Laboratoire Jean Kuntzmann EDP TEAM

Global Well-posedness and stabilization by saturated controllers of KdV equation on a Star-Shaped Network

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IX Partial differential equations, optimal design and numerics 2022.

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August 30, 2022

KdV equation and Networks

The Korteweg-de Vries (KdV) equation $u_t + u_x + u_{xxx} + uu_x = 0$ was to model the propagation of long water waves in a channel. We will study this equation on a Star Shaped Network.

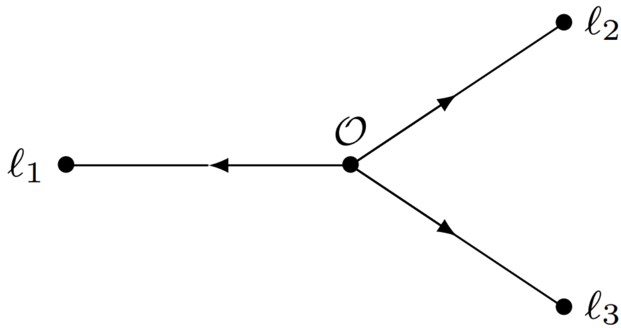

Figure: Star Shaped Network for $N = 3$.

- Ammari and Crépeau [2018] controllability and stabilization, Cerpa et al. [2020] controllability with less controls and Parada et al. [2022] stabilization with delayed feedback terms.

Saturation

We will consider a saturation (resp. **sat**) that could be any of the following cases:

- **sat** $=$ **sat**$_{loc}$: First consider the following scalar saturation,

$$\mathrm{sat}(f) = \begin{cases} -M, & \text{if } f \leq -M, \\ f, & \text{if } -M \leq f \leq M, \\ M, & \text{if } f \geq M, \end{cases}$$

  where $M > 0$ is given and denotes the saturation level. Then $\mathbf{sat}_{loc}(f)(x) = \mathrm{sat}(f(x))$.

- **sat** $=$ **sat**$_2$: For $f \in L^2(0,L)$ we define

$$\mathbf{sat}_2(f)(x) = \begin{cases} f(x), & \text{if } \|f\|_{L^2(0,L)} \leq M, \\ \dfrac{f(x)M}{\|f\|_{L^2(0,L)}}, & \text{if } \|f\|_{L^2(0,L)} \geq M. \end{cases} \tag{1}$$

Saturation

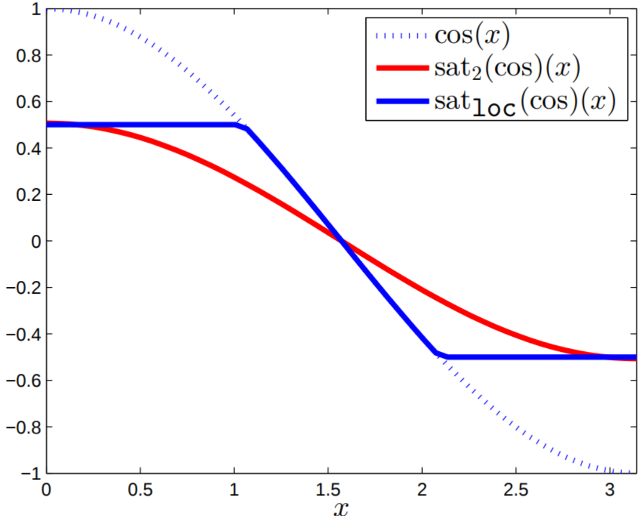

Figure: Saturation functions with saturation level $M = 0.5$, Marx et al. [2017].

System studied

Let $K = \{k_n : 1 \leq n \leq N\}$ be the set of the N edges of a network $\mathcal{T}$ described as the intervals $[0, \ell_n]$ with $\ell_n > 0$ for $n = 1, \cdots N$, the network $\mathcal{T}$ is defined by $\mathcal{T} = \bigcup_{n=1}^{N} k_n$. Specifically, we are going to consider the next evolution problem for the KdV equation,

$$\begin{cases} (\partial_t u_n + \partial_x u_n + u_n \partial_x u_n + \partial_x^3 u_n)(t,x) + \mathbf{sat}(a_n(x) u_n(t,x)) = 0, & \forall x \in (0, \ell_n),\ t > 0,\ n = 1, \cdots, N \\ u_n(t,0) = u_{n'}(t,0), & \forall n, n' = 1, \cdots N, \\ \displaystyle\sum_{n=1}^{N} \partial_x^2 u_n(t,0) = -\alpha u_1(t,0) - \frac{N}{3} u_1^2(t,0), & t > 0, \\ u_n(t,\ell_n) = \partial_x u_n(t,\ell_n) = 0, & t > 0,\ n = 1, \cdots, N, \\ u_n(0,x) = u_n^0(x), & x \in (0, \ell_n), \end{cases} \tag{KdV-S}$$

where $\alpha \geq \dfrac{N}{2}$.

