MULTILEVEL CONTROL

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Motivation

Motivation: Selective Harmonic Modulation (SHM)

This study has been conducted in the context of the research project **CONVADP** (Elkartek program of the Basque Government)

PARTICIPANTS:

- Fundación Deusto
- Universidad de Mondragón
- Tecnalia
- Ingeteam

Scope of CONVADP

To develop new technologies to increase the power density in electronic converters for high and low power applications, including energy extraction from eolic turbines or photovoltaic panels, drivers for boats and electrical vehicles.

Employment of a converter in an eolic turbine.
Source: nutechwindparts.com

A widely-employed technique is the **Selective Harmonic Modulation**.

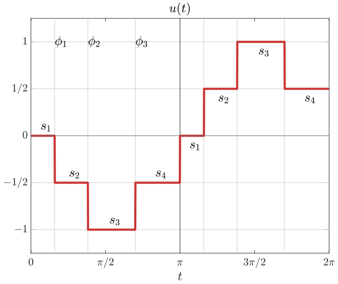
Objective of SHM

To generate a control signal with a desired harmonic spectrum by modulating some specific lower-order Fourier coefficients. This signal is constructed as a step function with a finite number of switches, taking values only in a given finite set.

IMPORTANT FEATURES:

Waveform: the sequence of values that the function takes in its domain.

Switching angles:, the sequence of points where the signal switches from one value to following one.



Mathematical formulation of the SHM problem

Mathematical formulation of SHM

Goal

Construct a signal

$$u(t) : [0, 2\pi) \to \mathcal{U}$$

in the form of a step function with a finite number of switches, such that some of its lower-order Fourier coefficients take specific values prescribed a priori.

$$\mathcal{U} = \{u_1, \dots, u_L\} \subset \mathbb{R}, \quad L \geq 2$$

$u_1 = -1,\ u_L = 1$ and $u_\ell < u_{\ell+1}$ for all $\ell \in \{1, \dots, L\}$

- $L = 2$: **bilevel** signal
- $L > 2$: **multilevel** signal

Half-wave symmetry

$$u(t + \pi) = -u(t) \quad \text{for all } t \in [0, \pi).$$

- $u \mapsto u|_{[0,\pi)}$
- $u(t) = \sum_{\substack{j \in \mathbb{N} \\ j \text{ odd}}} a_j \cos(jt) + \sum_{\substack{j \in \mathbb{N} \\ j \text{ odd}}} b_j \sin(jt)$

$$a_j = \frac{2}{\pi} \int_0^\pi u(\tau) \cos(j\tau)\, d\tau$$

$$b_j = \frac{2}{\pi} \int_0^\pi u(\tau) \sin(j\tau)\, d\tau$$

Mathematical formulation of SHM

Piecewise constant functions with a **finite number** of switches.

$$u(t) = \sum_{k=0}^{K} s_k \chi_{[\phi_k, \phi_{k+1})}(t), \quad K \in \mathbb{N}$$

Waveform

$$\mathcal{S} = \{s_k\}_{k=0}^{K} \text{ with } s_k \in \mathcal{U} \text{ and } s_k \neq s_{k+1} \text{ for all } k \in \{0, \ldots, K\}$$

Switching angles

$$\boldsymbol{\Phi} = \{\phi_k\}_{k=1}^{K} \text{ such that } 0 = \phi_0 < \phi_1 < \ldots < \phi_K < \phi_{K+1} = \pi$$

Mathematical formulation of SHM

In practical engineering applications, due to technical limitations, it is preferable to employ signals taking consecutive values in $\mathcal{U}$.

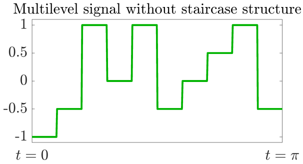
Staircase property

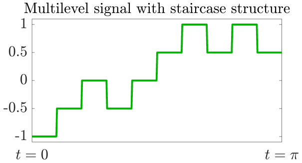
We say that a piecewise constant signal u fulfills the **staircase property** if its waveform $\mathcal{S}$ satisfies

$$(s_k^{min}, s_k^{max}) \cap \mathcal{U} = \varnothing$$

$$\text{for all } k \in \{0, \dots, K-1\},$$

where $s_k^{min} := \min\{s_k, s_{k+1}\}$ and $s_k^{max} := \max s_k, s_{k+1}$.





Remark

Note that when $\mathcal{U} = \{-1, 1\}$ (**bilevel** problem), this property is satisfied by any piece-wise linear function $u : [0, \pi) \to \mathcal{U}$.

Mathematical formulation of SHM

SHM problem - mathematical formulation

Let $\mathcal{E}_a$ and $\mathcal{E}_b$ be finite sets of odd numbers of cardinality $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively. For any two given vectors $\mathbf{a}_T \in \mathbb{R}^{N_a}$ and $\mathbf{b}_T \in \mathbb{R}^{N_b}$, we want to construct a function $u : [0, \pi) \to \mathcal{U}$ in staircase form such that the vectors $\mathbf{a} \in \mathbb{R}^{N_a}$ and $\mathbf{b} \in \mathbb{R}^{N_b}$, defined as

$$\mathbf{a} = (a_j)_{j \in \mathcal{E}_a} \quad \text{and} \quad \mathbf{b} = (b_j)_{j \in \mathcal{E}_b}$$

satisfy

$$\mathbf{a} = \mathbf{a}_T \quad \text{and} \quad \mathbf{b} = \mathbf{b}_T.$$

Remark

We gave a very general mathematical formulation of SHM. This formulation contains also the so-called **Selective Harmonic Elimination (SHE)** problem, in which the target vectors are such that

$$
\begin{aligned}
(\mathbf{a}_T)_1 \neq 0 \quad (\mathbf{a}_T)_{i\neq 1} = 0 \qquad &\text{for all } i \in \mathcal{E}_a \\
(\mathbf{b}_T)_1 \neq 0 \quad (\mathbf{b}_T)_{j\neq 1} = 0 \qquad &\text{for all } j \in \mathcal{E}_b.
\end{aligned}
$$

SHE is of great relevance in the electric engineering literature. Its objective is to generate a signal with amplitude

$$m_1 = \sqrt{a_1^2 + b_1^2}$$

and phase

$$\varphi_1 = \arctan\left(\frac{b_1}{a_1}\right),$$

removing some specific high-frequency components. In this way, SHE may be understood as a **generator of clean Fourier modes** through a staircase signal.

