The sloshing problem in 2D: spectral properties and control

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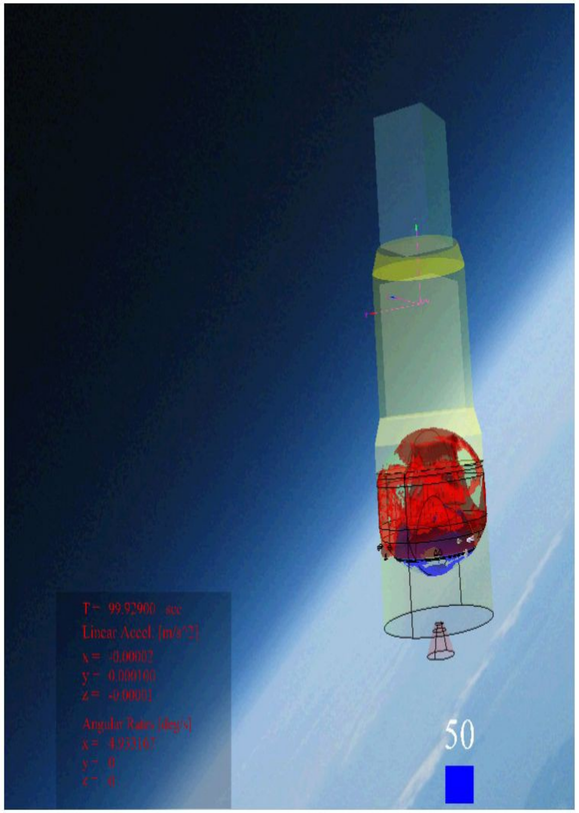
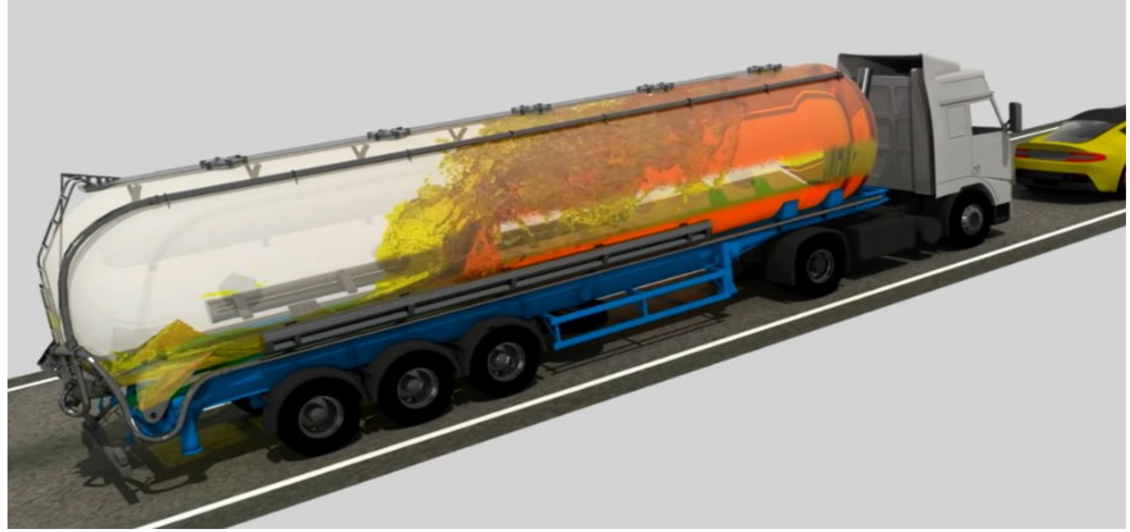
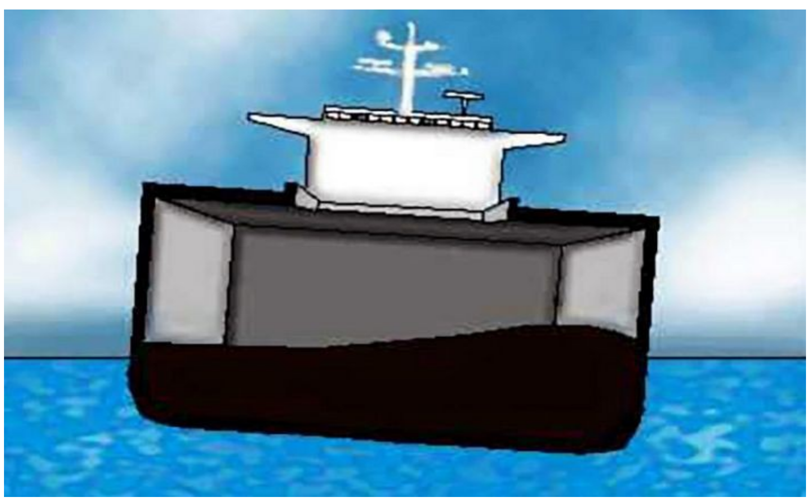
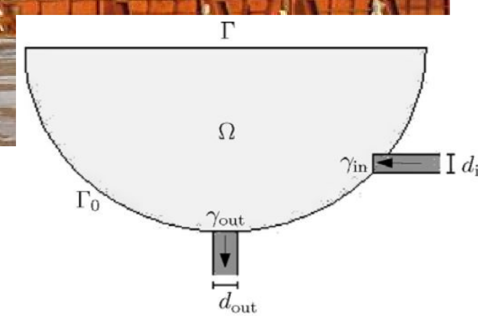