Central node boundary conditions

If we denote by u_n and v_n the dimensionless and scaled variables standing respectively for the deflection from rest position and the velocity on the branch n of long water waves by *Whitham, 1999*

$$\begin{cases} \partial_t u_n + \partial_x u_n + \partial_x^3 u_n + u_n \partial_x u_n = 0, & \forall x \in (0, \ell_n),\ t > 0,\ n = 1, \cdots, N, \\ v_n = u_n - \dfrac{1}{6} u_n^2 + 2\partial_x^2 u_n, & \forall x \in (0, \ell_n),\ t > 0,\ n = 1, \cdots, N. \end{cases}$$

Moreover, at the central node, we can suppose that the elevation of water is the same in all branches and that the sum of the flux is null , which implies:

$$\begin{cases} u_n(t,0) = u_{n'}(t,0), & \forall n, n' = 1, \cdots N, \\ \displaystyle\sum_{n=1}^{N} u_n(t,0) v_n(t,0) = 0, & t > 0. \end{cases}$$

Then we obtain the following problem,

$$\begin{cases} u_n(t,0) = u_{n'}(t,0), & \forall n, n' = 1, \cdots N, \\ \displaystyle\sum_{n=1}^{N} \partial_x^2 u_n(t,0) = -\frac{N}{2} u_1(t,0) + \frac{N}{12} u_1^2(t,0), & t > 0. \end{cases}$$

We adapt the boundary condition at the central node to have a decreasing energy.

Preliminary functional setting

Let $H_r^1(0,\ell_n) = \{v \in H^1(0,\ell_n),\ v(\ell_n) = 0\}$, where the index r is related to the null right boundary conditions, the space $\mathbb{H}_e^1(\mathcal{T})$ will be the cartesian product of $H_r^1(0,\ell_n)$ including the continuity condition on the central node $(u_n(0) = u_{n'}(0), \forall n, n' = 1, \cdots, N)$

$$\mathbb{H}_e^1(\mathcal{T}) = \left\{ \underline{u} = (u_1, \cdots, u_N)^T \in \prod_{n=1}^{N} H_r^1(0,\ell_n), u_n(0) = u_{n'}(0), \forall n, n' = 1, \cdots, N \right\},$$

Introduce also the state space

$$\mathbb{L}^2(\mathcal{T}) = \prod_{n=1}^{N} L^2(0,\ell_n), \quad \mathbb{B}_T = C([0,T], \mathbb{L}^2(\mathcal{T})) \cap L^2(0,T; \mathbb{H}_e^1(\mathcal{T})),$$

$$\mathbb{Y}_T = \left\{ \underline{v} \in \mathbb{B}_T :\ \partial_x^{\kappa} v_n \in L_x^{\infty}(0,\ell_n; H^{\frac{1-\kappa}{3}}(0,T)),\ \kappa = 0,1,2 \ \text{ and } \ n = 1, \cdots, N \right\}$$

Related results

$$\begin{cases} (\partial_t u_n + \partial_x u_n + u_n \partial_x u_n + \partial_x^3 u_n)(t,x) = 0, & \forall x \in (0,\ell_n),\ t>0,\ n = 1,\cdots,N, \\ u_n(t,0) = u_{n'}(t,0), & \forall n, n' = 1, \cdots N, \\ \displaystyle\sum_{n=1}^N \partial_x^2 u_n(t,0) = -\alpha u_1(t,0) - \frac{N}{3} u_1^2(t,0), & t>0, \\ u_n(t,\ell_n) = \partial_x u_n(t,\ell_n) = 0, & t>0,\ n=1,\cdots,N, \\ u_n(0,x) = u_n^0(x), & x \in (0,\ell_n), \end{cases} \tag{KdV-N}$$

In Ammari and Crépeau [2018], Cerpa et al. [2020] the next result was derived:

Theorem

Let $(\ell_n)_{n=1,\cdots N} \in (0,\infty)^N$, $\alpha \geq \frac{N}{2}$ and $T>0$. Then there exist $\epsilon > 0$ and $C>0$ such that for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq \epsilon$, there exists a unique solution of (KdV-N) that satisfies $\|\underline{u}\|_{\mathbb{B}_T} \leq C\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$.