SHM via finite-dimensional optimization

Finite-dimensional optimization for SHM

A typical approach to the SHM problem is to look for signals u with a specific waveform $\mathcal{S}$ a priori determined, **optimizing only over the location of the switching angles Φ**.

Remark

For a fixed waveform $\mathcal{S}$, the Fourier coefficients of u can be written in terms of the switching angles Φ in the following way:

$$a_j = a_j(\Phi) = \frac{2}{j\pi}\sum_{m=0}^{M} s_m \Big[\sin(j\phi_{m+1}) - \sin(j\phi_m)\Big]$$

$$b_j = b_j(\Phi) = \frac{2}{j\pi}\sum_{m=0}^{M} s_m \Big[\cos(j\phi_m) - \cos(j\phi_{m+1})\Big]$$

For two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ and any fixed $\mathcal{S}$, we can then define the functions

$$\mathbf{a}_{\mathcal{S}}(\Phi) := \big(a_j(\Phi)\big)_{j\in\mathcal{E}_a} \in \mathbb{R}^{N_a}, \qquad \mathbf{b}_{\mathcal{S}}(\Phi) := \big(b_j(\Phi)\big)_{j\in\mathcal{E}_b} \in \mathbb{R}^{N_b}$$

which associate, to any sequence of switching angles $\{\phi_m\}_{m=1}^{M}$, the corresponding Fourier coefficients.

Therefore, SHM can be cast as a finite-dimensional optimization problem in the following way.

Optimization problem for SHM

Let $\mathcal{E}_a$, $\mathcal{E}_b$, $\mathbf{a}_T$, and $\mathbf{b}_T$ be given. Let $\mathcal{S} := \{s_m\}_{m=0}^{M}$ be a fixed waveform satisfying the staircase property. We look for a sequence of switching angles $\mathbf{\Phi} = \{\phi_m\}_{m=1}^{M}$ solution to the following minimization problem:

$$\min_{\Phi \in [0,\pi]^M} \left(\left\| \mathbf{a}_{\mathcal{S}}(\Phi) - \mathbf{a}_T \right\|^2 + \left\| \mathbf{b}_{\mathcal{S}}(\Phi) - \mathbf{b}_T \right\|^2 \right)$$

$$\text{subject to: } 0 = \phi_0 < \phi_1 < \ldots < \phi_M < \phi_{M+1} = \pi$$

Finite-dimensional optimization for SHM

Optimal value

We call **optimal value** $V_{\mathcal{S}} : \mathbb{R}^{N_a} \times \mathbb{R}^{N_b} \to \mathbb{R}$, the function that takes as input variables the target vectors $\mathbf{a}_T$ and $\mathbf{b}_T$ and returns the optimal value of the optimization problem.

Solvable set

We define a **solvable set** $\mathcal{R}_{\mathcal{S}}$ as:

$$\mathcal{R}_{\mathcal{S}} = \left\{ (\mathbf{a}_T, \mathbf{b}_T) \in \mathbb{R}^{N_a + N_b} \ : \ V_{\mathcal{S}}(\mathbf{a}_T, \mathbf{b}_T) = 0 \right\}$$

Policy

We call **policy** any function $\Pi_{\mathcal{S}} : \mathcal{R}_{\mathcal{S}} \to [0, \pi]^M$ such that $\Phi^* = \Pi_{\mathcal{S}}(\mathbf{a}_T, \mathbf{b}_T)$, with Φ^* being the optimal switching angles, solutions to the SHM problem with target $(\mathbf{a}_T, \mathbf{b}_T)$.

Finite-dimensional optimization for SHM

With the aim of reconstructing the policy $\Pi_{\mathcal{S}}$, a typical approach is to solve numerically the optimization problem for a limited number of points in $\mathbb{R}^{N_a+N_b}$ and check that the optimal value is zero. Secondly, one interpolates the function $\Pi_{\mathcal{S}}$ in the convex set generated by the points previously obtained. Nevertheless, **this approach has several difficulties and drawbacks**.

Combinatory problem

In practice, **one does not dispose of a suitable waveform $\mathcal{S}$** which yields a solution to the SHM problem. A common approach to solve the SHM problem consists in fixing the number of switches M, and then solve the optimization problem for all the possible combinations of M elements of $\mathcal{U}$.

Taking into account that the number of possible M-tuples in $\mathcal{U}$ is of the order $(L-1)^M$, it is evident that the complexity of the above approach increases rapidly when $L > 1$.

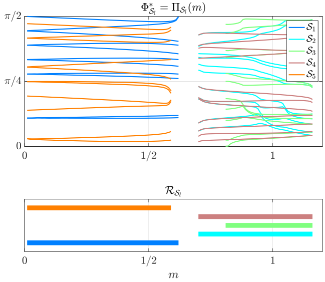
Finite-dimensional optimization for SHM

Solvable set problem

Given a waveform $\mathcal{S}$, the corresponding solvable set $\mathcal{R}_{\mathcal{S}}$ is usually very small, yielding to policies $\Pi_{\mathcal{S}}$ which are not very effective.

This issue is addressed by solving the optimization problem for a set of waveforms $\{\mathcal{S}_l\}_{l=1}^r$ and obtaining different policies $\{\Pi_{\mathcal{S}_l}\}_{l=1}^r$ and solvable sets $\{\mathcal{R}_{\mathcal{S}_l}\}_{l=1}^r$. By gathering them, one creates a new policy applicable in a wider range.