E. Godoy, A. Osses, J. H. Ortega, and A. Valencia. Modeling and control of surface gravity waves in a model of a copper converter. Applied Mathematical Modelling, 32(9):1696–1710, 2008.

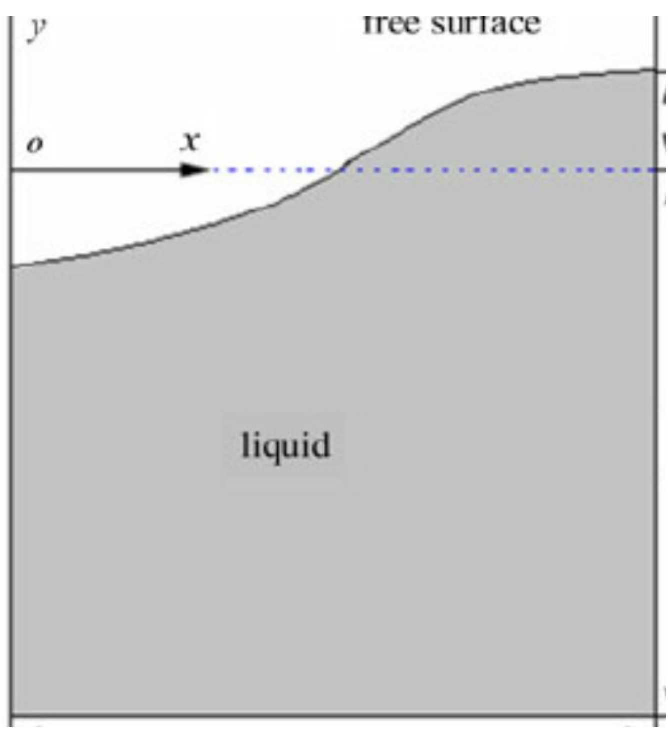
Euler equations:

$$\nabla \cdot \mathbf{u} = 0,$$

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] = -\nabla p - g e_2,$$

By considering the potential function φ of $\mathbf{u}$, so that $\mathbf{u} = \nabla \varphi$:

$$\Delta \varphi = 0,$$



$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}|\nabla \varphi|^2 + \frac{1}{\rho} p + gy = \text{const.}$$

$$\frac{\partial \varphi}{\partial n} = J(t, x), \quad \text{at the solid boundary}$$

$$\eta_t = \frac{\partial \varphi}{\partial n}, \quad |x| < 1.$$

Linearization about equilibrium

$$\begin{aligned} \Delta\phi &= 0, \\ \phi_t + \zeta &= 0, \quad \text{at } y = 0,\ |x| \le 1, \\ \zeta_t &= \frac{\partial \phi}{\partial n}, \quad \text{at } y = 0,\ |x| \le 1, \\ \frac{\partial \phi}{\partial n} &= j(t,x), \quad \text{on the solid walls.} \end{aligned}$$