- The main problem to get a global well-posedness result is the action of the nonlinear boundary condition on the central node.

Related results

Similar boundary conditions appear in the work Rosier [2004], Caicedo and Zhang [2017]. The system that they studied was the next one

$$\begin{cases} \partial_t u(t,x) + \partial_x u(t,x) + u(t,x)\partial_x u(t,x) + \partial_x^3 u(t,x) = 0, & \forall x \in (0,L),\ t>0, \\ \partial_x^2 u(t,0) = -u(t,0) + \frac{1}{6}u^2(t,0) + h(t), & t>0, \\ u(t,L) = \partial_x u(t,L) = 0, & t>0, \\ u(0,x) = \phi(x), & x \in (0,L), \end{cases} \quad (2)$$

Theorem

Let $T>0$ and $\gamma>0$ be given. There exists $T^* \in (0,T]$ such that for any $\phi \in L^2(0,L)$ and $h \in H^{-\frac{1}{3}}(0,T)$ satisfying, $\|\phi\|_{L^2(0,L)} + \|h\|_{H^{-\frac{1}{3}}(0,T)} \leq \gamma$. Then the problem (2) admits a unique solution $u \in C([0,T^*];L^2(0,L)) \cap L^2(0,T^*;H^1(0,L))$. Moreover, the corresponding solution map is Lipschitz continuous and the solution possesses the hidden regularities (the Sharp Kato smoothing properties) $\partial_x^\kappa u \in L_x^\infty(0,L;H^{\frac{1-\kappa}{3}}(0,T^*)),\ \kappa = 0,1,2.$

Well-posedness result

The first main result of our work is the following global in time well-posedness theorem.

Theorem

Let $(\ell_n)_{n=1,\cdots N} \in (0,\infty)^N$, $\alpha \geq \dfrac{N}{2}$ and $T > 0$. Then, for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, then there exists a unique solution $\underline{u} \in \mathbb{B}_T$ of (KdV-N). Moreover, there exists $0 < T^* \leq T$ and $C > 0$ such that $\underline{u} \in \mathbb{Y}_{T*}$ and $\|\underline{u}\|_{\mathbb{Y}_{T*}} \leq C\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$.

Note that our result generalize the previous well-posedness results obtained for (KdV-N) in the sense that the smallness assumption on the initial data is not needed.
The scheme of the proof is the next:

- Follow Caicedo and Zhang [2017] to obtain similar hidden regularity for a linear system.
- In order to deal with the nonlinear part, we use a fixed point argument to obtain local in time well-posedness in $\mathbb{Y}_{T*}$.
- Finally, we use an energy estimation to obtain a global well-posedness in time in $\mathbb{B}_T$.

Sketch of the proof WP

We start by considering the following linear system for the KdV equation on a star-shaped network $\mathcal{T}$ and formally we apply the usual Laplace Transform

$$\begin{cases} \partial_t u_n + \partial_x^3 u_n = 0, & n = 1, \cdots, N, \\ u_n(t,0) = u_{n'}(t,0), & \forall n, n' = 1, \cdots N, \\ \displaystyle\sum_{n=1}^{N} \partial_x^2 u_n(t,0) = h(t), & \\ u_n(t,\ell_n) = 0, \quad \partial_x u_n(t,\ell_n) = 0, & n = 1, \cdots, N, \\ u_n(0,x) = 0, & n = 1, \cdots, N, \end{cases} \qquad \begin{cases} s\hat{u}_n + \partial_x^3 \hat{u}_n = 0, & n = 1, \cdots, N, \\ \hat{u}_n(s,0) = \hat{u}_{n'}(s,0), & \forall n, n' = 1, \cdots N, \\ \displaystyle\sum_{n=1}^{N} \partial_x^2 \hat{u}_n(s,0) = \hat{h}(s), & \\ \hat{u}_n(s,\ell_n) = 0, \quad \partial_x \hat{u}_n(s,\ell_n) = 0, & n = 1, \cdots, N, \\ \hat{u}_n(0,x) = 0, & n = 1, \cdots, N, \end{cases} \tag{3}$$

where

$$\hat{u}_n(s,x) = \int_0^\infty e^{-st} u_n(t,x)dt, \quad \hat{h}(s) = \int_0^\infty e^{-st} h(t)dt, \quad \forall x \in (0,\ell_n).$$

Following Bona et al. [2003] we can see that the N component solutions to (3) can be written as

$$\hat{u}_n(s,x) = \sum_{j=1}^{3} c^N_{3(n-1)+j}(s) e^{\lambda_j(s)}, \tag{4}$$

where $\lambda_j(s)$, $j = 1, 2, 3$ are the solutions of the characteristic equation $s + \lambda^3 = 0$.

