Integral input-to-state stability of unbounded bilinear control systems

joint work with Birgit Jacob (Wuppertal) and Felix Schwenninger (Twente)
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IX Partial differential equations, optimal design and numerics
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Outline

The Fokker-Planck equation

$$\begin{cases} \frac{\partial \rho}{\partial t}(x,t) = \nu \Delta \rho(x,t) + \operatorname{div}\Big(\rho(x,t)\nabla V(x,t)\Big), \\ \rho(x,0) = \rho_0(x), \\ 0 = (\nu \nabla \rho + \rho \nabla V) \cdot \vec{n} \quad \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{FP}$$

- $x \in \Omega \subset \mathbb{R}^n$ (bounded open domain, $\partial\Omega$ smooth), $t > 0$,
- $\nu > 0$,
- V is a potential,
- ρ_0 denotes the initial probability distribution with $\int_\Omega \rho_0(x)\,\mathrm{d}x = 1$,
- $\vec{n}$ is the outward-pointing unit normal vector at the boundary.

The Fokker-Planck equation

Langevin equation: motion of a large set of dragged Brownian particles:

$$\mathrm{d}x(s) = y(s)\,\mathrm{d}(s), \quad \mathrm{d}y(s) = -\beta y(s)\,\mathrm{d}s + F(x,s)\,\mathrm{d}s + \sqrt{2\beta kT/m}\,\mathrm{d}B(s),$$

with friction parameter $\beta > 0$, Bolzmann constants k, temperature T, mass of the particle m, n-dimensional Brownian motion B and force F, assumed to be related to a potential V:

$$F = -\nabla V.$$

Smoluchowski equation: approximation for large friction

$$\mathrm{d}x(t) = -\nabla V(x,t)\,\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}B_t,$$

where $t = s/\beta$ and $\nu = \beta kT/m$.
The probability density function ρ is the solution of the Fokker-Planck equation.

The Fokker-Planck equation

[Breiten, Kunisch, and Pfeiffer, "Control Strategies for the Fokker–Planck Equation"]:

$$\begin{cases} \frac{\partial \rho}{\partial t}(x,t) = \nu \Delta \rho(x,t) + \operatorname{div}(\rho(x,t)\nabla W(x)) + \operatorname{div}(u(t)\rho(x,t)\nabla \alpha(x)), \\ \rho(x,0) = \rho_0(x), \\ 0 = (\nu \nabla \rho + \rho \nabla W) \cdot \vec{n} \quad \text{on } \partial\Omega \times (0,\infty) \end{cases} \tag{FP}$$

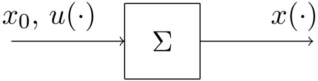
- interactions with the particle by means of an optical tweezer:

$$V(x,t) = W(x) + \alpha(x)u(t)$$

  with $W \in W^{2,\infty}(\Omega)$ and $\alpha \in W^{1,\infty}(\Omega) \cap H^2(\Omega)$ with $\nabla\alpha \cdot \vec{n} = 0$ on $\partial\Omega$,
- stabilization via feedback law.

Goal: Analyze stability & robustness of bilinear controlled systems

Input-to-state stability (ISS)



ISS = internal stability ($u = 0$) + robustness w.r.t. u

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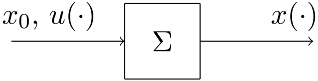

ISS = internal stability $(u = 0)$ + robustness w.r.t. u

Definition ([Sontag, 1989])

For all considered initial values x_0 and inputs u the unique (continuous) solution satisfies

$$\|x(t)\|_X \leq \beta(\|x_0\|, t) + \gamma(\|u\|_{L^\infty(0,t;U)}) \tag{ISS}$$

or

$$\|x(t)\|_X \leq \beta(\|x_0\|_X, t) + \theta\left(\int_0^t \mu(\|u(s)\|_U)\,\mathrm{d}s\right), \tag{integral ISS}$$

for some $\beta \in \mathcal{KL}, \gamma, \theta, \mu \in \mathcal{K}_\infty$ (Lyapunov classes).

