Uniform Turnpike Property and Singular Limits

IX Partial differential equations, optimal design and numerics

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Index

Index

Homogenization Theory

Composite materials are characterized by the fact that they contain two or more finely mixed material.

They have in general a "better" behavior than the average behavior of their individual material.

We can analyze the composite materials in two scales

- Microscopic (Describing the heterogeneities)
- Macroscopic (Describing the global behavior of the composite)

From the macroscopic point of view, the composite looks like a "homogeneous" material.

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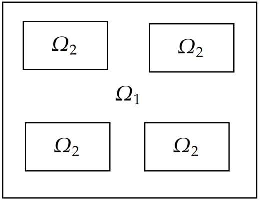
Example: Temperature of a composite material

Consider the heterogeneous material $\Omega = \Omega_1 \cup \Omega_2$



Also, consider $u(x)$ the temperature of Ω with define

$$\begin{cases} -div(a(x)\nabla u) = f(x), & \Omega = \Omega_1 \cap \Omega_2, \\ u = 0 & \partial\Omega. \end{cases}$$

with

$$a(x) = \begin{cases} a_1, & \text{if } x \in \Omega_1 \\ a_2, & \text{if } x \in \Omega_2 \end{cases}$$

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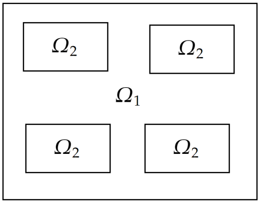

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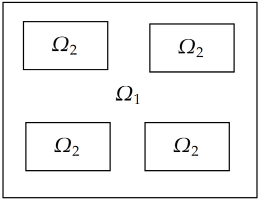

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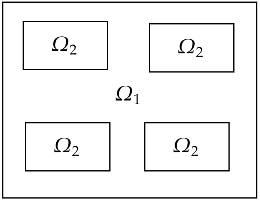

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Since we are interested in the case were the material are finely mixed, we can consider the equation

$$\begin{cases} -div\left(a\left(\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon}\right) = f(x), & \Omega, \\ u^{\varepsilon} = 0 & \partial\Omega. \end{cases}$$

Here $a(\cdot)$ can be a periodic function.

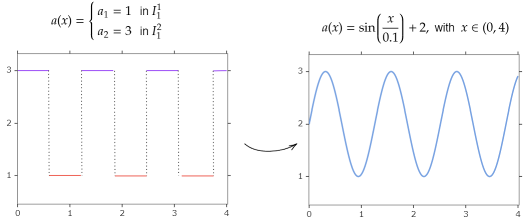

Figura: The function $\sin(\cdot)$ approximate the periodic piecewise constant function.

Example: Temperature of a composite material

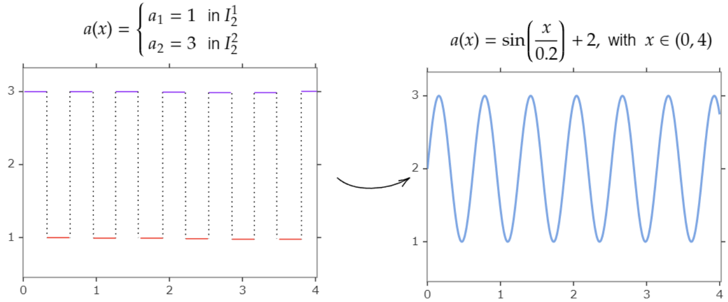

Figura: If the material are more mixed, we can increase the denominator of the sin(·).

Example: Temperature of a composite material

How does u^ε behave when $\varepsilon \to 0$?

If we consider a periodic function $a(x/\varepsilon) \in L^p(\mathbb{R})$ with period I, we have that

$$a(x/\varepsilon) \to \mathcal{M}(a) := \frac{1}{|I|}\int_I a(y)dy, \text{ weakly in } L^p$$

when $\varepsilon \to 0$. Then a natural candidate for u^ε when $\varepsilon \to 0$ could be $u_{\mathcal{M}}$ the solution of

$$\begin{cases} -div(\mathcal{M}(a)\nabla u_{\mathcal{M}}) = f(x), & \Omega, \\ u_{\mathcal{M}} = 0 & \partial\Omega. \end{cases}$$

However, this analysis is wrong!