$$\phi_{tt} + \frac{\partial \phi}{\partial n} = 0.$$

at the free interface

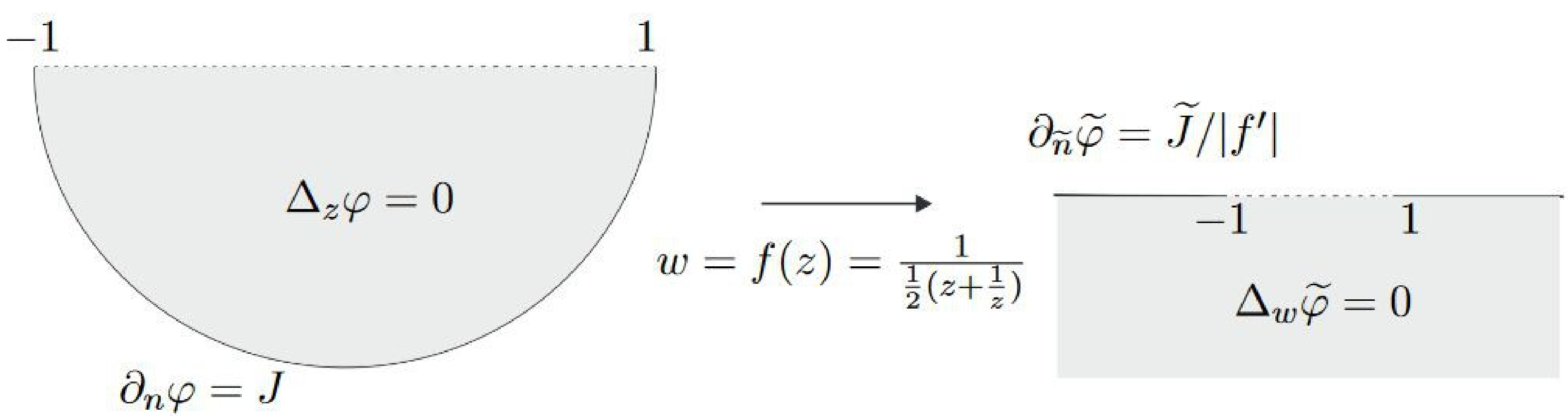


FIGURE 1. Geometry of the problem.

By definition

$$\psi(x,y) = \widetilde{\psi}(x'(x,y), y'(x,y)),$$

where $x' + iy' = w = f(z) = f(x+iy)$ and f is holomorphic. If the curve C in the z-plane is written $z = z(t)$ we have (see [25])

$$\frac{\partial \psi}{\partial n} = \frac{1}{\left|\frac{dz}{dt}\right|} \operatorname{Im}\left[\left(\frac{\partial \psi}{\partial x} - i\frac{\partial \psi}{\partial y}\right)\frac{dz}{dt}\right]. \tag{11}$$

Under the conformal mapping $w = f(z)$, C goes into the curve C^* : $w(t) = f(z(t))$. By the chain rule, if $\widetilde{\psi}(x', y') = \psi(x(x', y'), y(x', y'))$,

Now, $z = f^{[-1]}(w)$ implies $dz = f^{[-1]\prime}(w)dw$, and then by the Cauchy-Riemann equations 2)

$$\begin{aligned} \frac{dz}{dw} &= f^{[-1]\prime}(w) = \frac{\partial f^{[-1]}}{\partial x'}(w) \\ &= \frac{\partial x}{\partial x'} + i\frac{\partial y}{\partial x'} \\ &= \frac{\partial x}{\partial x'} - i\frac{\partial x}{\partial y'}. \end{aligned} \tag{13}$$

Then, replacing in (12) and using (11)

$$\left|\frac{dw}{dt}\right| \frac{\partial \widetilde{\psi}}{\partial \widetilde{n}} = \left|\frac{dz}{dt}\right| \frac{\partial \psi}{\partial n}.$$

Thus the normal derivative of $\widetilde{\psi}$ on C^* is given in terms of the normal derivative of ψ at the corresponding point of C by

$$\frac{\partial \widetilde{\psi}}{\partial \widetilde{n}} = \left|\frac{dz}{dw}\right| \frac{\partial \psi}{\partial n} = \frac{1}{|f'(z)|}\frac{\partial \psi}{\partial n}. \tag{14}$$

By taking Fourier transform in x' of (14), denoted by $\widetilde{\Phi}$, and using convention

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(x') e^{-ikx'} dx',$$

we have

$$\widetilde{\Phi}_{y'y'} - k^2 \widetilde{\Phi} = 0,$$

which implies

$$\widetilde{\Phi}(t, k, y') = \widetilde{\Phi}(t, k, 0) e^{|k| y'}$$

By taking inverse Fourier transform

$$\widetilde{\phi}(t, x', y') = -\frac{y'}{\pi} \int_{-\infty}^{+\infty} \frac{\widetilde{\phi}(t, \xi, 0)}{(x' - \xi)^2 + y'^2} d\xi.$$

Since we want to establish properties on the normal derivative, $\widetilde{\phi}_{y'}$, taking the y' derivative in (17) instead, and evaluating at $y' = 0$ we find

$$\begin{aligned} \widetilde{\Phi}_{y'}(t, k, 0) &= \frac{1}{i} \operatorname{sgn}(k)(ik) \widetilde{\Phi}(t, k, 0), \\ &= \widehat{\frac{\sqrt{2}}{\sqrt{\pi} x'}} \widetilde{\Phi}_{x'}(t, k, 0). \end{aligned} \tag{18}$$

Then, taking inverse Fourier transform

$$\widetilde{\phi}_{y'}\Big|_{y'=0} = \frac{1}{\pi} P.V. \int_{-\infty}^{+\infty} \frac{\widetilde{\phi}_{x'}(t, \xi, 0)}{x' - \xi} d\xi = H\left(\widetilde{\phi}_{x'}\Big|_{y'=0} \right). \tag{19}$$