Applications:

- Robust feedback design
- Stability of coupled systems/networks

Literature:

- Sontag, "Smooth stabilization implies coprime factorization", 1989
- Sontag, "Comments on integral variants of ISS", 1998
- Karafyllis and Krstic, *Input-to-state stability for PDEs*, 2019
- Mironchenko and Prieur, "Input-to-State Stability of Infinite-Dimensional Systems: Recent Results and Open Questions", 2020
- Schwenninger, "Input-to-state stability for parabolic boundary control: Linear and Semi-linear systems", 2020
- Jacob, Nabiullin, Partington, and Schwenninger, "Infinite-dimensional input-to-state stability and Orlicz spaces", 2018
- Mironchenko and Ito, "Characterizations of integral input-to-state stability for bilinear systems in infinite dimensions", 2016

Linear sytsems & (integral-) ISS

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \ t \geq 0, \\ x(0) = x_0, \end{cases} \qquad (\Sigma(A,B))$$

- state space X, input space U (Banach spaces),
- A generates a C_0-semigroup $T(\cdot)$ on X,

$$X_{-1} = (X, \|(\beta - A)^{-1} \cdot \|_X,)^{\sim}, \quad \beta \in \rho(A),$$

$\rightsquigarrow$ extensions A_{-1}, $T_{-1}(\cdot)$,

- $B \in \mathcal{L}(U, X_{-1})$.

Linear sytsems & (integral-) ISS

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \ t \geq 0, \\ x(0) = x_0, \end{cases} \qquad (\Sigma(A,B))$$

A mild solution of $\Sigma(A,B)$ is a function $x \in C([0,\infty); X)$ such that

$$x(t) = T(t)x_0 + \underbrace{\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s}_{\in X \sim B \text{ admissible}}.$$

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Theorem (Jacob, Nabiullin, Partington, Schwenninger 2018)

For linear systems

- *integral ISS* $\Rightarrow$ *ISS* $\Leftrightarrow$ $T(\cdot)$ *exp. stable &* B L^∞*-admissible,*
- *integral ISS* $\Leftrightarrow$ $T(\cdot)$ *exp. stable &* B E_Φ*-admissible for some Orlicz space* E_Φ.

Idea: Interpret the integral in the (integral ISS)-estimate as a norm.

Young functions

Definition (Young function)

A continuous, increasing and convex function $\Phi : [0, \infty) \to [0, \infty)$ is called Young function if Φ satisfies

$$\lim_{s \to 0} \frac{\Phi(s)}{s} = 0 \quad \text{and} \quad \lim_{s \to \infty} \frac{\Phi(s)}{s} = \infty.$$

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Examples

1. $\Phi(s) = s^p, \qquad 1 < p < \infty,$
2. $\Phi(s) = (1+s)\ln(1+s) - s,$
3. $\Phi(s) = \mathrm{e}^{s^p} - 1, \qquad 1 < p < \infty.$

Orlicz spaces

Definition (Orlicz space)

Let Φ be a Young function

"Orlicz space":

$$\mathrm{L}_\Phi(I;U) := \left\{ u : I \to U \,\middle|\, \int_I \Phi\left(\frac{\|u(s)\|_U}{k}\right) \mathrm{d}s < \infty \text{ for some } k > 0 \right\}$$

Luxemburg norm:

$$\|u\|_{\mathrm{L}_\Phi(I;U)} := \inf\left\{ k > 0 \,\middle|\, \int_I \Phi\left(\frac{\|u(s)\|_U}{k}\right) \mathrm{d}s \le 1 \right\}$$

Orlicz space:

$$\mathrm{E}_\Phi(I;U) := \overline{\{u \in \mathrm{L}^\infty(I;U) \mid \operatorname{ess\,supp} u \text{ is bounded}\}}^{\|\cdot\|_{\mathrm{L}_\Phi(I;U)}}$$

Orlicz spaces

- $\mathrm{L}_\Phi(I;U)$ and $\mathrm{E}_\Phi(I;U)$ are Banach spaces with $\|\cdot\|_{L_\Phi}$,
- $E_\Phi = \mathrm{L}_\Phi \Leftrightarrow \Phi \in \Delta_2$ (growth condition),
- Let Φ, $\widetilde{\Phi}$ be complementary Young functions ($\sim$ Hölder conjugates),

Orlicz-Norm: $$\|u\|_{\Phi,(I;U)} = \sup\left\{ \int_I \|u(s)\||v(s)|\,\mathrm{d}s \,\middle|\, \int_I \widetilde{\Phi}(|v(s)|)\,\mathrm{d}s \leq 1 \right\},$$

- Generalized Hölder inequality:

$$\int_I \|u(s)\|_U \|v(s)\|_V \,\mathrm{d}s \leq 2\|u\|_{\mathrm{L}_\Phi(I;U)} \|v\|_{\mathrm{L}_{\tilde{\Phi}}(I;V)}.$$

Bilinear systems

$$\begin{cases} \dot{x}(t) &= Ax(t) + B_1 F(x(t), u_1(t)) + B_2 u_2(t),\ t \geq 0, \\ x(0) &= x_0, \end{cases} \qquad (\Sigma(A, [B_1, B_2], F))$$

- state space X, input spaces U_1, U_2,
- A generates a C_0-semigroup $T(\cdot)$ on X,
- control operators $B_1 \in \mathcal{L}(\tilde{X}, X_{-1})$ and $B_2 \in \mathcal{L}(U_2, X_{-1})$,
- non-linear mapping $F : X \times U_1 \to \tilde{X}$.