This union of policies may give rise to regions where the solution for the same target $(\mathbf{a}_T, \mathbf{b}_T)$ is not unique, or even generate regions with no solution at all.



Finite-dimensional optimization for SHM

Policy problem

Due to the complexity of a policy generated by the union of different waveforms, the continuity of the switching angles cannot be guaranteed. This is a well known problem in the SHM community.

SHM as an optimal control problem

Optimal control for SHM

We propose to formulate the SHM problem as an optimal control one.

The Fourier coefficients of the signal $u(t)$ are identified with the terminal state of a controlled dynamical system of $N_a + N_b$ components defined in the time-interval $[0, \pi)$.

The control of the system is the signal $u(t)$, defined as a function $[0, \pi) \to \mathcal{U}$, which has to steer the state from the origin to the desired values of the prescribed Fourier coefficients.

D. J. Oroya-Villalta, C. Esteve-Yagüe and U.B. - Multilevel Selective Harmonic Modulation via optimal control, 2021.

Step 1: dynamical system for the Fourier coefficients

For all $u \in L^\infty([0,\pi);\mathbb{R})$ we have $a_j = y_a(\pi)$ and $b_j = y_b(\pi)$ with

$$y_a(t) = \frac{2}{\pi}\int_0^t u(\tau)\cos(j\tau)d\tau \in C([0,\pi);\mathbb{R})$$
$$y_b(t) = \frac{2}{\pi}\int_0^t u(\tau)\sin(j\tau)d\tau \in C([0,\pi);\mathbb{R})$$

Fundamental theorem of calculus

The functions $y_a(\cdot)$ and $y_b(\cdot)$ are the unique solutions to the differential equation

$$\begin{cases} \dot{y}_a(t) = \dfrac{2}{\pi}\cos(jt)u(t), \ \ t \in [0,\pi) \\ y_a(0) = 0 \end{cases} \qquad \begin{cases} \dot{y}_b(t) = \dfrac{2}{\pi}\sin(jt)u(t), \ \ t \in [0,\pi) \\ y_b(0) = 0 \end{cases}$$

Step 1: dynamical system for the Fourier coefficients

Hence, for $\mathcal{E}_a$, $\mathcal{E}_b$, $\mathbf{a}_T$, and $\mathbf{b}_T$ given, the SHM problem can be reduced to:

SHM problem - dynamical system formulation

Find a staircase control function u such that the corresponding solution $\mathbf{y} \in C([0,\pi); \mathbb{R}^{N_a+N_b})$ to the dynamical system

$$\begin{cases} \dot{\mathbf{y}}(t) = \dfrac{2}{\pi}\mathbf{D}(t)u(t), & t \in [0,\pi) \\ \mathbf{y}(0) = 0 \end{cases}$$

satisfies $\mathbf{y}(\pi) = [\mathbf{a}_T; \mathbf{b}_T]^\top$, where

$$\mathbf{D}(t) = \begin{bmatrix} \mathbf{D}^a(t) \\ \mathbf{D}^b(t) \end{bmatrix}, \quad \mathbf{D}^a(t) = \begin{bmatrix} \cos(e_a^1 t) \\ \cos(e_a^2 t) \\ \vdots \\ \cos(e_a^{N_a} t) \end{bmatrix} \in \mathbb{R}^{N_a}, \quad \mathbf{D}^b(t) = \begin{bmatrix} \sin(e_b^1 t) \\ \sin(e_b^2 t) \\ \vdots \\ \sin(e_b^{N_b} t) \end{bmatrix} \in \mathbb{R}^{N_b}$$

$$\mathcal{E}_a = \{e_a^1, e_a^2, e_a^3, \ldots, e_a^{N_a}\}, \quad \mathcal{E}_b = \{e_b^1, e_b^2, e_b^3, \ldots, e_b^{N_b}\}$$

Step 2: time reversion

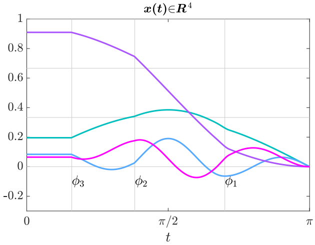
We can reverse the time using the transformation $\mathbf{x}(t) = \mathbf{y}(\pi - t)$. In this way, the SHM problem turns into the following null controllability one.

SHM via null controllability

Let $\mathcal{U}$, $\mathcal{E}_a$, $\mathcal{E}_b$ and the targets $\mathbf{a}_T$ and $\mathbf{b}_T$ be given. We look for a staircase function $u : [0, \pi) \to [-1, 1]$ such that the solution to the initial-value problem

$$\begin{cases} \dot{\mathbf{x}}(t) = \dfrac{2}{\pi}\mathbf{C}(t)u(t), & t \in [0, \pi) \\ \mathbf{x}(0) = [\mathbf{a}_T, \mathbf{b}_T]^\top =: \mathbf{x}_0 \end{cases}$$

with $\mathbf{C} = -\mathbf{D}$ satisfies $\mathbf{x}(\pi) = 0$.



Evolution in the time horizon $[0, \pi)$ of the dynamics $\mathbf{x}$ with $\mathcal{E}_a = \mathcal{E}_b = \{1, 3\}$.

Optimal control for SHM

Step 3: optimal control problem for SHM

$$\mathcal{A}_{ad} := \left\{ u : [0, \pi) \to [-1, 1] \text{ measurable satisfying the staircase property } \right\}$$

Optimal control problem for SHM

Let $\mathcal{U}$, $\mathcal{E}_a$, $\mathcal{E}_b$ and the targets $\mathbf{a}_T$ and $\mathbf{b}_T$ be given. We look for an admissible control $u \in \mathcal{A}_{ad}$ solution to the following optimal control problem:

$$\min_{u \in \mathcal{A}_{ad}} \frac{1}{2} \|\mathbf{x}(\pi)\|^2. \qquad (OCP1)$$

Remark

The cost functional (OCP1) is quadratic and the existence of at least one minimizer is ensured for any target $[\mathbf{a}_T, \mathbf{b}_T]^\top$.