Since $HH = -I$ (see hochstadt2011integral [17]):

$$\widetilde{\phi}_{x'}\Big|_{y'=0} = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\widetilde{\phi}_{y'}(t, \xi, 0)}{x' - \xi} d\xi = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\frac{\partial \widetilde{\phi}}{\partial \widetilde{n}}(t, \xi, 0)}{x' - \xi} d\xi. \tag{20}$$

For instance:

$$\omega = f(z) = \frac{1}{\frac{1}{2}\left(z + \frac{1}{z}\right)},$$

$$\Delta\widetilde{\phi} = 0, \quad \text{for } y' < 0,$$

$$\widetilde{\phi}_{tt} + |f'(x')|\frac{\partial\widetilde{\phi}}{\partial\widetilde{n}} = 0, \quad \text{at } y' = 0,\ |x'| \le 1,$$

$$\frac{\partial\widetilde{\phi}}{\partial\widetilde{n}} = \frac{\widetilde{j}(t,x')}{|f'(x')|}, \quad \text{at } y' = 0,\ |x'| > 1,$$

Neumann to Dirichlet (ND) operator:

$$\widetilde{\phi}_{x'}\Big|_{y'=0} = -\frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{\widetilde{\phi}_{y'}(t,\xi,0)}{x'-\xi}d\xi = -\frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{\frac{\partial\widetilde{\phi}}{\partial\widetilde{n}}(t,\xi,0)}{x'-\xi}d\xi.$$

Dirichlet to Neumann (DN) operator:

$$\widetilde{\phi}_{y'}(x') = \partial_{x'}\left(\sqrt{1-x'^2}\frac{1}{\pi}P.V.\int_{-1}^{1}\frac{1}{\sqrt{1-\xi^2}}\frac{\widetilde{\phi}(\xi)}{x'-\xi}d\xi\right)$$

Integrodifferential PDE:

$$\widetilde{\phi}_{tt} = -\left|f'(x')\right| \partial_{x'} \left(\sqrt{1-x'^2} \frac{1}{\pi} P.V. \int_{-1}^{1} \frac{\widetilde{\phi}(t,\xi)}{\sqrt{1-\xi^2}(x'-\xi)} d\xi \right).$$

Eigenvalue problem:

$$\lambda \frac{\widetilde{\phi}}{|f'(x')|} = \partial_{x'} \left(\sqrt{1-x'^2} P.V. \frac{1}{\pi} \int_{-1}^{1} \frac{\widetilde{\phi}(\xi)}{\sqrt{1-\xi^2}(x'-\xi)} d\xi \right)$$

With operator

$$\mathcal{A}\widetilde{\phi} \equiv \partial_{x'} \left(\sqrt{1-x'^2} \frac{1}{\pi} P.V. \int_{-1}^{1} \frac{\widetilde{\phi}(\xi)}{\sqrt{1-\xi^2}(x'-\xi)} d\xi \right).$$

Tchebyshev polinomials:

$$T_n(\cos\theta) = \cos(n\theta),$$
$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

Form a complete set with orthogonality relations

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0, & \text{if } n \neq m, \\ \pi & \text{if } n = m = 0, \\ \frac{\pi}{2} & \text{if } n = m \neq 0, \end{cases}$$

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}dx = \begin{cases} 0, & \text{if } n \neq m, \\ \frac{\pi}{2} & \text{if } n = m. \end{cases}$$

They also satisfy

$$\frac{1}{\pi}P.V.\int_{-1}^{1} \frac{T_r(\xi)}{\sqrt{1-\xi^2}(x'-\xi)}d\xi = -U_{r-1}(x'),$$
$$\frac{d}{dx'}\left(\sqrt{1-x'^2}U_{r-1}(x')\right) = -r\frac{T_r(x')}{\sqrt{1-x'^2}}.$$

By writing

$$\widetilde{\phi}(\xi) = \sum_{n=0}^{\infty} a_n T_n(\xi),$$

and using the identities above, we find

$$\partial_{x'} \left(\sqrt{1-x'^2} \frac{1}{\pi} P.V. \int_{-1}^{1} \frac{\widetilde{\phi}(\xi)}{\sqrt{1-\xi^2}(x'-\xi)} d\xi \right) = \sum_{n=1}^{\infty} n a_n \frac{T_n(x')}{\sqrt{1-x'^2}}.$$

So that

$$(\mathcal{A}\widetilde{\phi}, \widetilde{\psi})_{L^2} = \int_{-1}^{1} \left(\sum_{m=0}^{\infty} b_m T_m(x') \right) \left(\sum_{n=1}^{\infty} n a_n \frac{T_n(x')}{\sqrt{1-x'^2}} \right) dx' = \frac{\pi}{2} \sum_{n=1}^{\infty} n a_n b_n,$$