Sketch of the proof WP

Using the boundary conditions, the coefficients $c^N = (c_k)^N_{k=1,\cdots,3N}$ solves the following linear system $A_N c^N = \hat{h} e_1$. Where, $A_N \in M_{3N}$ can be decomposed in blocks as:

$$A_N = \begin{bmatrix} A_{N-1} & B_N \\ C_N & D_N \end{bmatrix}, \tag{5}$$

for an appropriate choice of B_N, C_N and D_N.
To compute $c^N_{3(n-1)+j}$, let $\Delta^N(s)$ be the determinant of A_N and $\Delta^N_{3(n-1)+j}(s)$ be the determinant of the matrix that is obtained by replacing the column $3(n-1)+j$ of the matrix A_N by $[1\ 0 \ldots\ 0]^T$. If $\Delta^N(s) \neq 0$, Cramer's Rule implies that

$$c^N_{3(n-1)+j}(s) = \frac{\Delta^N_{3(n-1)+j}(s)}{\Delta^N(s)} \hat{h}(s). \tag{6}$$

Formally, taking the inverse of the Laplace Transform of $\hat{u}_n$ we get $t > 0$ and $x \in (0, \ell_n)$, we get

$$u_n(t,x) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} \frac{\Delta^N_{3(n-1)+j}(s)}{\Delta^N(s)} \hat{h}(s) e^{\lambda_j(s)x} ds.$$

Sketch of the proof WP

Notation: $f^+(\rho) = f(i\rho^3)$ and $f^-(\rho) = \overline{f^+(\rho)}$. Thus, $u_n(t,x)$ can be seen as

$$u_n(t,x) = \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^\infty e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \frac{\Delta^{N,+}_{3(n-1)+j}(\rho)}{\Delta^{N,+}(\rho)} \hat{h}^+(\rho) 3\rho^2 d\rho$$
$$+ \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^\infty e^{-i\rho^3 t} e^{\lambda_j^-(\rho)x} \frac{\Delta^{N,-}_{3(n-1)+j}(\rho)}{\Delta^{N,-}(\rho)} \hat{h}^-(\rho) 3\rho^2 d\rho$$

Our idea now is to obtain estimates for u_n, for that we are going to prove some asymptotic properties for $\frac{\Delta^{N,+}_{3(n-1)+j}(\rho)}{\Delta^{N,+}(\rho)}$, the following proposition collects these properties.

Proposition

The following asymptotic properties hold, for $\rho \to \infty$

$$\frac{\Delta^{N,+}_{3(n-1)+1}}{\Delta^{N,+}} \sim -\delta_N \rho^{-2} e^{-\frac{1}{2}\rho\sqrt{3}\ell_n - i\frac{3}{2}\rho\ell_n}, \quad \frac{\Delta^{N,+}_{3(n-1)+2}}{\Delta^{N,+}} \sim \delta_N \rho^{-2} e^{-\rho\sqrt{3}\ell_n + i\frac{\pi}{3}}, \quad \frac{\Delta^{N,+}_{3(n-1)+3}}{\Delta^{N,+}} \sim \delta_N \rho^{-2} e^{-i\frac{\pi}{3}},$$
$$\sum_{j=1}^{3} \frac{\Delta^{N,+}_{3(n-1)+j}}{\Delta^{N,+}} \sim \delta_N \rho^{-2} e^{-i\frac{\pi}{3}}, \quad n = 1, \cdots, N, \tag{7}$$

where $\delta_N > 0$ only depends on N.