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Theorem (Bensoussan, Lions and Papanicolaou [1978])

Assume that $a(\cdot)$ is an elliptic and periodic function. Consider the homogenized system

$$\begin{cases} -div\left(\left(\mathcal{M}(a^{-1})\right)^{-1}\nabla u_H\right) = f(x), & \Omega, \\ u_H = 0 & \partial\Omega. \end{cases}$$

The we have that

$$u^\varepsilon \to u_H \text{ weakly in } H_0^1(\Omega).$$

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Problem Setting

Let us consider

$$\begin{cases} y_t^\varepsilon - Div(a(\frac{x}{\varepsilon})\nabla y^\varepsilon) = \chi_\omega f^\varepsilon & (x,t) \in \Omega \times (0,T), \\ y^\varepsilon(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T), \\ y^\varepsilon(x,0) = y_0 & x \in \Omega, \end{cases}$$

with initial condition $y_0 \in L^2(\Omega)$ and the control $f^\varepsilon \in L^2(0,T;\Omega)$ for each ε. Also $\omega \subset \Omega$ is a nonempty set. We assume that the function $a(\cdot)$ satisfy the elliptic condition

$$0 < a_0 \le a(x) \le a_1 < \infty \quad \text{a.e. in } \mathbb{R}^n.$$

Then the equation admits a unique solution y^ε in the class $W(0,T)$ for each $\varepsilon > 0$ *(see Lions, J. Optimal control of systems governed by partial differential equations.)*.

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Uniform null Controllability

In addition, if we assume that $a \in C^1(\mathbb{R}^n)$ has uniformly bounded Lipschitz constant

$$\|Da(x)\|_{L^\infty(\mathbb{R}^n)} \leq a_2,$$

by the classical Carleman estimation developed in

Fursikov, A. & Imanuvilov, O. Controllability of evolution equations (1996).
Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul.

We can ensure that the heat equation is uniform null controllable.

Uniform null controllability

Theorem (Uniform null controllability)

Assume that $a \in C^1(\mathbb{R}^n)$ is an elliptic function with uniformly bounded Lipschitz constant. Let $T > 0$. Then for any $y_0 \in L^2(\Omega)$ and $\varepsilon > 0$, the solution highly oscillation heat equation satisfy the null controllability property. That is, there exists a control $f_0^\varepsilon \in L^2(0, T; \Omega)$ such that

$$y^\varepsilon(x, T) = 0, \quad \forall x \in \Omega.$$

Furthermore, the control satisfy

$$\|f_0^\varepsilon\|_{L^2(0,T;\Omega)} \leq C\|y_0\|_{L^2(\Omega)},$$

for every ε, with a constant $C > 0$ independent of ε.

Uniform null controllability

- In the one dimensional case ($\Omega \subset \mathbb{R}$), we can relax the hypotheses under $a(\cdot)$. The later theorem still holds only assuming that $a \in L^\infty(\mathbb{R})$ is bounded.
- The uniform null controllability going to be a main ingredient in order to conclude the uniform turnpike property.

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- The uniform null controllability going to be a main ingredient in order to conclude the uniform turnpike property.

Contenidos

Evolutive control problem

We are interested in analyze this problem in the context of homogenization, consider the following optimal control problem

$$\min_{f^{\varepsilon}\in L^2(0,T;\Omega)}\left\{J_{\varepsilon}^{T}(f^{\varepsilon})=\frac{1}{2}\int_0^T \|f^{\varepsilon}(\cdot,t)\|^2_{L^2(\Omega)}dt+\|y^{\varepsilon}(\cdot,t)-y_d(\cdot)\|^2_{L^2(\Omega)}dt\right\},$$

where $y_d \in L^2(\Omega)$ is a given target, and y^{ε} solution of the heat equation.

Stationary optimal control

In order to analyze the turnpike property consider the optima control problem time independent

$$\min_{f^{\varepsilon}\in L^2(\Omega)}\left\{J_{\varepsilon}^{s}(f^{\varepsilon})=\frac{1}{2}\left(\|f^{\varepsilon}(\cdot)\|^2_{L^2(\Omega)}+\|y(\cdot)-y_d(\cdot)\|^2_{L^2(\Omega)}\right)\right\}.$$

Here $y_d \in L^2(\Omega)$ is the same target as in the previous optimization problem, and y satisfy the elliptic equation

$$\begin{cases}-Div(a(\frac{x}{\varepsilon})\nabla y)=\chi_{\omega}f & x\in\Omega,\\ y(x)=0 & x\in\partial\Omega.\end{cases}$$

which has a unique solution $y \in H_0^1(\Omega)$.

