad \text{in } \Omega \quad , \quad u = 0 \quad \text{in } \partial\Omega \qquad (*)$$

- $\mathcal{E}_{\Omega,f}(a) := \frac{1}{2} \int f u$ is the compliance (convex functional of a)

Then we want to solve for given $m > 0$:

$$\mathcal{I}(m) = \inf \left\{ \mathcal{E}_{\Omega,f}(a) \ : \ \int_\Omega a \le m \right\} \qquad \text{(MOP)}$$

Optimality and relaxation issues

A smooth pair (a, u) solving state equation (*) is optimal if

$$|\nabla u| \leq C \quad \text{a.e. in } \Omega \,, \; |\nabla u| = C \quad \text{on } \{a > 0\} \qquad (**)$$

However:

- Existence ? minimizing (a_n) may concentrate (no L^∞ bound)
- Case of discrete loads ? $(f \notin H^{-1}(\Omega))$

Existence is obtained by passing from smooth $a(x)$ to measures $\mu \in \mathcal{M}_+(\overline{\Omega})$ while defining

$$\mathcal{E}_{\Omega,f}(\mu) := \sup\left\{ \int fv - \frac{1}{2}\int_\Omega |\nabla v|^2 \, d\mu \ : \ v \in \mathcal{D}(\Omega) \right\}.$$

Second step: rewrite eikonal eq. (**) for optimal μ.

Tangential calculus with respect to a measure

Back to years 1995: if u is Lipschitz, then $\nabla_\mu u$ can be defined in L^∞_μ so that $\nabla_\mu u \in T_\mu(x), \mu$-a.e. (tangent space to μ) and $\nabla_\mu u = \mathrm{Proj}_{T_\mu(x)}(\nabla u)$ for smooth u.
[G.Buttazzo, P. Seppecher, GB: COCV (1995)], [T.Champion, C. Jimenez,GB (2005)], [I.Fragala, GB: JFA (2006)] for $\nabla^2_\mu u$

THM: For every measure $f \in \mathcal{M}(\Omega)$, $\exists$ optimal (μ, u, q) in $\mathcal{M}_+(\Omega) \times \mathrm{Lip}(\Omega) \times \mathcal{M}(\Omega, \mathbb{R}^d)$ such that

$$q = (\nabla_\mu u)\,\mu\ ,\ -\operatorname{div} q = f \quad \text{in } \mathcal{D}'(\Omega),\ u = 0 \text{ in } \partial\Omega \qquad (*)$$

$$|\nabla u| \le 1 \quad \text{a.e. in } \Omega\ ,\ |\nabla_\mu u| = 1 \quad \mu\text{-a.e.} \qquad (**)$$

Connexion with Monge optimal transport

[G.Buttazzo, P. Seppecher, GB: CRAS (1997)] [G.Buttazzo, GB: JEMS (2001)]

Let $\mu, \nu \in \mathcal{M}_+(\overline{\Omega})$ such that $\mu(\overline{\Omega}) = \nu(\overline{\Omega})$; Then the Monge distance is given by

$$W_1(\mu,\nu) := \min \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} \overbrace{|x-y|}^{\text{euclidean distance}} \gamma(dxdy) \ : \ \gamma \in \Gamma(\mu,\nu) \right\}$$

(Kantorovich relaxation of $\inf\{\int_{\overline{\Omega}} |x - Tx| \, \mu(dx) : T^\sharp(\mu) = \nu\}$).

We set $W_1(\mu,\nu) = +\infty$ if $\mu(\overline{\Omega}) \neq \nu(\overline{\Omega})$

WHY the euclidean metric ?

$$|\nabla u| \leq C \quad \text{a.e. in } \Omega \quad \Longleftrightarrow \quad |u(x)-u(y)| \leq |x-y| \quad \forall (x,y) \in \Omega^2$$

Key duality identities

We consider the linear programming problem:

$$\mathcal{I}_0(f,\Omega) := \sup\left\{\langle f,u\rangle \ : \ u \in \mathrm{Lip}_1(\Omega),\ u = 0 \text{ in } \partial\Omega\right\}.$$

where $f \geq 0$. The Monge distance from f to $\partial\Omega$ defined by

$$W_1(f,\partial\Omega) = \min\{W_1(f,g) : g \in \mathcal{M}_+(\partial\Omega)\}$$

.

THM The following equalities hold

(i) $\mathcal{I}_0(f,\Omega) = W_1(f,\partial\Omega)$ (Kantorovich-Rubinstein duality)

(ii) $\mathcal{I}(m) = \dfrac{(\mathcal{I}_0(f,\Omega))^2}{2m}$ (inf sup = sup inf argument)

Remark: extension to signed measure f by setting:

$$W_1(f,\partial\Omega) = \min\left\{W_1(f_+ + \mu, f_- + \nu) : \mu,\nu \in \mathcal{M}_+(\partial\Omega)\right\}$$

($\exists$ optimal μ,ν s.t. $\int \mu \leq \int f_-$ and $\int \nu \leq \int f_+$)

Recovering optimal u, γ, g, q, μ

Assume that $f \geq 0$ and denote for every $x \in \Omega$:

$$p_{\partial\Omega}(x) = \{y \in \partial\Omega : |x - y| = d(x, \partial\Omega)\}.$$

Then:

- $u(x) = d(x, \partial\Omega)$ optimal for $\mathcal{I}_0(f, \Omega)$ (visco. sol. of $|\nabla u| = 1$).
- Let $\{\gamma^x\}$ be a family in $\mathcal{P}(\mathbb{R}^d)$ and γ given by

$$\langle \gamma, \varphi \rangle = \int_\Omega \left(\int_{\partial\Omega} \varphi(x, y)\, \gamma^x(dy) \right) f(dx) \quad (\forall \varphi : \overline{\Omega}^2 \to \mathbb{R}).$$

Then: γ optimal $\iff$ $\mathrm{spt}(\gamma^x) \subset p_{\partial\Omega}(x)$.