We introduce the natural functional spaces:

$$L^2_w \equiv \left\{ u : \int_{-1}^{1} \sqrt{1-x'^2}\, |u|^2 \, dx' < \infty \right\}.$$

$$H^{\frac{1}{2}}_{w^{-1}} \equiv \left\{ \widetilde{\psi} : \left\| \widetilde{\psi} \right\|_{H^{\frac{1}{2}}_{w^{-1}}} = \left\| \widetilde{\psi} \right\|_{L^2_{w^{-1}}} + \left(\sum_{n=1}^{\infty} n \left(\int_{-1}^{1} \frac{T_n(x')}{\sqrt{1-x'^2}} \widetilde{\psi}(x') dx' \right)^2 \right)^{1/2} < \infty \right\}$$

Under the assumption

$$\left|f'(x')\right| \geq C\sqrt{1-x'^2}, \quad x' \in [-1,1],$$

implying

$$\int_{-1}^{1} \sqrt{1-x'^2}\frac{v^2(x')}{|f'(x')|^2}dx' \leq \frac{1}{C}\int_{-1}^{1} \frac{v^2(x')}{|f'(x')|}dx' \leq \frac{1}{C^2}\int_{-1}^{1} \frac{v^2(x')}{\sqrt{1-x'^2}}dx',$$

Theorem 1. *Let Ω be a domain such that the interior angles between the free liquid interface and the solid are greater or equal than $\pi/2$. There exist a Hilbert basis $\{e_n\}_{n\geq 1}$ of $L^2_{|f'-1}$ and a sequence $\{\lambda_n\}_{n\geq 1}$ of real numbers with $\lambda_n > 0\ \forall n$ and $\lambda_n \to +\infty$ such that*

$$e_n \in H^{\frac{1}{2}}_{w^{-1}},$$

$$\mathcal{A}e_n = \lambda_n \frac{e_n}{|f'(x')|}.$$

Step 1:

$$\mathcal{A}\widetilde{\phi} = u,$$

Lemma 2. *Let $u \in L^2_w$. Then, there exists a unique weak solution $\widetilde{\phi} \in H^{1/2}_{w^{-1}}$*

$$\sum_{n=1}^{\infty} n a_n b_n = \sum_{n=1}^{\infty} u_n b_n,$$

$$|(\mathcal{A}\widetilde{\phi}, \widetilde{\psi})_{L^2}| = \frac{\pi}{2}\left|\sum_{n\geq 1} n a_n b_n\right| \leq \frac{\pi}{2}\left(\sum_{n\geq 1} n|a_n|^2\right)^{1/2}\left(\sum_{n\geq 1} n|b_n|^2\right)^{1/2}$$
$$\leq \frac{2}{\pi}\|\widetilde{\phi}\|_{H^{1/2}_{w^{-1}}}\|\widetilde{\psi}\|_{H^{1/2}_{w^{-1}}}.$$

$(\mathcal{A}\widetilde{\phi}, \widetilde{\psi})_{L^2}$ is also coercive (the $L^2_{w^{-1}}$ part of the norm of $\widetilde{\phi}$ is trivially bounded by $\sum_{n=1}^{\infty} n a_n^2$,

Finally, $u \in L^2_w$ defines a linear continuous functional on $H^{1/2}_{w^{-1},0}$:

$$\left|\int_{-1}^{1} u\widetilde{\phi}dx'\right| \leq \int_{-1}^{1} \sqrt[4]{1-x'^2}|u|\frac{\widetilde{\phi}}{\sqrt[4]{1-x'^2}}dx'$$
$$\leq \|u\|_{L^2_w}\|\widetilde{\phi}\|_{L^2_{w^{-1}}}$$
$$\leq \|u\|_{L^2_w}\|\widetilde{\phi}\|_{H^{1/2}_{w^{-1}}}.$$

Step 2:

$$\mathcal{A}\widetilde{\phi} = \frac{v}{|f'(x')|}$$

where v is such that

$$\int_{-1}^{1} \sqrt{1-x'^2}\frac{v^2(x')}{|f'(x')|^2}dx' < \infty \qquad (62)$$

has also a unique solution $\widetilde{\phi} \in H^{\frac{1}{2}}_{w^{-1}}$. Note that (62) is satisfied if $v \in L^2_{|f'(x')|^{-1}}$ provided there exists a constant C such that

$$|f'(x')| \geq C\sqrt{1-x'^2}, \quad x' \in [-1,1], \qquad (63)$$

.