Classical results

The evolutive and stationary problems admits a unique solution by virtue of the direct method in the calculus of variations. Let us call this optimal variables as $(y^\varepsilon, f^\varepsilon)$ and $(\overline{y}^\varepsilon, \overline{f}^\varepsilon)$ respectively.

Let us begin by introducing the adjoint states, which allows us to characterize the optimal control of the problems.

Lemma (Adjoint stationary state)

The optimal control $\overline{f}^\varepsilon$ is characterized by $\overline{f}^\varepsilon = -\chi_\omega \overline{\psi}^\varepsilon$ with $\overline{\psi}^\varepsilon$ solution of

$$\begin{cases} -Div(a(\frac{x}{\varepsilon})\nabla\overline{\psi}^\varepsilon) = \overline{y}^\varepsilon - y_d & x \in \Omega, \\ \overline{\psi}^\varepsilon(x) = 0 & x \in \partial\Omega, \end{cases}$$

where $\overline{y}^\varepsilon$ is the optimal state associate to $\overline{f}^\varepsilon$.

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where $\overline{y}^\varepsilon$ is the optimal state associate to $\overline{f}^\varepsilon$.

Classical results

Lemma (Adjoint evolutive state)

The optimal control f^ε can be characterized by the identity $f^\varepsilon = -\chi_\omega \psi^\varepsilon$, where ψ^ε satisfy

$$\begin{cases} -\psi_t^\varepsilon - Div(a(\frac{x}{\varepsilon})\nabla\psi^\varepsilon) = y^\varepsilon - y_d & (x,t) \in \Omega \times (0,T), \\ \psi^\varepsilon(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T), \\ \psi^\varepsilon(x,T) = 0 & x \in \Omega, \end{cases}$$

with y^ε the optimal state associate to f^ε.

Our goal

We aim to prove the exponential turnpike property with uniform constant. That is prove the inequality

$$\|y^{\varepsilon}(\cdot,t)-\overline{y}^{\varepsilon}(\cdot)\|_{L^2(\Omega)}+\|f^{\varepsilon}(\cdot,t)-\overline{f}^{\varepsilon}(\cdot)\|_{L^2(\Omega)}\leq C(e^{-\mu t}+e^{-\mu(T-t)}),$$

for every $t\in(0,T)$, where C and μ are two positive constants independent of T and ε.

Contenidos

Integral Turnpike Property

Theorem:

Assume that $a \in C^1(\mathbb{R}^n)$ is a function satisfy the elliptic condition and has uniformly bounded Lipschitz constant. Consider the optimal pairs $(y^\varepsilon, f^\varepsilon)$ and $(\overline{y}^\varepsilon, \overline{f}^\varepsilon)$. Then we have

$$\left\| \frac{1}{T}\int_0^T y^\varepsilon(\cdot,t)dt - \overline{y}^\varepsilon(\cdot) \right\|_{L^2(\Omega)} + \left\| \frac{1}{T}\int_0^T f^\varepsilon(\cdot,t)dt - \overline{f}^\varepsilon(\cdot) \right\|_{L^2(\Omega)} \leq \frac{C}{T},$$

with C a positive constant independent of T and ε.

Proof Idea: Use energy estimations.

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Exponential Turnpike Property

Theorem (Exponential turnpike property)[M.H, E.Zuazua]

Assume that $a \in C^1(\mathbb{R}^n)$ is a function satisfy the elliptic condition and has uniformly bounded Lipschitz constant. Let us consider the optimal pairs $(y^\varepsilon, f^\varepsilon)$ and $(\overline{y}^\varepsilon, \overline{f}^\varepsilon)$. Then there exist two constants $C > 0$ and $\mu > 0$ independent of T and ε such that

$$\|y^\varepsilon(\cdot, t) - \overline{y}^\varepsilon(\cdot)\|_{L^2(\Omega)} + \|f^\varepsilon(\cdot, t) - \overline{f}^\varepsilon(\cdot)\|_{L^2(\Omega)} \leq C(e^{-\mu t} + e^{-\mu(T-t)}),$$

for every $t \in (0, T)$.