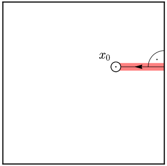
- For such γ, we get optimal μ and q in sliced form:

$$\mu = \int_\Omega \mathcal{H}^1 \llcorner [x, y]\, \gamma(dxdy) \quad , \quad q = -\iint \lambda^{x,y}\, \gamma(dxdy) \ ,$$

where the vector measure $\lambda^{x,y} = \mathcal{H}^1 \llcorner [x, y]\, \frac{y-x}{|y-x|}$.

Example with one Dirac mass

Take $f = \delta_{x_0}$ and Ω a square domain.



Then:

- $p_{\partial\Omega}(x_0)$ is a singleton $\{y_0\}$;
- unique optimal $\gamma = \delta_{x_0} \otimes \delta_{y_0}$;
- optimal $\mu = \mathcal{H}_1 \llcorner [x_0, y_0]$;
- optimal flux $q = -\binom{1}{0}\,\mu$ $\;(\,\nabla u = -\binom{1}{0}$ on $[x_0, y_0])$
- Uniqueness of μ.

Ex: what happens if x_0 is moved horizontally up to the first diagonal? then to the center of the square ?

II- From free material design problem to optimal membrane problem

The free material design Problem (FMD)

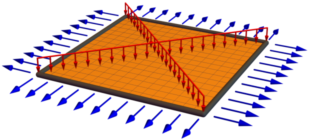
An anisotropic variant of (MOP) is obtained by enlarging the set $\{\mu \in \mathcal{M}_+(\overline{\Omega}) : \int \mu \leq m\}$ to positive tensors $\sigma \in \mathcal{S}_+^{d\times d}(\overline{\Omega})$ (new unknown) to which we associate the compliance

$$\mathcal{C}_{\Omega,f}(\sigma) := \sup \left\{ \int fv - \frac{1}{2} \int_\Omega \langle \sigma, \nabla v \otimes \nabla v \rangle \ : \ v \in \mathcal{D}(\Omega) \right\}.$$

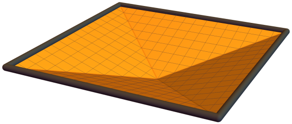
to be minimized over $\left\{ \sigma \in \mathcal{S}_+^{d\times d}(\overline{\Omega}) : \int \frac{1}{d} \mathrm{Tr} \sigma \leq m \right\}$.

- $\mathcal{C}_{\Omega,f}(\mathrm{Id}\, \mu) = \mathcal{E}_{\Omega,f}(\mu) \implies \tilde{\mathcal{I}}(m) := \inf(\mathrm{FMD}) \ \leq \ \mathcal{I}(m)$
- Optimal σ for (FMD) are shown to be rank-one . Thus $\mathcal{I}(m) = d\, \tilde{\mathcal{I}}(m)$ [Bolbotowski-Lewinski (COCV-2022)]

The pre-stressed membrane model



pre-stressed membrane



after loading

- $\Omega \subset \mathbb{R}^2$ stands for an horizontal domain
- The design $\sigma \in \mathcal{M}(\overline{\Omega}; S^{2\times 2})$ is the in plane stress of a thin membrane placed in Ω subject to:
  - a horizontal pre-load on the boundary (job of the designer)
  - a vertical pressure $f \in \mathcal{M}(\overline{\Omega})$
  - u represents the deflection of the membrane (pinned vertically on $\partial\Omega$).

The equilibrium of such a membrane requires two conditions:

- $\mathrm{Div}\sigma = 0$ in $\mathcal{D}'(\Omega)$ (in plane load supported in $\partial\Omega$)
- $\sigma \geq 0$ (membrane in tension only)

It is convenient to write the optimal membrane Pb with a normalized Lagrange multiplier of the trace constraint:

$$Z_0 := \min_{\sigma \in \mathcal{M}(\overline{\Omega}; \mathcal{S}_+^{d\times d})} \left\{ \mathcal{C}_{\Omega,f}(\sigma) + \int \mathrm{Tr}\sigma \ : \ \mathrm{Div}\sigma = 0 \right\} \qquad (OM).$$

Dropping $\mathrm{Div}\sigma = 0$, gives the (FMD) counterpart

$$Z := \min_{\sigma \in \mathcal{M}(\overline{\Omega}; \mathcal{S}_+^{d\times d})} \left\{ \mathcal{C}_{\Omega,f}(\sigma) + \int \mathrm{Tr}\sigma \right\} \leq Z_0$$

Remark: $Z = \tilde{\mathcal{I}}(m)$ for $m = \sqrt{\tilde{\mathcal{I}}(1)}$ (thanks to $\mathcal{C}(t\sigma) = \frac{1}{t}\mathcal{C}(\sigma)$).

