Nodal Profile Control for 1-D Wave Equation

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Chair
DYNAMICS, CONTROL
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FAU

Plan of the presentation

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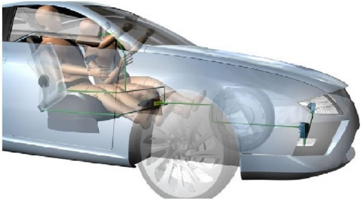
Control theory

Control theory is an interdisciplinary field of research that relates mathematical laws with rich applications in engineering, economics, and other sciences. It is nowadays a rich crossing point of engineering and mathematics.

Control theory is used to enhance production, efficiency and safety in many areas such as agriculture, military, nuclear power plants, radar tracking system, food processing, economics, traffic system, biology, medicine, radiotherapy, oncology etc.

Control theory



Control theory is especially used in mechanical engineering.

Presentation of Problem

Consider the following variable coefficients controlled 1-d wave equation:

$$\begin{cases} \rho(x)y_{tt} - (a(x)y_x)_x = 0, & 0 < x < L,\ 0 < t < T \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), & 0 < x < L \\ y(0,t) = u(t),\ y(L,t) = 0, & 0 < t < T. \end{cases} \tag{1}$$

In (1), $0 < T < \infty$ stands for the length of the time-horizon, L is the length of the string, $y = y(x,t)$ is the state and $u = u(t)$ is a control that acts on the system through the extreme $x = 0$.

The goal is to answer the following control problem:

Given an initial data $\{y_0(x), y_1(x)\}$ and a target $p(t)$, we want to find a control $u = u(t)$ such that the corresponding solution fulfills:

$$y_x(L,t) = p(t), \quad t \geq 0 \tag{2}$$

Presentation of Problem

Because of the finite-velocity of propagation one does not expect this result to hold for all $T > 0$, but rather only for $T > \tau$ large enough, so that the action of the control at $x = 0$ can reach the other extreme $x = L$ along characteristics, $\tau > 0$ being this waiting.

We assume that the coefficients ρ and a are in BV and to be uniformly bounded above and below by positive constants, *i.e.*

$$0 < \rho_0 \leq \rho(x) \leq \rho_1, \;\; 0 < a_0 \leq a(x) \leq a_1 \;\; \text{a.e. in} \;\; (0, L) \tag{3}$$

and

$$\rho, a \in BV(0, L). \tag{4}$$

Presentation of Problem

This is a non-standard controllability problem since, most often, controllability refers to the possibility of steering the solution to a target in the final time $t = T$.

But our aim is rather to assure that a given trace, the given profile $p = p(t)$, is achieved on the boundary after a waiting time by means of the boundary control.

A list of classical and recent works

- Gugat, Herty and Schleper proposed a new kind of exact boundary controllability, called the nodal profile control (tracking control or sidewise profile control). The goal is to assure that the state fits a given profile on one or some nodes of the network, after a waiting time, by means of boundary controls.
  - M. Gugat, M. Herty, V. Schleper(2011)
- For the purpose of some practical applications, this analysis was extended to 1-D quasilinear hyperbolic systems by a constructive method employing the method of characteristics.
  - T. Li (2010)
  - K. Wang, Q. Gu (2014)
  - T. Li, K. Wang, Q. Gu (2016)
  - K. Zhuang, G. Leugering, T. Li (2019)

Main result

For any given (y_0, y_1) with $y_0 \in L^2(0,L)$ and $\rho y_1 \in H^{-1}(0,L)$ and any $u \in L^2(0,T)$, the system admits a unique solution y, enjoying the regularity property

$$y \in C([0,T]; L^2(0,L)), \ \ \rho y_t \in C([0,T]; H^{-1}(0,L)).$$

The solutions of the system in the above regularity class fulfill the added boundary regularity condition

$$y_x(L,t) \in H^{-1}(0,T).$$

Main result

Theorem (S.-Zuazua, JOTA, 2022)

Consider

$$\begin{cases} \rho(x)y_{tt} - (a(x)y_x)_x = 0, & 0 < x < L,\ 0 < t < T \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), & 0 < x < L \\ y(0,t) = u(t),\ y(L,t) = 0, & 0 < t < T. \end{cases}$$

with coefficients $\rho, a \in BV(0,L)$ *satisfying*

$$0 < \rho_0 \le \rho(x) \le \rho_1,\ \ 0 < a_0 \le a(x) \le a_1 \ \ a.e.\ in\ \ (0,L).$$

Let $T > L\beta$ *with*

$$\beta = \operatorname{ess}\sup_{x\in[0,L]} \sqrt{\frac{\rho}{a}}. \qquad (5)$$

Then, for any $p \in H_*^{-1}(L\beta, T)$ *there exists a control* $u \in L^2(0,T)$ *such that*

$$y_x(L,t) = p(t)\ \ for\ all\ \ t \in (L\beta, T).$$

Main result

- Note that in the present formulation of the sidewise controllability problem the velocity of propagation plays an important role.
- The sidewise controllability property is only guaranteed when $T > L\beta$.
  This is the natural minimal time to achieve our aim since, otherwise, because of the finite velocity of propagation, the action on $x = 0$ will not reach the extreme $x = L$.
- The tracking condition is only assured in the time sub-interval $(L\beta, T)$.

Main result

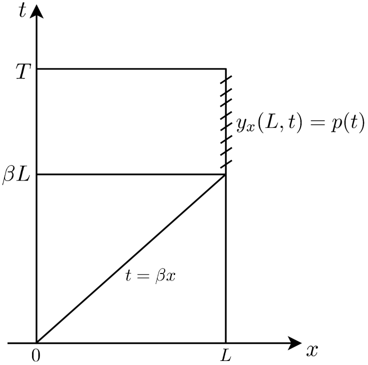


Figure: Sidewise controllability: The figure, which corresponds to the constant coefficient case, represents the time needed for characteristics emanating from $x = 0$ to reach $x = L$, and to assure that the control of the trace to the given profile $p = p(t)$ is achieved for $T \geq L\beta$. The situation is similar for variable coefficients.