Proof Sketch:

Porretta, A. & Zuazua, E.
Long time versus steady state optimal control (2013).
SIAM J. Control Optim..

The proof is based on a decoupling strategy, for which is necessary to introduce Riccati operators. The main step is prove the following proposition

Proposition

Denote by $\mathcal{E}^{\varepsilon}(t)$ and $\hat{E}^{\varepsilon}$, the evolutive and stationary Riccati operators. Then there exist two positive constant μ and C independent of T and ε such that

$$\|\mathcal{E}^{\varepsilon}(t) - \hat{E}^{\varepsilon}\|_{\mathcal{L}(L^2(\Omega))} \leq Ce^{-\mu t},$$

for any $t > 0$.

To obtain uniform constant above, is essential the uniform null controllability.

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Contenidos

Singular Limit of the Optimal Control Problems

We are interested in address the question if we can pass the limit in the exponential turnpike inequality. Consider the homogenized equation (Evolutive and stationary)

$$\begin{cases} y_t - Div(A\nabla y) = f & (x,t) \in \Omega \times (0,T), \\ y(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T), \\ y(x) = y_0(x) & x \in \Omega, \end{cases}, \quad \begin{cases} -Div(A\nabla y) = f & x \in \Omega, \\ y(x) = 0 & x \in \partial\Omega. \end{cases}$$

Let us denote by (y, f) and $(\overline{y}, \overline{f})$ the optimal variables of the evolutive and stationary problem, subject to the previous equation (respectively).

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$$\begin{cases} y_t - Div(A\nabla y) = f & (x,t) \in \Omega \times (0,T), \\ y(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T), \\ y(x) = y_0(x) & x \in \Omega, \end{cases}, \quad \begin{cases} -Div(A\nabla y) = f & x \in \Omega, \\ y(x) = 0 & x \in \partial\Omega. \end{cases}$$

Let us denote by (y, f) and $(\overline{y}, \overline{f})$ the optimal variables of the evolutive and stationary problem, subject to the previous equation (respectively).

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Then we are interested in prove that

$$\|y^{\varepsilon}(\cdot,t)-\overline{y}^{\varepsilon}(\cdot)\|_{L^2(\Omega)}+\|f^{\varepsilon}(\cdot,t)-\overline{f}^{\varepsilon}(\cdot)\|_{L^2(\Omega)} \to \|y(\cdot,t)-\overline{y}(\cdot)\|_{L^2(\Omega)}+\|f(\cdot,t)-\overline{f}(\cdot)\|_{L^2(\Omega)}$$

when $\varepsilon \to 0$.

Singular Limit of the Optimal Control Problems

Theorem (Brahim-Otsmane, Francfort, Murat.)[1992]

Let us consider in the previous equation, and sequence of controls $f^\varepsilon \in L^2(0,T;\Omega)$. Then for y the solution of

$$\begin{cases} u_t - div\left(\left(\mathcal{M}(a^{-1})\right)^{-1}\nabla u\right) = \chi_\omega f^\varepsilon(x), & \Omega, \\ u = 0 & \partial\Omega. \end{cases}$$

we have

1. If f^ε weakly converges in $L^2(0,T;\Omega)$ to f, the solutions u^ε satisfy

$$u^\varepsilon \rightharpoonup u \text{ weakly-* in } L^\infty(0,T;L^2(\Omega)) \text{ as } \varepsilon \to 0.$$

2. If f^ε strongly converges in $L^2(0,T;\Omega)$ to f, the solutions u^ε satisfy

$$u^\varepsilon \to u \text{ strongly in } C([0,T];L^2(\Omega)) \text{ as } \varepsilon \to 0.$$

Singular Limit of the Optimal Control Problems

Lemma

Let us consider (y, f) and $(\overline{y}, \overline{f})$ the optimal pairs of the optimization problems, subject to the homogenized equation respectively. Then we have that

$$f^{\varepsilon} \to f \text{ strongly in } L^2(0, T; \Omega), \text{ and } y^{\varepsilon} \to y \text{ strongly in } C([0, T]; L^2(\Omega)).$$