Comparison between (FMD) and (OM) problems

We expect that $Z_0 < Z$ since condition $\mathrm{Div}\sigma = 0$ in (OM) rules out many competitors:

- $\sigma = \mathrm{Id}\,\mu$ admissible $\iff$ $\mu = p\,dx$ for a constant $p > 0$
- $\sigma = p(x)\,\tau_C \otimes \tau_C\,\mathcal{H}^1 \llcorner C$ (for a curve C) $\iff$ C is a straight line connecting two points of $\partial\Omega$ and p is constant.
- **Truss like structures**: let $\tau^{x,y} := \frac{y-x}{|y-x|}$, $\gamma \in \mathcal{M}_+(\overline{\Omega} \times \overline{\Omega})$ and set

$$\sigma^\gamma := \iint \sigma^{x,y}\,\gamma(dxdy) \quad (\text{with } \sigma^{x,y} := \tau^{x,y} \otimes \tau^{x,y}\,\mathcal{H}^1 \llcorner [x,y])$$

Then σ^γ admissible

$\iff$ $\iint \langle w(x) - w(y), \tau^{x,y}\rangle\,\gamma(dxdy) = 0 \ \ \forall w \in \mathcal{D}(\Omega;\mathbb{R}^2)$.

A geometrical condition for the equality $Z_0 = Z$

We introduce the the high ridge of Ω defined by

$$M(\Omega) := \{x \in \Omega \,:\, d(x, \partial\Omega) \geq d(z, \partial\Omega) \ , \ \forall z \in \Omega\}$$

THM: Assume $f \geq 0$. Then

$$Z_0 = Z \iff \operatorname{spt} f \subset M(\Omega).$$

Proof:

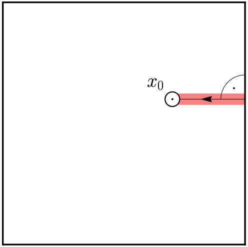
- Any optimal σ for (FMD) is rank one of the form σ^γ being γ an optimal plan for $W_1(f, \partial\Omega)$.
- If $f \geq 0$, $\gamma = \int_\Omega \nu^x(dy)\, f(dx)$ with $\nu^x \in \mathcal{P}(P_{\partial\Omega}(x))$.
- $\mathrm{Div}\sigma^\gamma = 0 \iff \int y\, \nu^x(dy) = x \ , \ \forall x \in \operatorname{spt}(f)$

hence the conclusion since

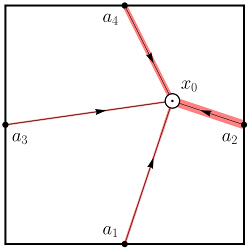
$$M(\Omega) = \{x \in \Omega : x \in \operatorname{co}(P_{\partial\Omega}(x))\}$$

$\square$

Back to the one Dirac mass in a square



(FMD)



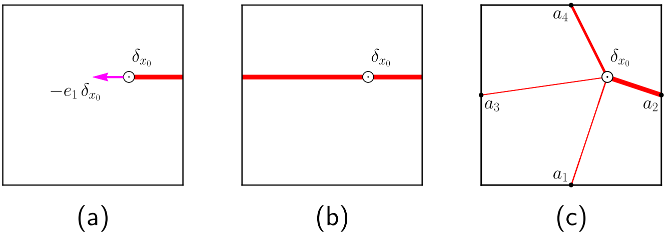
(OM)

(a_i are centres of the square's sides)

Remark: the high ridge of the square reduces to its center.

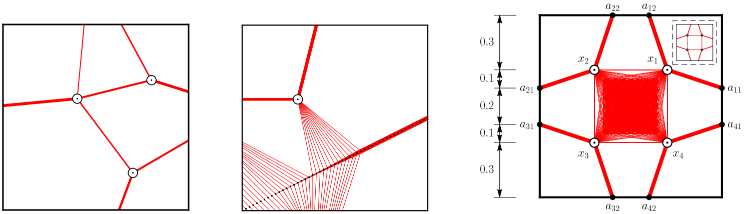
Examples of stress measures σ

$$\sigma = \iint \sigma^{x,y}\, \gamma(dxdy) \quad , \quad \sigma^{x,y} := \tau^{x,y} \otimes \tau^{x,y}\, \mathcal{H}^1 \llcorner [x,y].$$



(a) $\operatorname{div}\sigma \neq 0$; (b) $\operatorname{div}\sigma = 0$ not optimal; (c) optimal

Some (numerical) optimal stress measures



(a) (b) (c)

(a) optimal σ for three asymmetric point forces;
(b) optimal σ for point force and force distributed along a line;
(c) optimal σ for a four points source (alternative solution in the top right corner)

III- Duality and PDE approach

Primal problem

Our duality scheme involves pairs $(q, \sigma) \in \mathcal{M}(\overline{\Omega} : \mathbb{R}^d \times \mathcal{S}_+^{d\times d})$ and $(u, w) \in \mathcal{D}(\Omega; \mathbb{R} \times \mathbb{R}^d)$, noticing that

$$-\operatorname{div} q = f,\ \operatorname{Div}\sigma = 0 \iff \langle q, \nabla u\rangle + \langle \sigma, e(w)\rangle = \langle f, u\rangle \quad \forall (u, w)$$