Sketch of the proof WP

Classical methods gives us:

$$\begin{cases} \partial_t v_n(t,x) + \partial_x^3 v_n(t,x) = f_n(t,x) & \forall x \in (0,\ell_n),\ t>0,\ n=1,\cdots,N, \\ v_n(t,0) = v_{n'}(t,0), & \forall n,n' = 1,\cdots N, \\ \sum_{n=1}^N \partial_x^2 v_n(t,0) = h(t), & t>0, \\ v_n(t,\ell_n) = 0, \quad \partial_x v_n(t,\ell_n) = 0, & t>0,\ n=1,\cdots,N, \\ v_n(0,x) = v_n^0, & x \in (0,\ell_n). \end{cases} \tag{8}$$

Proposition

Let $T>0$ be given, then, for any $\underline{v}^0 \in \mathbb{L}^2(\mathcal{T})$, $h \in H^{-\frac{1}{3}}(0,T)$ and $\underline{f} \in L^1(0,T;\mathbb{L}^2(\mathcal{T}))$, then the problem (8) admits a unique solution $\underline{v} \in \mathbb{Y}_T$. Moreover, there exists $C>0$ such that

$$\|\underline{v}\|_{\mathbb{Y}_T} \leq C\left(\|h\|_{H^{-\frac{1}{3}}(0,T)} + \|\underline{f}\|_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))} + \|\underline{v}^0\|_{\mathbb{L}^2(\mathcal{T})}\right).$$

Sketch of the proof WP

Lemma

(Lemma 3.1 Bona et al. [2003])There exists a constant $C > 0$ such that for any $T > 0$ and u, $v \in Y_T$

$$\int_0^T \|u(t,\cdot)\partial_x v(t,\cdot)\|_{L^2(0,L)} dt \leq C(T^{1/2} + T^{1/3})\|u\|_{Y_T}\|v\|_{Y_T}.$$

where Y_T is $\mathbb{Y}_T$ for $N = 1$.

Lemma

(Lemma 3.2 Jia and Zhang [2012]) There exists a constant C, $\beta > 0$ such that for any $T > 0$ and $g_1, g_2 \in H^{\frac{1}{3}}(0,T)$, $g_1 g_2 \in H^{-\frac{1}{3}}(0,T)$ and

$$\|g_1 g_2\|_{H^{-\frac{1}{3}}(0,T)} \leq C T^{\beta} \|g_1\|_{H^{\frac{1}{3}}(0,T)} \|g_2\|_{H^{\frac{1}{3}}(0,T)}.$$

- Using the new regularity, we can prove the local in time well-posedness of the nonlinear equation (KdV-N) using a fixed point approach $\mathbb{Y}_{T*}$.
- Finally, using multipliers estimates and integrations by parts we derive our global in time result $\mathbb{B}_T$. $\square$

Stabilization

- Perla Menzala et al. [2002] exponential stability if $L \notin \mathcal{N}$ where $\mathcal{N}$ is called the set of critical lengths defined by
$$\mathcal{N} = \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}},\ k, l \in \mathbb{N}^* \right\},$$
.
- Perla Menzala et al. [2002], Pazoto [2005] internal stabilization using localized damping in the case, $L \in \mathcal{N}$.
- Marx et al. [2017] where the saturated internal stabilization of a single KdV equation was studied.
- Parada et al. [2022] stability of the KdV equation in a star-shaped network with delayed internal feedback terms.

Saturated stabilization

In what follow, **sat** will correspond either $\mathbf{sat}_{loc}$ or $\mathbf{sat}_2$. In order to consider the saturated stabilization problem, we will study the next system

$$\begin{cases} (\partial_t u_n + \partial_x u_n + u_n \partial_x u_n + \partial_x^3 u_n)(t,x) + \mathbf{sat}(a_n(x) u_n(t,x)) = 0, & x \in (0,\ell_n),\ t>0,\ n=1,\cdots,N, \\ u_n(t,0) = u_{n'}(t,0), & \forall n, n' = 1, \cdots N, \\ \sum_{n=1}^N \partial_x^2 u_n(t,0) = -\alpha u_1(t,0) - \frac{N}{3} u_1^2(t,0), & t>0, \\ u_n(t,\ell_n) = \partial_x u_n(t,\ell_n) = 0, & t>0,\ n=1,\cdots,N, \\ u_n(0,x) = u_n^0(x), & x \in (0,\ell_n). \end{cases} \tag{KdV-S}$$

where the damping terms $(a_n)_{n=1,\cdots,N} \in \prod_{n=1}^N L^\infty(0,\ell_n)$ and $a_n \geq c_n > 0$ in an open nonempty set ω_n of $(0,\ell_n)$ for all $n=1,\cdots,N$. We consider $E(t) = \frac{1}{2}\|\underline{u}\|^2_{\mathbb{L}^2(\mathcal{T})}$.