Such a minimizer solves the SHM problem if and only if the minimum of (OCP1) is zero. Otherwise, the target $[\mathbf{a}_T, \mathbf{b}_T]^\top$ is unreachable.

Due to the limitations on the size of controls ($u \in \mathcal{U}$) and the time horizon ($T = \pi$), not every target $[\mathbf{a}_T, \mathbf{b}_T]^\top \in \mathbb{R}^{N_a + N_b}$ is reachable.

Step 4: penalized optimal control problem for SHM

The optimal control problem (OCP1) is defined on the non-convex set $\mathcal{A}_{ad}$ to take into account the staircase constraints on u.

In order to have a convex optimal control problem, we add a penalization term for the control to the cost functional, and remove the staircase constraint on the control.

Penalized optimal control problem for SHM

Fix $\varepsilon > 0$ and a convex function $\mathcal{L} \in C([-1,1];\mathbb{R})$. Let $\mathcal{E}_a$, $\mathcal{E}_b$ and the targets $\mathbf{a}_T$ and $\mathbf{b}_T$ be given. We look for a control

$$u \in \mathcal{A} := \left\{ u : [0,\pi) \to [-1,1] \ \text{ measurable } \right\}$$

solution to the following optimal control problem:

$$\min_{u \in \mathcal{A}} \left(\frac{1}{2} \|\mathbf{x}(\pi)\|^2 + \varepsilon \int_0^\pi \mathcal{L}(u(t))dt \right). \qquad (OCP2)$$

The multilevel and staircase form of u can be both ensured by a suitable choice of the penalization function $\mathcal{L}$.

Multilevel SHM

Theorem

Let $\mathcal{U}$ and $\mathbf{x}_0$ be given. For any $\alpha > 0$ and $\beta \in \mathbb{R}$, set the function

$$\mathcal{P}(u) = \alpha(u - \beta)^2.$$

Consider (OCP2) with

$$\mathcal{L}(u) = \begin{cases} \lambda_\ell(u) & \text{if } u \in [u_\ell, u_{\ell+1}) \\ \mathcal{P}(1) & \text{if } u = u_L \end{cases} \quad \text{for all } \ell \in \{1, \dots, L-1\},$$

$$\lambda_\ell(u) := \frac{(u - u_\ell)\mathcal{P}(u_{\ell+1}) + (u_{\ell+1} - u)\mathcal{P}(u_\ell)}{u_{\ell+1} - u_\ell}.$$

Assume that $\mathcal{L}$ has a unique minimum in $[-1, 1]$. Then, (OCP2) admits a unique minimizer u_ε which has the multilevel and staircase structure. Moreover, u_ε is continuous with respect to $\mathbf{x}_0$ in the strong topology of $L^1(0, \pi)$. Finally, the associated optimal trajectory $\mathbf{x}_\varepsilon$ satisfies $\|\mathbf{x}_\varepsilon(\pi)\|^2_{\mathbb{R}^N} \leq 4\pi\varepsilon\|\mathcal{L}\|_\infty$.

D. J. Oroya-Villalta, C. Esteves-Yagüe and U. B., Multilevel Selective Harmonic Modulation via optimal control, 2021.

Proof (sketch)

Existence and uniqueness of the minimizer: they can be obtained via a standard argument since the functional is convex with respect to the control, the admissible controls in $\mathcal{A}$ are uniformly bounded and the dynamical constraints are linear.

Proof (sketch)

Existence and uniqueness of the minimizer: they can be obtained via a standard argument since the functional is convex with respect to the control, the admissible controls in $\mathcal{A}$ are uniformly bounded and the dynamical constraints are linear.

Continuity of solutions: the argument uses the fact that the optimal solutions are uniformly bounded in $BV(0,\pi) \hookrightarrow L^1(0,\pi)$ with compact embedding. More details can be found in

Proof (sketch)

Multilevel structure and staircase property: introduce the Hamiltonian

$$\mathcal{H}(t, \mathbf{p}, u) = \varepsilon \mathcal{L}(u) - \mu(t)u(t), \quad \mu(t) := \frac{2}{\pi}(\mathbf{p}(t) \cdot \mathbf{D}(t))$$

and derive the optimality conditions via Pontryagin's Maximum Principle.

1. The adjoint system reads as

$$\begin{cases} \dot{\mathbf{p}}^*(t) = -\nabla_x \mathcal{H}(u(t), \mathbf{p}^*(t), t) = 0, \ t \in [0, \pi) \\ \mathbf{p}^*(\pi) = \mathbf{x}^*(\pi) \end{cases} \qquad \rightarrow \qquad \mathbf{p}^*(t) = \mathbf{x}^*(\pi).$$

2. Optimality condition:

$$u^*(t) \in \underset{|u| \leq 1}{\operatorname{argmin}} \left[\varepsilon \mathcal{L}(u) - \mu^*(t) u \right]$$

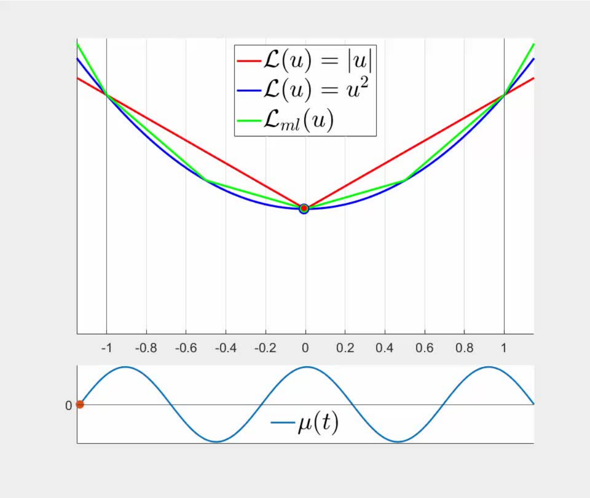
$$\mu^*(t) := \frac{2}{\pi}(\mathbf{x}^*(\pi) \cdot \mathbf{D}(t)) = \sum_{j \in \mathcal{E}_a} a_j^*(\pi) \cos(jt) + \sum_{j \in \mathcal{E}_b} b_j^*(\pi) \sin(jt).$$

With our choice of $\mathcal{L}(u)$, the above argmin is a singleton for a.e. $t \in [0, \pi)$, except for a finite number of times (the switching angles).