(sym. gradient $e(w)$ acts as lagrange multiplier of $\operatorname{Div}\sigma = 0$)

The (OM) problem is recast as the primal problem

$$(\mathcal{P}) \qquad \min \Big\{ \underbrace{\int \operatorname{Tr}\sigma + \frac{1}{2}\langle \sigma^{-1} q, q\rangle}_{J(q,\sigma)} \ :\ -\operatorname{div} q = f,\ \operatorname{Div}\sigma = 0 \Big\}$$

($J(q, \sigma) = \int \chi_C^*(q, \sigma)$ is a convex, 1-homogeneous on measures).

Dual problem on Lipschitz funtions

The previous convex C is given by:

$$C := \left\{ (z, M) \in \mathbb{R}^d \times \mathcal{S}^{d\times d} : \frac{1}{2} z \otimes z + M \leq \mathrm{Id} \right\}$$

Accordingly the dual problem reads

$$(\mathcal{P}^*) \qquad \sup_{(u,w)\in \mathrm{Lip}_0(\Omega;\mathbb{R}^{1+d})} \left\{ \langle f, u\rangle \ : \ \frac{1}{2}\nabla u \otimes \nabla u + e(w) \leq \mathrm{Id} \quad \text{a.e.} \right\}$$

Remark: The existence of solutions for $(\mathcal{P}^*)$ is an open issue. A relaxed version will involves pairs $(u, w) \in W^{1,2}(\Omega) \times BV(\Omega; \mathbb{R}^d)$

No gap result and optimality conditions

Theorem

i) $\min(\mathcal{P}) = \sup(\mathcal{P}^*)$

ii) Admissible pairs (q, σ) and (u, w) are optimal iff $J(q, \sigma) = \langle f, u \rangle$.

The equality $J(q, \sigma) = \langle f, u \rangle$ can be localized after extending the duality via *μ-tangential differential calculus*. Let $\mu = \mathrm{Tr}\sigma$ and $S \in L^\infty_\mu(\Omega; \mathcal{S}^{d\times d}_+)$ such that $\sigma = S\,\mu$. Then

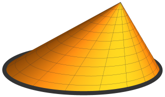
$$J(q, \sigma) < +\infty \implies \exists \theta \in (L^1_\mu)^d \ : \ q = (S\theta)\,\mu.$$

Optimality system

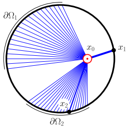
The pairs $(q, S\mu) \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^d \times \mathcal{S}_+^{d\times d})$ and $(u, w) \in \mathrm{Lip}(\Omega; \mathbb{R}^{1+d})$ are optimal for $(\mathcal{P})$ and $(\mathcal{P}^*)$ respectively if and only if all the following conditions are satisfied:

$$
\begin{cases}
(i) & u = 0 \ \text{ on } \partial\Omega, \quad w = 0 \ \text{ on } \partial\Omega, \\
(ii) & -\operatorname{div} q = f \ \text{ in } \mathcal{D}'(\Omega), \quad \mathrm{Div}(S\,\mu) = 0 \ \text{ in } \mathcal{D}'(\Omega), \\
(iii) & \frac{1}{2}\nabla u \otimes \nabla u + e(w) \leq \mathrm{Id} \quad \text{a.e. in } \Omega, \\
(iv) & q = S\,\nabla_\mu u \quad \mu\text{-a.e.}, \\
(v) & \left\langle \frac{1}{2}\nabla_\mu u \otimes \nabla_\mu u + e_\mu(w), S \right\rangle = \mathrm{Tr} S \quad \mu\text{- a.e.}, \\
(vi) & \mu(\partial\Omega) = 0.
\end{cases}
$$

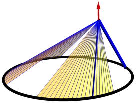
Family of solutions for $\Omega = \{|x| \leq R_0\}$ and $f = \delta_{x_0}$



optimal u



optimal σ



mixed 1d-2d structure

Let $x_0 \in \Omega$ and $d_0 = \sqrt{(R_0^2 - |x_0|^2)}$. Then for every $\nu \in \mathcal{P}(\partial\Omega)$ satisfying $x_0 = \int_{\partial\Omega} x\, \nu(dx)$, we get a solution to $(\mathcal{P})$:

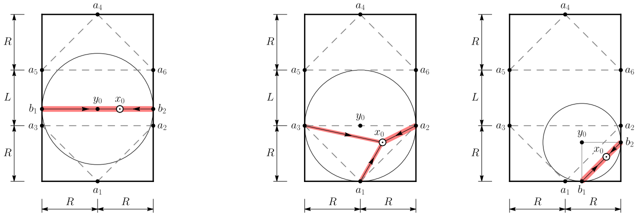
$$q = \int_{\partial\Omega} \lambda^{x,x_0}\, \nu(dx) \quad , \quad \sigma = \frac{1}{d_0} \int_{\partial\Omega} |x - x_0|\, \sigma^{x,x_0}\, \nu(dx)$$

while a Lipschitz solution (u, w) to $(\mathcal{P}^*)$ is given by

$$u(x) = \sqrt{2} d_0\ h(x) \quad , \quad w(x) = 2\, x_0\ h(x)\ ,$$

the graph of h is the cone of vertex $(x_0, 1)$ with basis $\partial\Omega \times \{0\}$.