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Assumptions on $F : X \times U_1 \to \tilde{X}$:

1. $s \mapsto F(g(s), h(s))$ is measurable if $g : I \to X$, $h : I \to U_1$ are measurable,
2. $\forall X_{\mathsf{b}} \subset X$ bounded $\exists L_{X_{\mathsf{b}}} > 0 \ \forall x, \tilde{x} \in X_{\mathsf{b}}, u \in U_1$:
$$\|F(x,u) - F(\tilde{x}, u)\|_{\tilde{X}} \leq L_{X_{\mathsf{b}}} \|u\|_{U_1} \|x - \tilde{x}\|_X,$$
3. $\exists m > 0 \ \forall x \in X, u \in U_1$:
$$\|F(x,u)\|_{\tilde{X}} \leq m \|x\|_X \|u\|_{U_1}.$$

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Examples for F

(a) $\tilde{X} = X$, $U_1 = \mathbb{C}$ and $F(x,u) = xu$,

(b) $\tilde{X} = U_1 = X$, $f \in X^*$, $F(x,u) = f(x)u$,

(c) $\tilde{X} = \mathbb{C}$, $U_1 = X^*$, $F(x,u) = \langle x, u \rangle$.

Bilinear systems

$$\begin{cases} \dot{x}(t) &= Ax(t) + B_1F(x(t), u_1(t)) + B_2u_2(t),\ \ t \geq 0, \\ x(0) &= x_0, \end{cases} \qquad (\Sigma(A, [B_1, B_2], F))$$

Definition

$x \in C([0, \infty); X)$ is called a mild solution of $\Sigma(A, [B_1, B_2], F)$ if for all $t \geq 0$

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)[B_1F(x(s), u_1(s)) + B_2u_2(s)]\,\mathrm{d}s.$$

Proof

Theorem (H., Jacob, Schwenninger 2022)

1. *$\Sigma(A, B_1)$ and $\Sigma(A, B_2)$ are integral ISS $\Rightarrow \Sigma(A, [B_1, B_2], F)$ is integral ISS.*
2. *$\Sigma(A, [B_1, B_2], F)$ is integral ISS $\Rightarrow \Sigma(A, B_2)$ is integral ISS.*

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$$\Sigma(A, B_1),\ \Sigma(A, B_2) \text{ integral ISS} \Leftrightarrow \begin{cases} \|T(t)\| \le M\mathrm{e}^{-\omega t}, \omega > 0, \\ B_1 \text{ is } \mathrm{E}_\Phi\text{-adm}, \\ B_2 \text{ is } \mathrm{E}_\Psi\text{-adm}. \end{cases}$$

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$\Sigma(A, B_1)$, $\Sigma(A, B_2)$ integral ISS $\Leftrightarrow \begin{cases} \|T(t)\| \leq M\mathrm{e}^{-\omega t}, \omega > 0, \\ B_1 \text{ is } \mathrm{E}_\Phi\text{-adm}, \\ B_2 \text{ is } \mathrm{E}_\Psi\text{-adm}. \end{cases}$

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$\Rightarrow$ (admissibility)

$$\mathrm{e}^{\frac{\omega}{2}t}\|x(t)\| \leq \cdots + C_{B_1}\|\mathrm{e}^{\frac{\omega}{2}\cdot} F(x(\cdot), u_1(\cdot))\|_{E_\Phi(0,t;\tilde{X})} + \dots$$

$\Rightarrow$ (boundedness of F + (equivalent) Orlicz-norm $\rightsquigarrow \varepsilon, g_\varepsilon, \|g_\varepsilon\|_{\mathrm{L}_{\tilde{\Phi}}} \leq 1$)

$$\mathrm{e}^{\frac{\omega}{2}t}\|x(t)\| \lesssim \dots + \int_0^t \|u_1(s)\|\,|g_\varepsilon(s)| \left(\mathrm{e}^{\frac{\omega}{2}s}\|x(s)\|\right) \mathrm{d}s + \varepsilon + \dots$$

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$$\mathrm{e}^{\frac{\omega}{2}t}\|x(t)\| \lesssim \cdots + \int_0^t \|u_1(s)\| \, |g_\varepsilon(s)| \left(\mathrm{e}^{\frac{\omega}{2}s}\|x(s)\|\right) \mathrm{d}s + \varepsilon + \dots$$