The dual sidewise observability problem

Let us now consider the adjoint system:

$$\begin{cases} \rho(x)\psi_{tt} - (a(x)\psi_x)_x = 0, & 0 < x < L,\ 0 < t < T \\ \psi(x,T) = 0,\ \psi_t(x,T) = 0, & 0 < x < L \\ \psi(0,t) = 0,\ \psi(L,t) = s(t), & 0 < t < T \end{cases} \tag{6}$$

where the boundary data is of the form

$$s(t) = \begin{cases} s_0(t), & L\beta \le t \le T \\ 0, & 0 \le t \le L\beta \end{cases} \tag{7}$$

with $s_0 \in H^1(L\beta, T)$, $s_0(L\beta) = 0$.

This system admits a unique finite-energy solution ψ such that

$$(\psi, \frac{\partial \psi}{\partial t}) \in C([0,T], H^1(0,L) \times L^2(0,L))$$

and

$$\psi_x(0,.) \in L^2(0,L).$$

Observability inequality

The question is reduced to analyze whether the observability inequality is true.

$$\|s_0(t)\|_{H^1(L\beta,T)} \leq C_1 \left\|\psi_x(0,t)\right\|_{L_2(0,T)} \tag{8}$$

The dual version of the control problem leads to a non-standard observability inequality for the adjoint wave equation.

The observability inequality involves a non-homogeneous boundary condition at x = L that needs to be estimated out of measurements done at x = 0.

Observability inequality

Proposition (S.-Zuazua, JOTA, 2022)

Let $T > L\beta$ (β is given as in (5)).
Then, there exists $C_1 > 0$ such that

$$\|s_0(t)\|_{H^1(L\beta,T)} \leq C_1 \left\|\psi_x(0,t)\right\|_{L_2(0,T)} \tag{9}$$

is satisfied for every solution of the adjoint system.

The proof of the proposition can be obtained by using the sidewise energy estimates similar as in (Fernández-Cara and Zuazua, 2002) and (Cox and Zuazua, 1995).

Construction of the Control

The control is

$$u(t) = -a(0)\psi_x(0,t)$$

where ψ is the solution of the adjoint system corresponding to boundary condition $s(t) \in H^1_*(0,T)$ minimizing the functional

$$J(s) = \frac{1}{2}\int_0^T \left\{(a\psi_x)(0,t)\right\}^2 dt - a(L) < p(t), s_0(t) >_{H^{-1}\times H^1_*} \qquad (10)$$

in the space $H^1_*(0,T)$ ($H^1_*(0,T)$ is a subspace of the space $H^1(0,T)$ constituted by the functions vanishing in the time sub-interval $(0, L\beta)$).

The solution of the system corresponding to the control $u = u(t)$ fulfills:

$$y_x(L,t) = p(t), \quad L\beta \leq t \leq T, \qquad (11)$$

when the initial data $y_0 \equiv y_1 \equiv 0$.

Construction of the Control

Note that J is convex. The continuity of J is guaranteed by the fact that $\psi_x(0,t) \in L^2(0,T)$.

The observability inequality above guarantees that the functional is also coercive. The Direct method of the Calculus of Variations then ensures that J has a unique minimizer.

Construction of the Control

Remark

Once the control is built for $y_0 \equiv y_1 \equiv 0$, using the linear superposition of solutions of the wave equation, the control for arbitrary initial data can be built.
The functional J above can be also modified so to lead directly the control corresponding to non-trivial initial data.

Other boundary conditions

Our techniques apply to some similar problems with other boundary conditions.

One could for instance consider the same model with Neumann boundary conditions and control:

$$\begin{cases} \rho(x)y_{tt} - (a(x)y_x)_x = 0, & 0 < x < L,\ 0 < t < T, \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), & 0 < x < L, \\ y_x(0,t) = u(t),\ y_x(L,t) = 0, & 0 < t < T. \end{cases} \tag{12}$$

The aim is to find a control $u = u(t)$ such that

$$y(L,t) = p(t), \quad t \geq 0 \tag{13}$$

for a given function $p = p(t)$.
Our methods apply in this case too, leading to similar results with minor changes.

Optimal control

Instead of considering the sidewise controllability problem one could adopt a more classical optimal control approach.

The problem could be formulated as that in which one minimizes a functional of the form

$$\frac{1}{2}\left[\int_0^T u^2(t)dt + \kappa\|y_x(L,t) - p(t)\|^2_{H^{-1}(L\beta,T)}\right],$$

depending on $u \in L^2(0,T)$, with $\kappa > 0$ any penalty parameter.

Optimal controls for this problem exist for all $T > 0$. This is simply due to the quadratic structure of the functional to be minimized, its coercivity and continuity.

Semilinear wave equation

Consider the following system

$$\begin{cases} y_{tt} - y_{xx} + f(y) = 0, & in \ \ (0,L)\times(0,T) \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), & for \ \ x \in (0,L) \\ y(0,t) = u(t),\ y(L,t) = 0, & for \ \ t \in (0,T) \end{cases} \tag{14}$$

where T is a given positive number and f is a given function.

Our aim is to answer the following control problem:

Given a time-horizon $T > 0$, an initial data $(y_0(x), y_1(x))$ and a target $p(t)$, we want to find $u(t)$ such that

$$y_x(L,t) = p(t), \quad t \geq 0$$

under nonlinearity assumptions.

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Thank You!