Theorem

Let $(\ell_n)_{n=1}^N \subset (0,\infty)$ and $R>0$, then for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ with $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$, there exist $C(R)>0$ and $\mu(R)>0$ such that the energy of any solution of (KdV-S) *satisfies $E(t) \leq C(R)E(0)e^{-\mu(R)t}$ for all $t>0$.*

Saturated stabilization

Our idea is to prove the following observability inequality following the works Marx et al. [2017], Parada et al. [2022].

$$\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} \leq C\left((2\alpha - N)\int_0^T |u_1(t,0)|^2 dt + \sum_{n=1}^{N}\int_0^T |\partial_x u_n(t,0)|^2 dt \right. \\ \left. + \sum_{n=1}^{N}\int_0^T\int_0^{\ell_n} \textbf{sat}(a_n u_n) u_n dx dt\right). \tag{Obs}$$

Arguing by contradiction and using compactness ideas, we obtain a function $\underline{v}$ that satisfies $\|\underline{v}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1$ and the following equation for $\lambda \geq 0$

$$\begin{cases} \partial_t v_n + \partial_x v_n + \partial_x^3 v_n + \lambda v_n \partial_x v_n = 0, & \forall x \in (0,\ell_n),\ t>0,\ n = 1,\cdots,N, \\ v_n(t,\ell_n) = \partial_x v_n(t,\ell_n) = \partial_x v_n(t,0) = 0, & t \in (0,T), \forall n = 1,\cdots N, \\ (2\alpha - N) v_n(t,0) = 0, & t \in (0,T), \\ v_n(t,x) = 0, & (t,x) \in (0,T)\times\omega_n. \end{cases} \tag{9}$$

1. If $\lambda = 0$, the system satisfied by $\underline{v}$ is linear, then we can use Holmgrem's Theorem to conclude that $\underline{v} = 0$.
2. If $\lambda > 0$, in this case, we use the unique continuation property of Saut and Scheurer [1987] to derive $\underline{v} = 0$.

In any case, we have a contradiction which gives us the observability and hence the exponential stability. □

Acts only in critical lengths

Let $I_c = \{n \in \{1, \cdots, N\};\ \ell_n \in \mathcal{N}\}$ be the set of critical lengths and I_c^* be the subset of, I_c where we remove one index. We now consider the following problem.

$$\begin{cases} (\partial_t u_n + \partial_x u_n + \partial_x^3 u_n)(t,x) + \mathbf{sat}(a_n(x)u_n(t,x)) = 0, & x \in (0,\ell_n),\ t > 0,\ n = 1,\cdots,N, \\ u_n(t,0) = u_{n'}(t,0), & \forall n, n' = 1,\cdots N, \\ \sum_{n=1}^N \partial_x^2 u_n(t,0) = -\alpha u_1(t,0), & t > 0, \\ u_n(t,\ell_n) = \partial_x u_n(t,\ell_n) = 0, & t > 0,\ n = 1,\cdots,N, \\ u_n(0,x) = u_n^0(x), & x \in (0,\ell_n). \end{cases} \tag{LKdV-S}$$

where the damping $(a_n)_{n=1,\cdots,N} \in \prod_{n=1}^N L^\infty(0,\ell_n)$ satisfy

$$\begin{cases} a_n = 0 \ \text{ for } \ n \in \{1\cdots,N\}\backslash I_c^*, \\ a_n \geq c_n \ \text{ in an open nonempty set } \omega_n \text{ of } (0,\ell_n),\ \text{for all } n \in I_c^*, \\ \text{and } c_n > 0 \text{ is a constant.} \end{cases} \tag{10}$$

Then we are able to prove the following global stabilization result.

Theorem

Assume that the damping terms $(a_n)_{n=1,\cdots,N}$ satisfies (10). Let $(\ell_n)_{n=1}^N \subset (0,\infty)$, then for all $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, there exists $C > 0$ and $\mu > 0$ such that the energy of any solution of (LKdV-S) satisfy $E(t) \leq CE(0)e^{-\mu t}$ for all $t > 0$.

Other results for KdV

- Asymptotic behavior of KdV equation in a star-shaped network with bounded and unbounded lengths. (Regularity problems, stabilization using less feedbacks terms than equations). *Work in progress*.
- Null controllability with N controls in the central condition. Null controllability with N internal controls. *Work in progress.*
- Stability analysis of a Korteweg-de Vries equation with saturated boundary feedback. (single equation). *CPDE 2022, Kiel, Germany*.

Thank you for your attention

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