Proof (sketch)

Staircase property:

$$\mathcal{H}(u) = \varepsilon\mathcal{L}(u) - \mu(t)u$$



Adjoint formulation

Adjoint formulation of the SHM problem

Applying the **Fenchel-Rockafellar theory**, we can build the following dual problem

$$\mathbf{p}_{\varepsilon,\pi} = \operatorname*{argmin}_{\mathbf{p}_\pi \in \mathbb{R}^N} \mathcal{J}_\varepsilon(\mathbf{p}_\pi)$$

$$\mathcal{J}_\varepsilon(\mathbf{p}_\pi) = \int_0^\pi \mathcal{L}^\star\left(\mathbf{C}^\top(t)\mathbf{p}_\pi\right) dt + \frac{\varepsilon}{2}\left\|\mathbf{p}_\pi\right\|^2_{\mathbb{R}^N} + \langle \mathbf{x}_0, \mathbf{p}_\pi \rangle,$$

where

$$C(\mathbb{R}) \ni \mathcal{L}^\star(v) = \sup_{u\in\mathbb{R}} \Big(uv - \mathcal{L}(u)\Big)$$

is the **convex conjugate** of $\mathcal{L}$ and is still a piece-wise linear function.

Adjoint formulation of the SHM problem

Theorem

For any $\varepsilon > 0$, there exists a unique minimizer $\mathbf{p}_{\varepsilon,\pi} \in \mathbb{R}^N$ of the functional $\mathcal{J}_\varepsilon$. Moreover, this minimizer is related with the minimizer u_ε of (OCP2) through the formulas

$$u_\varepsilon(t) \in \partial\mathcal{L}^\star\left(\mathbf{C}^\top(t)\mathbf{p}_{\varepsilon,\pi}\right), \qquad \text{for a.e. } t \in [0,\pi)$$

and

$$\mathbf{x}_\varepsilon(\pi) = -\varepsilon\mathbf{p}_{\varepsilon,\pi}.$$

U. B. and E. Zuazua, Selective Harmonic Modulation by duality, 18th IFAC workshop on Control Applications of Optimization, Gif-sur-Yvette, France, July 18-22, 2022.

Numerical experiments

Numerical experiments

Control set

Test 1: $\mathcal{U} = \{-1, 0, 1\}$

Test 2: $\mathcal{U} = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$

Common parameters

$\mathcal{E}_a = \mathcal{E}_b = \{1, 5, 7, 11, 13\}$

$\mathbf{a}_T = \mathbf{b}_T = (m, 0, 0, 0, 0)$
$m \in [-0.8, 0.8]$

$\varepsilon = 10^{-6}$

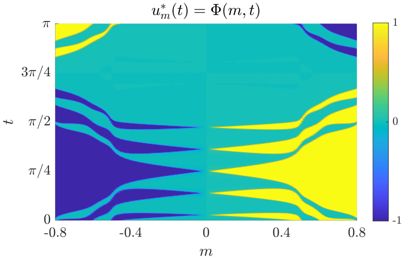
For all the experiments, we plot the function

$$\begin{array}{rccc} \Phi: & [-0.8, 0.8] \times [0, \pi] & \longrightarrow & \mathcal{U} \\ & (m, t) & \longmapsto & u_m^*(t), \end{array}$$

where for each $m \in [-0.8, 0.8]$, $u_m^*(\cdot)$ represents the solution to the SHM problem with the desired target frequencies.

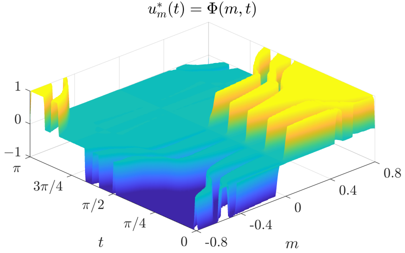
To solve the minimization problem we implement the interior point method via the optimization software `IPOPT`, coupled with the open-source tool for nonlinear optimization and algorithmic differentiation `CasADi`.

Test case 1



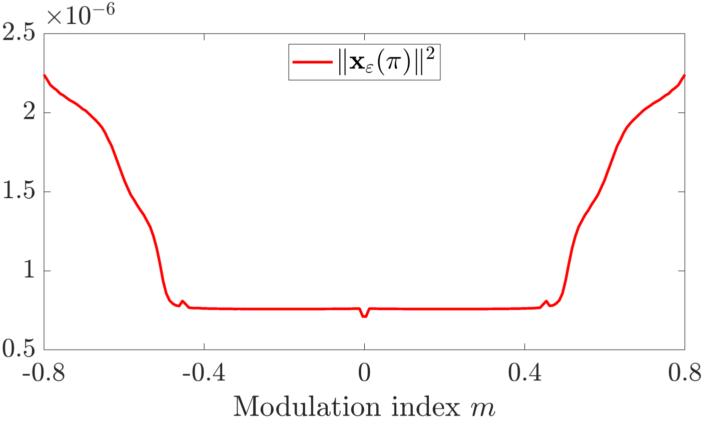
Top view of the 3-level control.

Side view of the 3-level control.

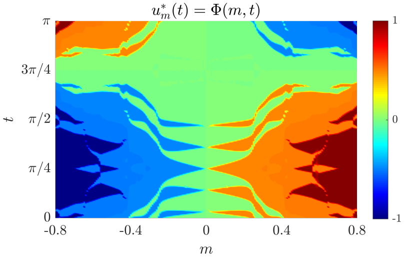


Test case 1 - error

$\|\mathbf{x}_\varepsilon(\pi)\|^2_{\mathbb{R}^{10}}$ for all values of the modulation index $m \in [-0.8, 0.8]$.