Also, for the stationary states we have that

$$\overline{f}^{\varepsilon} \to \overline{f}, \text{ and } \overline{y}^{\varepsilon} \to \overline{y} \text{ strongly in } L^2(\Omega).$$

Singular Limit of the Turnpike Property

Corollary:

Let (y, f) and $(\overline{y}, \overline{f})$ the optimal variable of the optimization problems evolutive ans stationary variables subject to the homogenized equations respectively. Then there exist two positive constants C and μ, independents of T and ε such that

$$\|y(\cdot, t) - \overline{y}(\cdot)\|_{L^2(\Omega)} + \|f(\cdot, t) - \overline{f}(\cdot)\|_{L^2(\Omega)} \leq C(e^{-\mu t} + e^{-\mu(T-t)}),$$

for every $t \in (0, T)$.

In other words, the previous corollary ensures that we can pass the limit in the turnpike property when $\varepsilon \to 0$.

Contenidos

Numerical Simulations

Let us consider the following optimal control problem

$$\min_{f^{\varepsilon}\in L^2(0,T;(0,1))}\left\{J^T_{\varepsilon}(f^{\varepsilon}) = \frac{1}{2}\int_0^T \|f^{\varepsilon}(\cdot,t)\|^2_{L^2(0,1)}dt + \|y^{\varepsilon}(\cdot,t)-1\|^2_{L^2(0,1)}dt\right\}.$$

where y^{ε} is the solution of the highly oscillatory heat equation with $T{=}100$.

We introduce the corresponding stationary optimization problem

$$\min_{\overline{f}^{\varepsilon}\in L^2(0,1)}\left\{J^s(\overline{f}^{\varepsilon}) = \frac{1}{2}\left(\|\overline{f}^{\varepsilon}(\cdot)\|^2_{L^2(0,1)} + \|\overline{y}^{\varepsilon}(\cdot)-1\|^2_{L^2(0,1)}\right)\right\}.$$

where in this case $\overline{y}^{\varepsilon}$ is the solution of the respective stationary system.

The previous problems with the periodic function

$$a(x) = \sin^2(x\pi) + 0{,}5.$$

Numerical Simulations

Let us consider the following optimal control problem

$$\min_{f^\varepsilon \in L^2(0,T;(0,1))} \left\{ J_\varepsilon^T(f^\varepsilon) = \frac{1}{2}\int_0^T \|f^\varepsilon(\cdot,t)\|^2_{L^2(0,1)} dt + \|y^\varepsilon(\cdot,t) - 1\|^2_{L^2(0,1)} dt \right\}.$$

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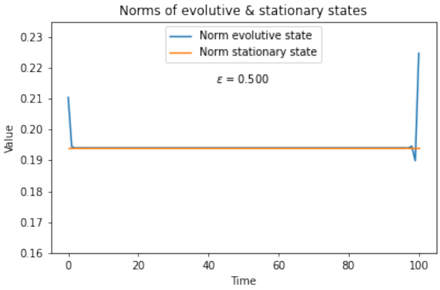
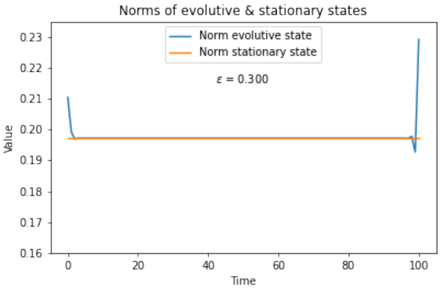
Numerical Simulations

Using Gekko library in Python, we obtain the following simulations

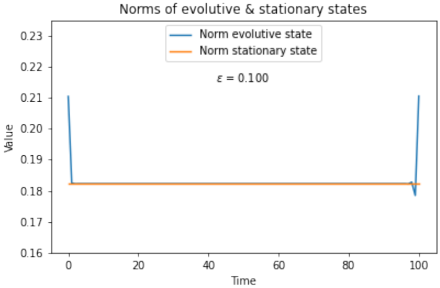


(a) Evolutive vs Stationary with $\varepsilon = 0{,}5$.