$$\int_{-1}^{1} \sqrt{1-x'^2}\frac{v^2(x')}{|f'(x')|^2}dx' \leq \frac{1}{C}\int_{-1}^{1} \frac{v^2(x')}{|f'(x')|}dx' \leq \frac{1}{C^2}\int_{-1}^{1} \frac{v^2(x')}{\sqrt{1-x'^2}}dx',$$

$$\|\varphi\|^2_{L^2_{|f'|^{-1}}} \leq \frac{1}{C}\|\varphi\|^2_{L^2_{w^{-1}}}.$$

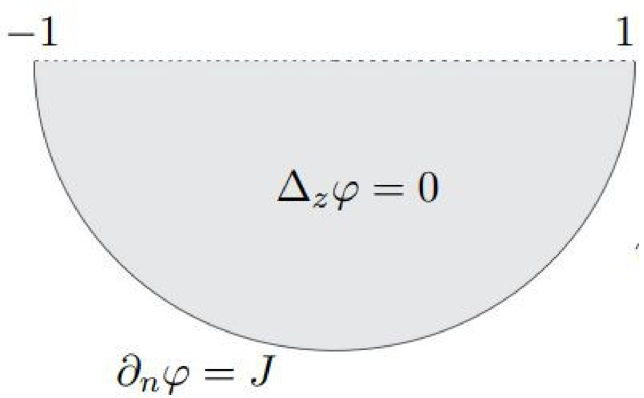
Step 3: We define now the operator $\mathcal{T}$ such that $\widetilde{\phi} = \mathcal{T}v$.

$\mathcal{T}$ as an operator from $L^2_{|f'|^{-1}}$ to $L^2_{|f'|^{-1}}$

Spectral decomposition Thm

Selfadjoint, compact,....

$$\lambda F^T D E \mathbf{b} = \pi(i)_{ii}\mathbf{b},$$ Matrix equation



–	$\lambda_1 = 1.363$	$\lambda_2 = 4.681$	$\lambda_3 = 7.886$	$\lambda_4 = 11.096$	$\lambda_5 = 14.325$
a_1	9.81E-01	7.70E-01	1.39E-01	4.48E-02	-6.98E-03
a_2	-1.70E-01	5.03E-01	9.30E-01	4.94E-01	1.91E-01
a_3	7.24E-02	-2.92E-01	2.34E-03	7.62E-01	7.50E-01
a_4	-4.07E-02	1.84E-01	-1.55E-01	-3.68E-01	3.13E-01
a_5	2.64E-02	-1.26E-01	1.66E-01	1.18E-01	-4.11E-01
a_6	-1.86E-02	9.13E-02	-1.48E-01	1.49E-03	3.00E-01
a_7	1.38E-02	-6.93E-02	1.27E-01	-5.57E-02	-1.83E-01
a_8	-1.07E-02	5.44E-02	-1.07E-01	7.83E-02	9.63E-02
a_9	8.56E-03	-4.38E-02	9.14E-02	-8.56E-02	-3.76E-02
a_{10}	-7.01E-03	3.61E-02	-7.83E-02	8.55E-02	-4.48E-04

TABLE 1. Eigenvalues for the antisymmetric eigenfunctions after a 100×100 approximation, the free-end case. Eigenfunctions are such that $\sum a_i^2 = 1$.