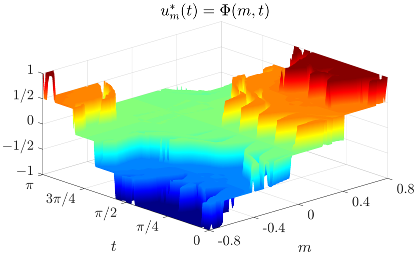


Test case 2



Top view of the 5-level control.

Side view of the 5-level control.



Multilevel control for ODE systems

Multilevel control for ODE systems

The concept of multilevel control can be generalized to linear finite-dimensional controlled systems

$$\begin{cases} \mathbf{x}'(t) = A\mathbf{x}(t) + Bu(t), & t \in (0,T) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{1}$$

satisfying the Kalman rank condition.

U. B. and E. Zuazua, Multilevel control by duality, 2021.

We call **multilevel control** a piece-wise constant function $u \in L^\infty(0,T;\mathcal{R})$ with finitely-many jumps, taking values in a **set of levels** $\mathcal{R} := \{\rho_1, \dots, \rho_L\} \subset \mathbb{R}$ with

$$\rho_k < \rho_{k+1}, \qquad \text{for all } k \in \{1, \dots, L-1\}$$

and such that the corresponding solution to (1) satisfies $\mathbf{x}(T) = 0$.

Conservative or dissipative dynamics: we can construct a multilevel and staircase control solving

$$\mathbf{p}^*_{T,ml} = \underset{\mathbf{p}_T \in \mathbb{R}^N}{\operatorname{argmin}} \mathcal{J}_{ml}(\mathbf{p}_T) \qquad \begin{cases} -\mathbf{p}'(t) = A^\top \mathbf{p}(t), & t \in (0,T) \\ \mathbf{p}(T) = \mathbf{p}_T \end{cases}$$

$$\mathcal{J}_{ml}(\mathbf{p}_T) = \int_0^T \mathcal{L}(B^\top \mathbf{p}(t))\,dt + \langle \mathbf{x}_0, \mathbf{p}(0)\rangle_{\mathbb{R}^N}$$

with a **suitably chosen penalization function** $\mathcal{L}$, provided that the time horizon T **is large enough**.

Choice of the penalization function $\mathcal{L}$

We design $\mathcal{L}$ as a **piece-wise affine interpolation of** $\mathcal{P}(u) = u^2$.

Let $\mathcal{I} := [\varpi_i, \varpi_f] \subset \mathbb{R}$ with $0 \in \mathcal{I}$, and let $L \geq 2$. On $\mathcal{I}$, we introduce an $L+1$-points partition $\mathcal{U}$ defined as

$$\mathcal{U} = \{u_1, \ldots, u_{L+1}\} \text{ with } u_1 = \varpi_i < 0, \quad u_{L+1} = \varpi_f > 0 \tag{2a}$$

$$u_k < u_{k+1}, \quad \text{for all } k \in \{1, \ldots, L\} \tag{2b}$$

$$u_k \neq 0, \quad \text{for all } k \in \{1, \ldots, L+1\} \tag{2c}$$

$$\lambda_k(u) := (u_{k+1} + u_k)u - u_{k+1}u_k, \quad \text{for all } k \in \{1, \ldots, L\} \tag{3}$$

straight line joining (u_k, u_k^2) and (u_{k+1}, u_{k+1}^2), with slope $\lambda_k'(u) = u_{k+1} + u_k$.

Penalization $\mathcal{L}$

$$\mathcal{L}(u) := \begin{cases} \lambda_1(u) & \text{if } u < u_1 \\ \lambda_k(u) & \text{if } u \in [u_k, u_{k+1}], \quad k \in \{1, \ldots, L\} \\ \lambda_L(u), & \text{if } u > u_{L+1} \end{cases} . \tag{4}$$

Main result - I

Theorem

Assume that $A \in \mathbb{R}^{N \times N}$ with $\mathrm{rank}(A) = N$ defines either a conservative or a dissipative dynamics, and let $B \in \mathbb{R}^N$ be such that the pair (A, B) fulfills the Kalman rank condition. For a given $\mathbb{N} \ni L \geq 2$, let $\mathcal{U} = \{u_k\}_{k=1}^{L+1}$ be defined as in (2a)-(2c) and let $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ be constructed as in (3)-(4). Then, the following facts hold:

1. There exists a positive time $T_* = T_*(\mathbf{x}_0, \mathcal{L}) > 0$ such that, for all $T \geq T_*$, $J_{ml}(\mathbf{p}_T)$ admits a minimizer $\mathbf{p}^*_{T,ml} \in \mathbb{R}^N$.
2. Let $\mathbf{p}^*_{T,ml} \in \mathbb{R}^N$ be a minimizer of J_{ml} and let $\mathbf{p}^*_{ml}$ denote the associated solution of the adjoint equation. Let $\mathcal{R} = \{\rho_k\}_{k=1}^{L}$ with $\rho_k = u_{k+1} + u_k$ for all $k \in \{1, \ldots, L\}$. Then, the function u^*_{ml} defined through

$$u^*_{ml} \in \partial\Big(\mathcal{L}(B^\top \mathbf{p}^*_{ml})\Big)$$

   is a multilevel and staircase control taking values in $\mathcal{R}$ such that, for any initial datum $x_0 \in \mathbb{R}^N$, the corresponding solution x to (1) fulfills $\mathbf{x}(T) = 0$.