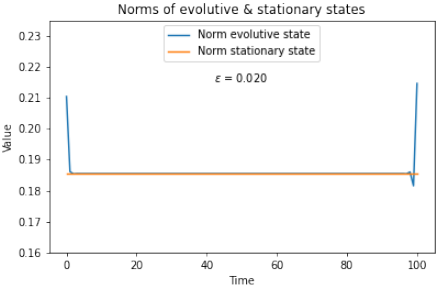


(b) Evolutive vs Stationary with $\varepsilon = 0{,}3$.

Numerical Simulations



(a) Evolutive vs Stationary with $\varepsilon = 0{,}1$.



(b) Evolutive vs Stationary with $\varepsilon = 0{,}02$.

Numerical Simulations

$$\|y^\varepsilon(t) - \overline{y}^\varepsilon\|_{L^2(\Omega)} + \|f^\varepsilon(t) - \overline{f}^\varepsilon\|_{L^2(\Omega)} \leq K(e^{-\mu t} + e^{-\mu(T-t)})$$

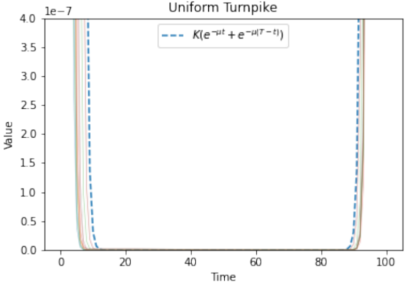


Figura: Left-hand side of the turnpike exponential inequality, for different values of $\varepsilon \in (0,1)$.

Here the constants are fixed. The value is $K = 0{,}1$ and $\mu = 1{,}5$.

Numerical Simulations

Let analyze the homogenized system. The coefficient is given by

$$\mathcal{M}(a^{-1})^{-1} = \left(\frac{1}{|\Omega|} \int_{\Omega} \frac{1}{a(x)} dx \right)^{-1}$$

In our example $\mathcal{M}(a^{-1})^{-1} \approx 0{,}86603$.

Solving the optimal control problems (evolutive and stationary) subject to the homogenized equations with coefficient a_H, we have

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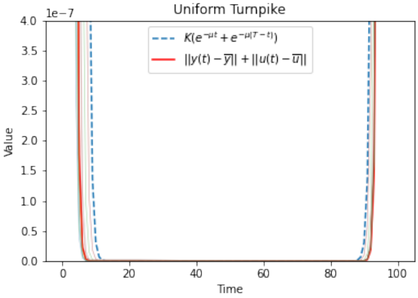
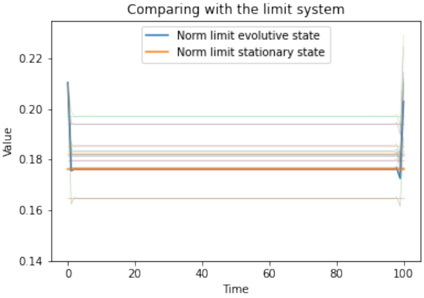


(a) Exponential turnpike property with constant $K = 0{,}1$ and $\mu = 1{,}5$.



(b) Optimal states with different values of ε.

We can observe that even in the limit case when $\varepsilon \to 0$, the turnpike property still holds with the same constants. In particular, we can observe the turnpike property for the homogenized system.

Index

Conclusions

The main conclusions of this work are the followings:

1. In our setting, uniform controllability property implies the uniform turnpike property.
2. Once we have a uniform turnpike property and if we know results from homogenization theory that guarantee that we can pass the limit in the ε, then we can conclude the turnpike property for the limit system.
3. The conclusion of this work can be extended easily to some general lineal parabolic equations.

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Open problems

1. It would be interesting to investigate similar questions or the equations

$$\begin{cases} \varepsilon y^\varepsilon_{tt} + y^\varepsilon_t - \Delta y^\varepsilon + y^\varepsilon = f^\varepsilon & \text{in } \Omega \times (0,T), \\ y^\varepsilon = 0 & \text{on } \partial\Omega \times (0,T), \\ y^\varepsilon(x,0) = y_0(x), \quad y^\varepsilon_t(x,0) = y_1(x) & \text{in } \Omega. \end{cases}$$

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Thanks for your attention