$\Rightarrow$ (Gronwall + generalized Hölder + $ab \leq a^2/2 + b^2/2$)

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma_1(\|u_1\|_{E_\Phi(0,t;U_1)}) + \gamma_2(\|u_2\|_{E_\Psi(0,t;U_2)})$$

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Lemma (H., Jacob, Schwenninger 2022)

For every Young function Φ there exists $\tilde{\theta}, \mu \in \mathcal{K}$ such that

$$\|u\|_{\mathrm{E}_\Phi(0,t;U)} \leq \tilde{\theta}\left(\int_0^t \mu(\|u(s)\|)\,\mathrm{d}s\right)$$

Hence,

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \theta_1\left(\int_0^t \mu_1(\|u_1(s)\|)\,\mathrm{d}s\right) + \theta_2\left(\int_0^t \mu_2(\|u_2(s)\|)\,\mathrm{d}s\right)$$

□

Back to Fokker-Planck

$$\begin{cases} \frac{\partial \rho}{\partial t}(x,t) = \nu \Delta \rho(x,t) + \operatorname{div}\Big(\rho(x,t)\nabla W(x)\Big) + \operatorname{div}\Big(u(t)\rho(x,t)\nabla\alpha(x)\Big), \\ \rho(x,0) \;= \rho_0(x), \\ \qquad 0 \;= (\nu\nabla\rho + \rho\nabla W)\cdot\vec{n} \quad \text{on } \partial\Omega\times(0,\infty). \end{cases} \tag{FP}$$

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(FP) is a bilinear control system on $X = \mathrm{L}^2(\Omega)$ with

$$\begin{aligned} Af &= \nu\Delta f + \operatorname{div}\Big(f\nabla W\Big), \\ D(A) &= \{f \in H^1(\Omega) \mid \Delta f \in L^2(\Omega), (\nu\nabla f + f\nabla W)\cdot\vec{n} = 0 \text{ on } \partial\Omega\}, \\ Bf &= \operatorname{div}(f\nabla\alpha), \\ D(B) &= H^1(\Omega), \\ F &: L^2(\Omega)\times\mathbb{C} \to L^2(\Omega), \quad (\rho,u)\mapsto \rho u. \end{aligned}$$

- A generates a bounded C_0-semigroup on X, $\sigma(A) = \sigma_p(A) \subseteq (-\infty, 0]$ is discrete and $\rho_\infty = \mathrm{e}^{-\ln \nu + \frac{W}{\nu}}$ is an eigenfunction to the simple eigenvalue 0.
- The state space shift $\mathcal{X} = X - \rho_\infty$, $y = \rho - \rho_\infty$ leads to

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$$\begin{cases} \dot{y}(t) &= \mathcal{A}y(t) + \mathcal{B}_1 F\left(y(t), u(t)\right) + \mathcal{B}_2 u(t),\ t \geq 0, \\ y(0) = \rho_0 - \rho_\infty, \end{cases}$$

where

$$\mathcal{A} : D(\mathcal{A}) = \mathcal{X} \cap D(A) \to \mathcal{X}, f \mapsto Af,$$
$$\mathcal{B}_1 : D(\mathcal{B}_1) \subset \mathcal{X} \to \mathcal{X}, f \mapsto Bf,$$
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We have:

- $\mathcal{A} \sim$ exp. stable semigroup on $\mathcal{X} = X - \rho_\infty$,
- $\mathcal{B}_1$ extends uniquely to L^2-admissible operator in $\mathcal{L}(\mathcal{X}, \mathcal{X}_{-1})$,
- $\mathcal{B}_2 \in \mathcal{L}(\mathbb{C}, \mathcal{X})$.

integral ISS for the bilinear Fokker-Planck equation

Theorem

There exists a constants $C, \omega > 0$ *such that for any* $\rho_0 \in L^2(\Omega)$ *with* $\int_\Omega \rho_0(x)\mathrm{d}x = 1$ *and* $u \in L^2(0,\infty;U)$*, the (continuous) mild solution* ρ *of the Fokker–Planck system satisfies for all* $t > 0$

$$\int_\Omega \rho(x,t)\,\mathrm{d}x = 1$$

and

$$\|\rho(t) - \rho_\infty\|_{L^2} \leq C\mathrm{e}^{-\omega t}\left(\|\rho_0 - \rho_\infty\|_{L^2} + \|\rho_0 - \rho_\infty\|_{L^2}^2\right) + \gamma\left(\int_0^t \|u(s)\|_U^2 \mathrm{d}s\right),$$

where $\gamma(r) = Cr\mathrm{e}^{Cr^{\frac{1}{2}}} + Cr^{\frac{1}{2}} + Cr$.

Thank you for your attention!