U. B. and E. Zuazua, Multilevel control by duality, 2021.

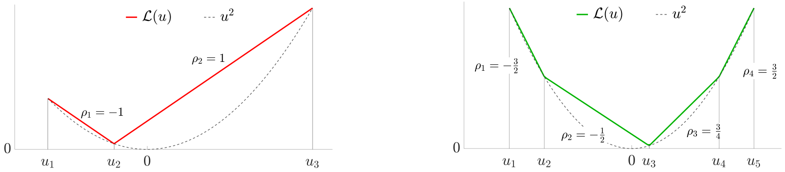
Main result - I

Remark

The choice of the interpolation points $\mathcal{U} = \{u_k\}_{k=1}^{L+1}$ determines the slopes of the affine branches $\lambda_k(u)$ composing the penalization function $\mathcal{L}$ and, therefore, the set of levels $\mathcal{R}$ for the multilevel control.

In some situations, it is possible to proceed the other way around, fixing a priori this set of levels $\mathcal{R} = \{\rho_k\}_{k=1}^{L}$ and then selecting the interpolation points $\mathcal{U} = \{u_k\}_{k=1}^{L+1}$ fulfilling (2a)-(2c) and such that

$$u_{k+1} + u_k = \rho_k, \qquad \text{for all } k \in \{1, \dots, L\}.$$



Particular examples of an admissible penalization $\mathcal{L}$ built prefixing the set of levels $\mathcal{R}$.

Proof

STEP 1: existence of a minimizer.

The existence of a minimizer $\mathbf{p}^*_{T,ml}$ is a consequence of the direct method of calculus of variations, since the functional J_{ml} is convex, continuous and coercive (if the dynamics is conservative or dissipative and provided that the time horizon T is large enough).

STEP 2: multilevel structure of the controls

The minimizers of J_{ml} are characterized by the optimality condition $0 \in \partial J_{ml}(\mathbf{p}^*_{T,ml})$. This is equivalent to the **Euler-Lagrange equation**

$$0 \in \int_0^T \partial\mathcal{L}(B^\top \mathbf{p}^*_{ml}(t))B^\top p(t)\,dt + \langle x_0, p(0)\rangle_{\mathbb{R}^N} \quad \text{for all } p_T \in \mathbb{R}^N,$$

$$\mathcal{I}_{ml} := \left\{ t \in (0,T) \;:\; B^\top \mathbf{p}^*_{ml}(t) = u_k,\ \exists\, k \in \{2,\dots,L\} \right\} \quad \Rightarrow \quad \mu(\mathcal{I}_{ml}) = 0$$

Hence, $u^*_{ml}(t) \in \partial\mathcal{L}(B^\top \mathbf{p}^*_{ml}(t))$ is a null-control such that $\mathbf{x}(T) = 0$.

- Multilevel structure: consequence of

$$\partial\mathcal{L}(u) = \begin{cases} \mathcal{L}'(u) & \text{if } u \neq u_k \in \mathcal{U} \text{ for all } k \in \{2,\dots,L\} \\ [\rho_{k-1}, \rho_k] & \text{if } u = u_k \in \mathcal{U} \text{ for some } k \in \{2,\dots,L\} \end{cases}$$

- Staircase structure: consequence of the continuity of $B^\top \mathbf{p}^*_{ml}$.

Main result - II

General dynamics: for general dynamics that satisfy the Kalman rank condition but are neither purely conservative nor purely dissipative, we can construct a multilevel and staircase control **for any** $T > 0$ solving

$$\mathbf{p}^*_{T,ml} = \underset{\mathbf{p}_T \in \mathbb{R}^N}{\operatorname{argmin}} \mathcal{J}_{ml}(\mathbf{p}_T)$$

$$\mathcal{J}_{ml}(\mathbf{p}_T) = \frac{1}{2}\left(\int_0^T \mathcal{L}(B^\top \mathbf{p}(t))\,dt\right)^2 + \langle \mathbf{x}_0, \mathbf{p}(0)\rangle_{\mathbb{R}^N}.$$

Control

$$u^*_{ml} \in \Lambda_{T,ml}\partial\Big(\mathcal{L}(B^\top \mathbf{p}^*_{ml})\Big) \quad \text{with} \quad \Lambda_{T,ml} := \int_0^T \mathcal{L}(B^\top \mathbf{p}^*_{ml}(t))\,dt$$

Numerical experiments

Numerical experiments

We discuss three situations.

Situation 1: conservative dynamics in large time horizon. We are able to compute multilevel and staircase controls for (1) by solving the first optimal control problem considered. We also show the failure of this approach in a short time horizon.

Situation 2: conservative dynamics in short time horizon. We are able to compute multilevel and staircase controls for (1) by solving the second optimal control problem considered.

Situation 3: general dynamics. We will consider a dynamics which is neither conservative nor dissipative to illustrate that multilevel and staircase controls can be computed by solving the second optimal control problem considered.

In all these numerical experiments, we have used a standard **gradient descent method** to carry out the minimization of the functionals.

Conservative dynamics in large time

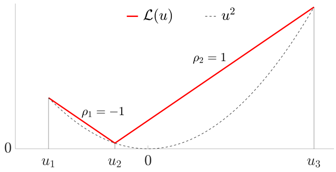
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} \qquad T = 4$$

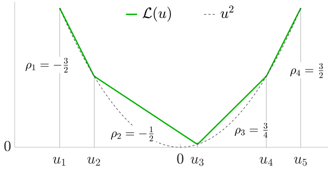
Control set

$$\mathcal{R} = \left\{ -1, 1 \right\}$$

Control set

$$\mathcal{R} = \left\{ -\frac{3}{2}, -\frac{1}{2}, \frac{3}{4}, \frac{3}{2} \right\}$$





Conservative dynamics in large time

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} \qquad T = 4$$

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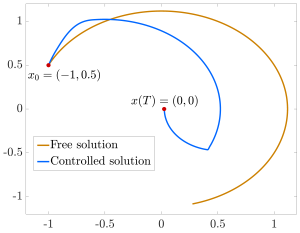
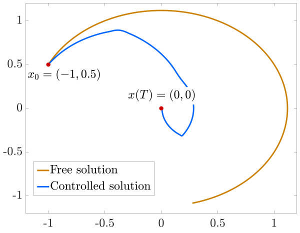