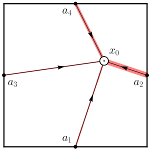
Solutions for Ω a rectangle and $f = \delta_{x_0}$



(a) (b) (c)

Only 3 cases (by symmetry): for each y_0 denotes the center of the disk passing through $\Sigma_0 := p_{\partial\Omega}(y_0)$ (2 or 3 points) ; the optimal (q, σ) in red stems from the OT plan $\gamma = \delta_{\mathbf{x_0}} \otimes \nu$ where $\nu \in \mathcal{P}(\Sigma_0)$ has barycenter x_0.

If $L = 0$, y_0 is the center of the square Ω and $\Sigma_0 := \{a_1, a_2, a_3, a_4\}$.



No existence result for dual problem

Compactness of maximizing $(u_n, w_n) \in \mathrm{Lip}_0(\Omega; \mathbb{R}^{1+d})$ thanks to

$$\frac{1}{2} \nabla u \otimes \nabla u + e(w) \leq \mathrm{Id} \qquad \text{(unilateral constraint)}$$

$$\Updownarrow$$

$$\frac{1}{2}|u(x) - u(y)|^2 \underbrace{+\langle w(x) - w(y), x - y\rangle \leq |x - y|^2}_{\text{(two-points condition)}} \quad \forall (x, y) \in \overline{\Omega}^2$$

- (u_n) is bounded in $C^{0,\frac{1}{2}}(\overline{\Omega}) \cap W^{1,2}(\Omega)$.
- (w_n) is bounded in $(W^{1,1} \cap L^\infty)(\Omega; \mathbb{R}^d)$

Bad new: $w_n \overset{*}{\rightharpoonup} w$ in $BV(\Omega; \mathbb{R}^d)$ (jumps of w are possible !)

Relaxation issue ?

Good new: Let $v(x) := \begin{cases} x - w(x) & \text{if } x \in \overline{\Omega} \\ x & \text{if } x \in \mathbb{R}^d \setminus \overline{\Omega} \end{cases}$.

Then v agrees a.e. with a maximal monotone map $\mathbf{v} : \mathbb{R}^d \mapsto \mathbb{R}^d$ and we have compactness for the Kuratowski convergence of graphs.

Two points of view are possible:

- build a metric $c_{\mathbf{v}}$ associated with $\mathbf{v}$ and conider OT with respect to this new cost $c_{\mathbf{v}}$
  (in (FMD) problem, $v \equiv \mathrm{id}$ and $c_{\mathbf{v}}(x, y) = |x - y|$)
- use a two-point duality scheme to get a truss-like reformulation of $(\mathcal{P})$.

IV- The geometric OT approach

Maximal monotoniciy of $v := \mathrm{id} - w$

Let (u, w) and set $v := \mathrm{id} - w$. Then (u, w) is admissible for $(\mathcal{P}^*)$ (i.e. fulfills the two-points condition) if and only if:

(i) v is maximal monotone and $v = \mathrm{id}$ on $\mathbb{R}^d \setminus \overline{\Omega}$

(ii) $u(x) - u(y) \le \ell_v(x, y) := \sqrt{2\langle v(y) - v(x), y - x\rangle}$

By [Alberti, Ambrosio 1999], (i) and (ii) are stable under the uniform convergence of u and the graph convergence of v provided we accept muti-valued maps $\mathbf{v}$. Accordingly we define:

$$\mathbf{M}_\Omega := \left\{ \mathbf{v} : \mathbb{R}^d \mapsto \mathbb{R}^d \ : \ \mathbf{v} \text{ max mono}, \ \mathbf{v} = \mathrm{id} \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega} \right\},$$

$$\ell_{\mathbf{v}}(x, y) := \min \left\{ \sqrt{2\langle x' - y', x - y\rangle} \ : \ x' \in \mathbf{v}(x), \quad y' \in \mathbf{v}(y) \right\}$$

($\ell_{\mathbf{v}}$ is not continuous in general but merely l.s.c)

Relaxed dual problem

By using a technical density argument, we get

Theorem

$$\sup(\mathcal{P}^*) = \max\Big\{\langle f, u\rangle \ :(u, \mathbf{v}) \in C_0(\Omega) \times \mathbf{M}_\Omega\ , \\ u(x) - u(y) \le \ell_{\mathbf{v}}(x, y)\ \forall(x, y) \in \overline{\Omega}^2\Big\}.$$

In the spirit of length spaces theory, we build a semi-metric associated with $\ell_{\mathbf{v}}$ given by:

$$c_{\mathbf{v}}(a, b) := \inf\left\{\sum_{i=1}^{N-1} \ell_{\mathbf{v}}(x_i, x_{i+1}) \ :\ x_1 = a,\ \ x_N = b,\ \ N \ge 2\right\}.$$

As $c_{\mathbf{v}}$ is the larger sub-additive function below $\ell_{\mathbf{v}}$, the inequality constraint above is equivalent to

$$u(x) - u(y) \le c_{\mathbf{v}}(x, y) \quad \forall(x, y) \in \overline{\Omega}^2.$$