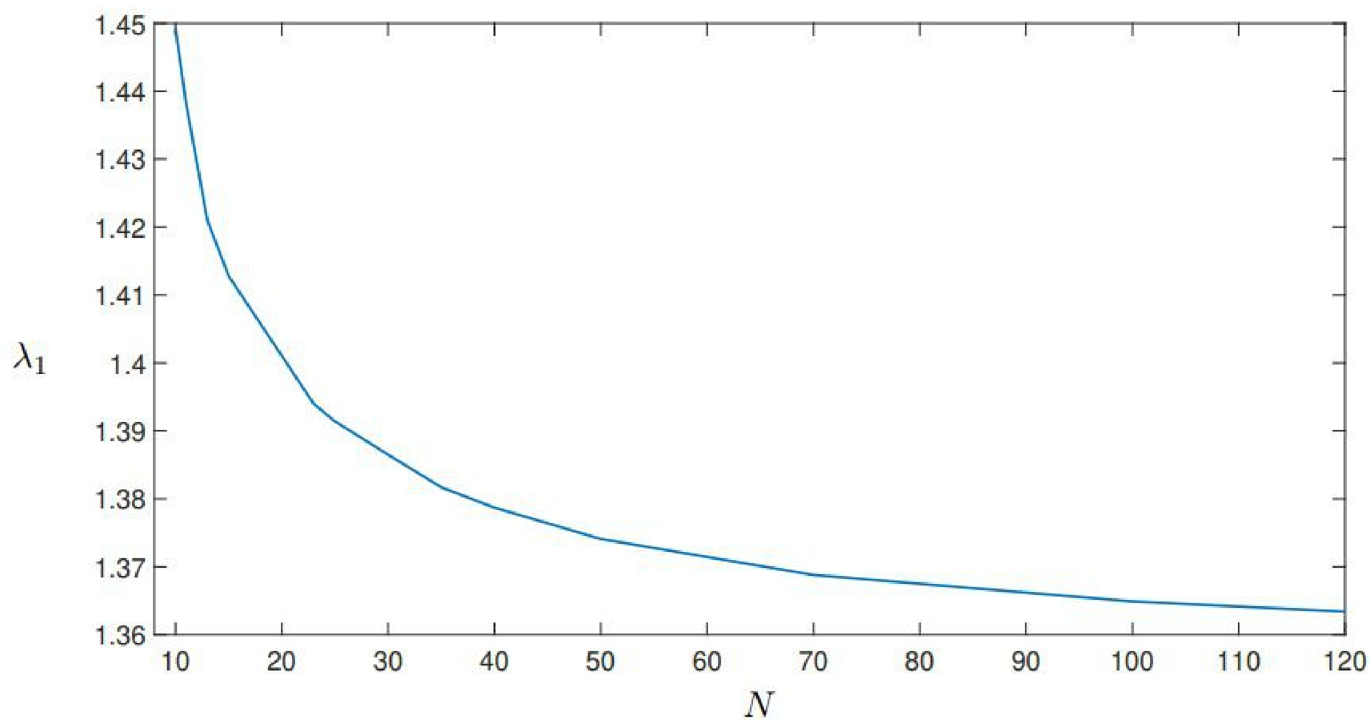


FIGURE 2. Error as a function of matrix size N.

–	$\lambda_1 = 2.695$	$\lambda_2 = 6.296$	$\lambda_3 = 9.5$	$\lambda_4 = 12.741$	$\lambda_5 = 16.001$
b_1	9.14E-01	8.76E-01	3.07E-01	1.01E-01	6.95E-03
b_2	-3.32E-01	2.61E-01	8.92E-01	6.44E-01	3.04E-01
b_3	1.74E-01	-2.58E-01	-2.13E-01	5.65E-01	7.80E-01
b_4	-1.09E-01	1.99E-01	-2.41E-02	-4.32E-01	7.62E-02
b_5	7.44E-02	-1.52E-01	9.84E-02	2.29E-01	-3.24E-01
b_6	-5.44E-02	1.18E-01	-1.17E-01	-9.40E-02	3.12E-01
b_7	4.16E-02	-9.42E-02	1.15E-01	1.52E-02	-2.35E-01
b_8	-3.28E-02	7.66E-02	-1.06E-01	2.87E-02	1.57E-01
b_9	2.66E-02	-6.34E-02	9.59E-02	-5.21E-02	-9.38E-02
b_{10}	-2.21E-02	5.33E-02	-8.58E-02	6.37E-02	4.73E-02

TABLE 2. Eigenvalues for the symmetric eigenfunctions after a 100×100 approximation, the free-end case. Eigenfunctions are such that $\sum b_i^2 = 1$.

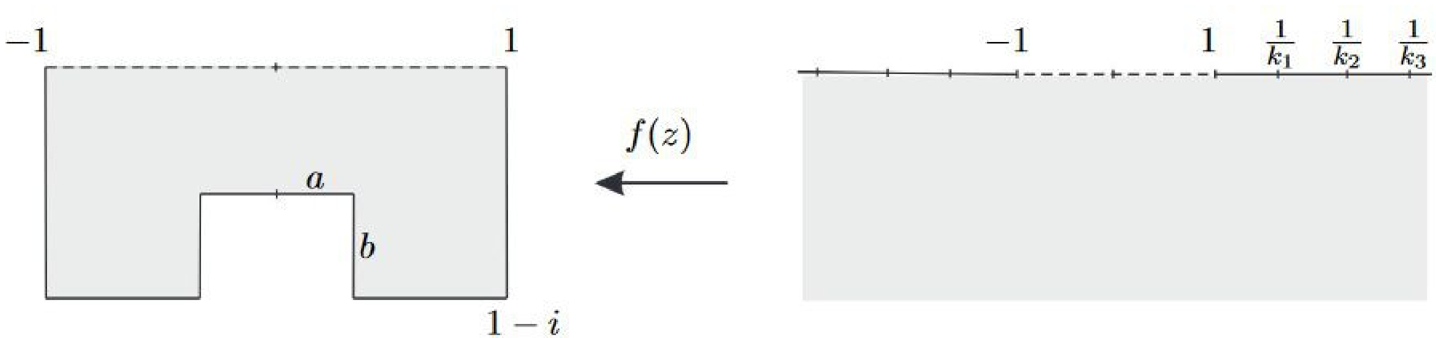


FIGURE 3. Geometry of the problem for a polygonal container mapped into the half-plane by the Schwarz-Christoffel map f.