Control set

$$\mathcal{R} = \left\{ -\frac{3}{2}, -\frac{1}{2}, \frac{3}{4}, \frac{3}{2} \right\}$$



Conservative dynamics in large time

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} \qquad T = 4$$

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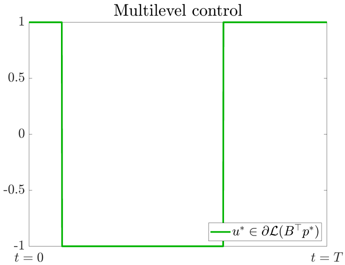
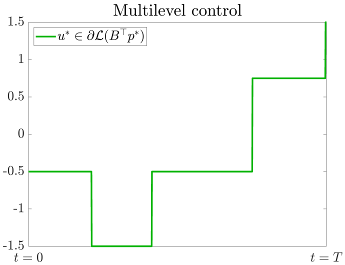


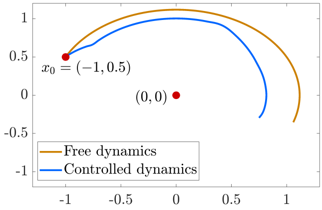


Conservative dynamics in short time

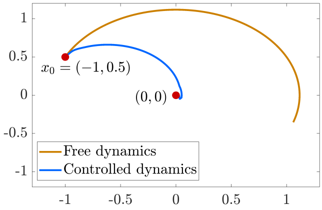
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} \qquad T = 3$$

$$\mathcal{R} = \left\{ -1, -\frac{1}{2}, 0, \frac{1}{2}, 1 \right\}$$

Functional J_{ml}



Functional $\mathcal{J}_{ml}$

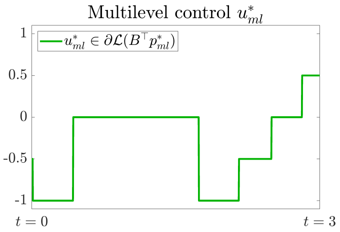


Conservative dynamics in short time

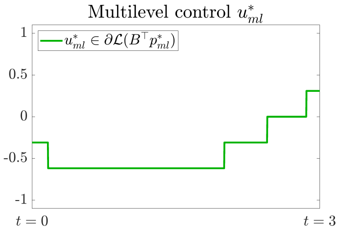
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} \qquad T = 3$$

$$\mathcal{R} = \left\{ -1, -\frac{1}{2}, 0, \frac{1}{2}, 1 \right\}$$

Functional J_{ml}



Functional $\mathcal{J}_{ml}$



$\Lambda_{T,ml} = 0,6181$

General dynamics

$$\begin{cases} x'(t) = x(t) + u(t), \quad t \in (0, T) \\ x(0) = x_0 \end{cases} \tag{5}$$

The dynamics is neither conservative nor dissipative. Hence, we cannot guarantee that the first optimization process would be capable of generating an effective multilevel and staircase control.

General dynamics

In fact, a multilevel control u^* generated through J_{ml} would be such that

$$u^*(t) \geq \rho_1, \qquad \text{for all } t \in (0, T).$$

The corresponding solution of (5) would then satisfy $x'(t) \geq x(t) + \rho_1$ for all $t \in (0, T)$, i.e.

$$x(t) \geq (x_0 + \rho_1)e^t - \rho_1, \qquad \text{for all } t \in (0, T).$$

Since $\rho_1 < 0$, this would imply that there can exist some time $T > 0$ such that $x(T) = 0$ if and only if $x_0 < -\rho_1$. If, instead, $x_0 \geq -\rho_1$, then

$$x(t) \geq -\rho_1 > 0, \qquad \text{for all } t \in (0, T).$$

Nevertheless, even if $x_0 < -\rho_1$, for this dynamics (5) we cannot guarantee that the functional J_{ml} is coercive. Therefore, the minimization of J_{ml} may not be successful.

All these issues disappear when looking for multilevel and staircase controls for (5) through the minimization of $\mathcal{J}_{ml}$. In fact, in this case we know that our multilevel control u^* would be such that

$$u^*(t) \geq \Lambda_{T,ml}\rho_1, \qquad \text{for all } t \in (0,T),$$

with

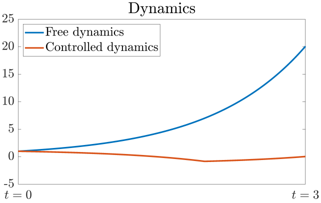
$$\Lambda_{T,ml} = \int_0^T \mathcal{L}(B^\top \tilde{\mathbf{p}}^*_{ml}(t))\, dt > 0$$

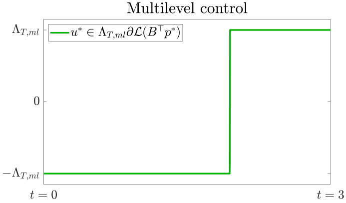
depending on the time horizon T and the optimal solution $\mathbf{p}^*_{T,ml}$.

During the optimization process, the intensity of the control is also optimized so to guarantee the controllability of (5) for all $x_0 \in \mathbb{R}$ and $T > 0$.

General dynamics

$$x_0 = 1 \quad \Rightarrow \quad x(t) = e^t$$
$$\mathcal{R} = \{-1, 1\} \quad \Rightarrow \quad u^*(t) \in \{-\Lambda_{T,ml}, \Lambda_{T,ml}\}$$
$$\text{for a.e. } t \in (0, T)$$





Open problems

Open problems

Minimal number of switching angles

In practical applications, to optimize the converters' performance, it is required to maintain the number of switches in the SHM signal the lowest possible.

Characterization of the solvable set

It would be interesting to have a full characterization of the solvable set for the SHM problem, thus determining the entire range of Fourier coefficients which can be reached by means of our approach.

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