Some properies of the semi-metric $c_{\mathbf{v}}$

- $c_{\mathbf{v}}(a,b) \leq M\,|b-a|^{\frac{1}{2}}$ for every $(a,b) \in \Omega \times \Omega$.
- If the infimum $c_{\mathbf{v}}(a,b)$ is attained by a finite set $\{x_i, 1 \leq i \leq N\}$, then the polygonal curve $C = \cup_{i=1}^{N-1}[x_i, x_{i+1}]$ is a geodesic joining a to b and $\mathbf{v}$ is tangentially affine on each $[x_i, x_{i+1}]$.
- $c_{\mathbf{v}}$ is a geodesic semi-distance on $\mathbb{R}^d$ (Riemann sums trick). If $\mathbf{v}(x) = \{v(x)\}$ with v Lipschitz and $\varphi_{\mathbf{v}}(x,z) = \limsup_{h \to 0+} \dfrac{1}{h}\, c_{\mathbf{v}}\big(x, x+hz\big)$ (Finsler), then $L_{\mathbf{v}}(\gamma) = \int_0^1 \varphi_{\mathbf{v}}\big(\gamma(t), \gamma'(t)\big)\, dt$ for every $\gamma \in \mathrm{Lip}([0,1];\mathbb{R}^d)$.
- Given (a,b), the evaluation map $\mathbf{v} \in (\mathbf{M}_\Omega, \mathbf{h}) \mapsto\ c_{\mathbf{v}}(a,b)$ is **concave and upper semicontinuous** ($\mathbf{h}$ = Hausdorff distance between graphs).

Kantorovich-Rubinstein distance

To the sub-additive cost $c_{\mathbf{v}}$, we associate a semi-distance between two measures $\mu, \nu \in \mathcal{M}_+(\overline{\Omega})$:

$$W_{c_{\mathbf{v}}}(\mu,\nu) \ := \ \min\left\{\iint c_{\mathbf{v}}(x,y)\,\gamma(dxdy)\ \gamma\in\Gamma(\mu,\nu)\right\}$$

As for the euclidean cost, we define the distance of f to $\partial\Omega$:

$$W_{c_{\mathbf{v}}}(f,\Omega) := \min\left\{W_{c_{\mathbf{v}}}(f_+ + \mu, f_- + \nu) : g \in \mathcal{M}_+(\partial\Omega)\right\}.$$

Remark: the minimum above is reached on the compact subset $\{(\mu,\nu)\in(\mathcal{M}_+(\partial\Omega))^2 : \int\mu \le \int f_-, \int\nu \le \int f_+\}$. In particular

$$f \ge 0 \implies W_{c_{\mathbf{v}}}(f,\Omega) := \min\left\{W_{c_{\mathbf{v}}}(f,g) : g\in\mathcal{M}_+(\partial\Omega)\right\}.$$

Looking for maximal semi-metrics

Following Kantorovich -Rubinstein, we get the duality indentity:

$$W_{c_{\mathbf{v}}}(f,\Omega) = \sup_{u\in C_0(\Omega)} \{\langle f,u\rangle \ : \ u(x)-u(y) \le c_{\mathbf{v}}(x,y) \text{ in } \Omega^2\}.$$

Then starting with the relaxed form of $(\mathcal{P}^*)$ and performing the supremum in u first, then in $\mathbf{v}$:

Theorem:

$$\sup(\mathcal{P}^*) \ = \ \max\left\{W_{c_{\mathbf{v}}}(f,\partial\Omega) \ : \ \mathbf{v}\in \mathbf{M}_\Omega\right\}.$$

$\rightsquigarrow$ existence of a maximimal semi-metric $c_{\mathbf{v}}$

Proof: Existence of optimal $\mathbf{v}$ follows from the concavity and upper semicontinuity of $\mathbf{v}\to W_{c_{\mathbf{v}}}(f,\Omega)$. $\square$

Remark: search for worse MK metrics in a different context : G.Buttazzo and all (2004) , B. Scwheizer and S. Conti (2011)

V- Two points scheme and truss-like solutions

Dualizing again !

In view of numerics, the case of discretized loads f is important. To that aim, we go back to the formulation of $(\mathcal{P}^*)$ as the supremum of $\langle f, u\rangle$ over smooth pairs (u, w) satisfying the two-point constraint

$$\frac{1}{2}\,(\xi(u))^2 + \zeta(w) \leq |x-y|^2 \text{ in } \overline{\Omega}^2$$

where: $\xi(u) = u(x) - u(y)$ and $\zeta(w) = \langle w(x) - w(y), \tau^{x,y}\rangle$

The new scheme involves two multipliers $\pi, \Pi \in \mathcal{M}(\overline{\Omega} \times \overline{\Omega})$ in duality with $(\xi(u), \zeta(w))$ and leads to another convex problem:

$$(\mathscr{P}) \qquad \inf\left\{\mathcal{J}(\pi, \Pi) \,:\, (\pi, \Pi) \in \mathscr{A}\right\}$$

here $\mathscr{A}$ denotes the class of pairs (π, Π) such that

$$\begin{cases} (i) & \int \big(u(y) - u(x)\big)\, \pi(dxdy) = \langle f, u \rangle \qquad \forall\, u \in C_{\Sigma_0}(\overline{\Omega}), \\ (ii) & \int \langle w(y) - w(x), \tau^{x,y} \rangle\, \Pi(dxdy) = 0 \qquad \forall\, w \in C_0(\Omega; \mathbb{R}^d), \\ (iii) & \Pi \geq 0 \end{cases}$$

$\mathcal{J}$ is a convex function on measures, finite only on (π, Π) such that $\Pi \geq 0$ and $\pi \ll \Pi$. If $\pi = \alpha\, \Pi$, then:

$$\mathcal{J}(\pi, \Pi) = \int_{\overline{\Omega} \times \overline{\Omega}} |x - y| \left(1 + \frac{\alpha^2}{2}\right) \Pi(dxdy).$$

A truss formulation for $(\mathcal{P})$

Each $(\pi, \Pi) \in \mathscr{A}$ encodes a truss-like measure (q^π, σ^Π) which is admissible for $(\mathcal{P})$ $(-\operatorname{div} q = f$ and $\mathrm{Div}\sigma = 0)$:

$$q^\pi = \iint \lambda^{x,y}\, \pi(dxdy) \quad , \quad \sigma^\Pi = \iint \sigma^{x,y}\, \Pi(dxdy).$$

In general, the following inequality is strict:

$$J(q^\pi, \Sigma^\Pi) \;\leq\; \mathcal{J}(\pi, \Pi)$$

Hence a priori, we expect merely $\inf(\mathscr{P}) \geq \min(\mathcal{P})$. In fact, by means of the dual problem, we can show

THM: $\inf(\mathscr{P}) = \sup(\mathcal{P}^*) = \min(\mathcal{P})$.

Existence issue for truss solutions

- Counter-examples show the non existence of solutions to $(\mathscr{P})$ for distributed source terms f
  - Despite $\sup_n \iint |x-y| d\Pi_n < +\infty$, minimizing sequences (Π_n) may blow up on the diagonal $x = y$
  - Sometimes curved stress lines (geodesics) appear as in Michell's truss problem.

- Existence for $(\mathscr{P})$ is expected if f is finitely supported

It relies on an extension property of monotone maps (conjecture) under wich competitors Π can be restricted to the class:

$$\mathrm{spt}(\Pi) \subset \ (\mathrm{spt}(f) \times \partial\Omega) \cup (\mathrm{spt}(f) \times \mathrm{spt}(f) \setminus \Delta) \quad (\Delta = \text{diagonal}).$$

(so that $|x-y| \geq \delta > 0$ holds Π-a.e.)

The extension conjecture

- Let $\mathbf{v}_0$ be a monotone map whose domain $D_0 := \mathrm{dom}(v_0)$ contains $\mathbb{R}^d \setminus \overline{\Omega}$. Then $\mathbf{v}_0$ admits at least one maximal monotone extension and any such extension is defined over whole $\mathbb{R}^d$.
- $\mathbf{v}_0$ induces a sub-additive function $c_{\mathbf{v}_0} : D_0 \times D_0 \to \mathbb{R}_+$ similary as before.

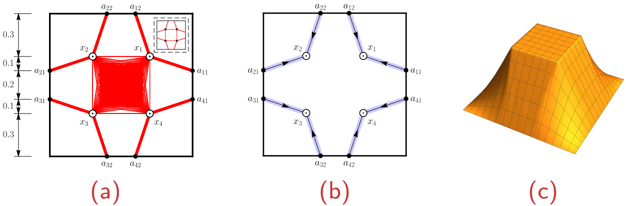
We will say that $\mathbf{v}_0$ has the *metric extension property* if there exits a maximal monotone map $\mathbf{v}$ of domain $\mathbb{R}^d$ such that $\mathbf{v} \supset \mathbf{v}_0$ and $c_{\mathbf{v}} = c_{\mathbf{v}_0}$ in $D_0 \times D_0$.

Conjecture: Let S be a finite subset of Ω and $D_0 = S \cup (\mathbb{R}^d \setminus \overline{\Omega})$. Then any monotone map $\mathbf{v}_0 : D_0 \to \mathbb{R}^d$ such that $\mathbf{v}_0 = \mathrm{id}$ in $\mathbb{R}^d \setminus \overline{\Omega}$ has the metric extension property.

Numerical simulations

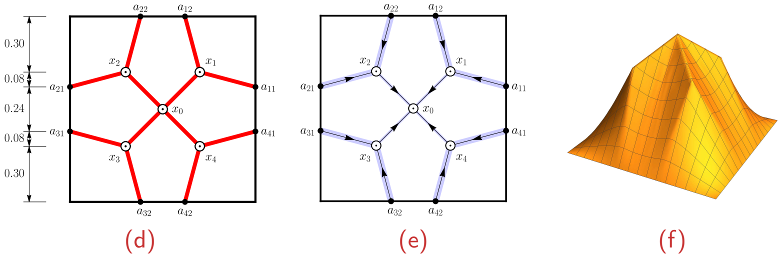
- Ω is the unite square $Q =]-1,1[^2$
- For $h > 0$ we use a grid $\mathrm{X}_h = \overline{\Omega} \cap \left\{(k_1 h, k_2 h) : (k_1, k_2) \in \mathbb{Z}^2\right\}$.
- The load f is discretized as $f_h = \sum_{x \in \mathrm{X}_h} f\big(Q_h(x)\big)\, \delta_x$ where $Q_h(x) = x + hQ$
- For $(\mathscr{P}_h)$, we narrow the search down to finite trusses spanned by X_h while for $(\mathscr{P}^*_h)$ the two-point constraint is checked on $K_k = X_h \times X_h$.
- Both problems $(\mathscr{P}_h)$ and $(\mathcal{P}^*_h)$ are handled as a pair of *conic quadratic programs* that we implement in MATLAB® with the use of the MOSEK® toolbox.

4 points source



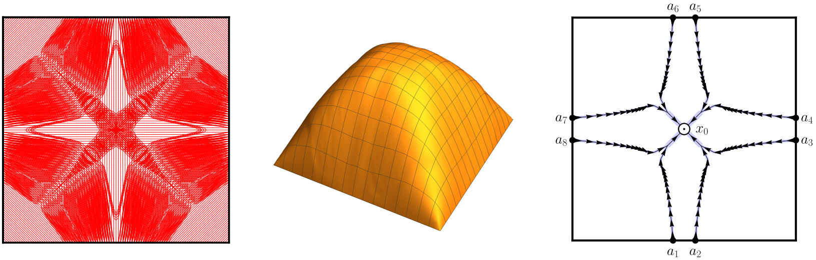
(a) optimal σ_Π (alternative solution in the top right corner); (b) optimal λ_π; (c) optimal u.

5 points source



(d) optimal σ_Π; (e) optimal λ_π; (f) optimal u.

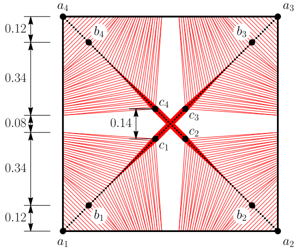
Uniform pressure load



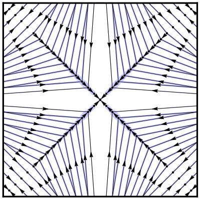
(g) (h) (i)

(g) optimal σ_Π; (h) optimal u;
(i) eight equivalent $c_{\mathbf{v}}$-geodesics from the central point x_0 to $\partial\Omega$ (computed for the numerical prediction of optimal $\mathbf{v}$).

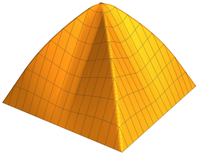
Diagonal load



(j)



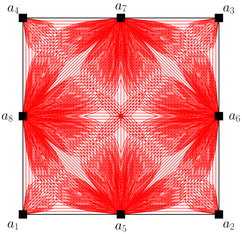
(k)



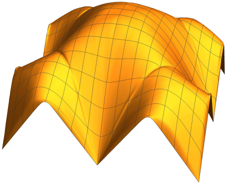
(l)

The discretized load is denoted by black dots: (j) optimal σ_Π (higher resolution); (k) optimal λ_π (lower resolution); (l) optimal u.

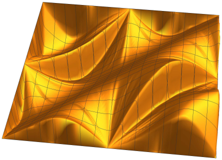
8 points Dirichlet condition and uniform pressure



(m)



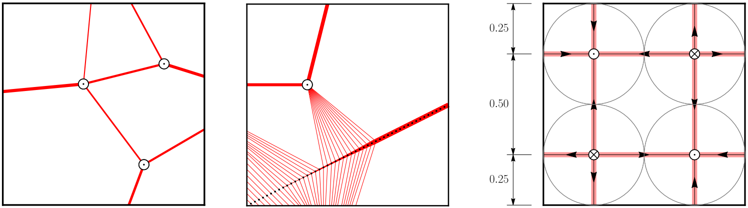
(n)



(o)

Dirichlet zone denoted by 8 solid squares:
(m) optimal σ_Π (fractal ?) ; (n) optimal u;
(o) component of w parallel to diagonal $[a_4, a_2]$ (discontinuous ?)

Miscellaneous



(p) (q) (r)

(p) optimal σ_Π for three asymmetric point forces;
(q) optimal σ_Π for point force and force distributed along a line;
(r) optimal σ_Π, λ_π for a signed load (here $Z = Z_0$)

Thank you for listening

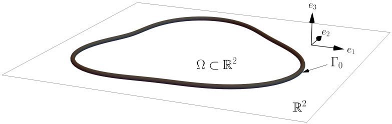
Vertically transmisible load and vault problem

Let $z \in C_0^1(\Omega)$ ($\Omega \subset \mathbb{R}^2$) and the associated graph $S_z \subset \mathbb{R}^3$. Then to any $f \in \mathcal{M}_+(\overline{\Omega})$, we associate a vertical load pressure F_z supported in S_z

$$F_z \ := \ -T_z^\sharp f\, e_3 \quad , \quad T_z(x) = (x, z(x))$$

The vault problem reads

$$\inf_{z,\sigma} \left\{ \int |\mathrm{Tr}\sigma| \ : \ \mathrm{spt}\,\sigma \subset S_z,\ \sigma \leq 0\ ,\ -\mathrm{Div}\sigma = F_z \right\}$$



e_3
e_2
e_1
$\Omega \subset \mathbb{R}^2$
Γ_0
$\mathbb{R}^2$