Let $0 < a, b < 1$. The Schwarz-Christoffel map $f(z)$, given by

$$f(z) = \alpha \int_0^z \frac{\sqrt{1-k_3^2\zeta^2}}{\sqrt{1-\zeta^2}\sqrt{1-k_1^2\zeta^2}\sqrt{1-k_2^2\zeta^2}} d\zeta + \beta,$$

with α and β two constants, is the conformal mapping of the half-plane $\mathrm{Im}\{w\} < 0$ onto the 8-gon of Figure 3. Note that ± 1, $\pm 1/k_1$, $\pm 1/k_2$, $\pm 1/k_3$ are mapped into the 8-gon vertex, symmetrically and $0 < k_1, k_2, k_3 < 1$.

$(a,b)/k$'s	k_1	k_2	k_3
$(0,0)$	0.7071	0.7071	0.7071
$(0.10, 0.10)$	0.7797	0.7795	0.7772
$(0.25, 0.10)$	0.77288	0.7634	0.75244
$(0.25, 0.25)$	0.8739	0.8727	0.8603
$(0.50, 0.10)$	0.7556	0.6762	0.6432
$(0.50, 0.25)$	0.84465	0.81758	0.76511
$(0.50, 0.50)$	0.9658	0.9633	0.9304
$(0.75, 0.10)$	0.7365	0.48396	0.4202
$(0.75, 0.25)$	0.80384	0.6726	0.54066
$(0.75, 0.50)$	0.92968	0.89989	0.74873
$(0.75, 0.75)$	0.996399	0.995244	0.95654

TABLE 5. Different values of k_1, k_2, k_3 corresponding to a, b fixed.

$(a,b)/\lambda$	λ_1	λ_2	λ_3	λ_4	λ_5
$(0,0)$	0.78745	2.5799	4.3205	6.0881	7.8892
$(0.10, 0.10)$	0.72188	2.4409	4.0931	5.7756	7.4958
$(0.25, 0.10)$	0.72092	2.4388	4.0895	5.7706	7.4895
$(0.25, 0.25)$	0.59508	2.1631	3.6446	5.1659	6.7372
$(0.50, 0.10)$	0.72615	2.4491	4.1058	5.7925	7.5165
$(0.50, 0.25)$	0.60269	2.178	3.6653	5.1917	6.7665
$(0.50, 0.50)$	0.34452	1.5712	2.7441	4.0016	5.3542
$(0.75, 0.10)$	0.74262	2.4831	4.1611	5.8682	7.611
$(0.75, 0.25)$	0.63473	2.2439	3.7693	5.3302	6.9345
$(0.75, 0.50)$	0.37822	1.6277	2.7885	4.0204	5.3328
$(0.75, 0.75)$	0.12212	1.043	2.0888	3.2623	4.5574

TABLE 6. First eigenvalues for different dimensions of the bottom. Antisymmetric, free end case, taking $N = 35$.

MA Fontelos, J López-Ríos, Gravity waves oscillations at semicircular and general 2D containers: an efficient computational approach to 2D sloshing problem ,Zeitschrift für angewandte Mathematik und Physik 71 (3) (2020), 1-24

Adjoint problem:

$$\frac{\widetilde{\phi}_{tt}}{|f'|} + \mathcal{A}\widetilde{\phi} = 0, \tag{62}$$

$$\widetilde{\phi} = \sum_{n=1}^{\infty} \left(A_n \cos(\theta_n t) + B_n \sin(\theta_n t)\right) e_n,$$

$$\widetilde{\phi}_0 = \sum_{n=1}^{\infty} A_n e_n, \quad \widetilde{\phi}_1 = \sum_{n=1}^{\infty} \theta_n B_n e_n,$$

$$\sum_{n=1}^{\infty} B_n^2 = \sum_n \left(\frac{1}{\theta_n} \int_{-1}^{1} \frac{\widetilde{\phi}_1 e_n}{|f'(x')|} dx' \right)^2 = \sum_n \frac{1}{\lambda_n} \left(\int_{-1}^{1} \frac{\widetilde{\phi}_1 e_n}{|f'(x')|} dx' \right)^2 .$$

Let us define, accordingly, the space

$$\widetilde{H}^{-\frac{1}{2}} = \left\{ \widetilde{\phi} : \sum_n \frac{1}{\lambda_n} \left(\int_{-1}^{1} \frac{\widetilde{\phi} e_n}{|f'(x')|} dx' \right)^2 < \infty \right\},$$

and its dual space

$$\widetilde{H}^{\frac{1}{2}} = \left\{ \widetilde{\phi} : \sum_n \lambda_n \left(\int_{-1}^{1} \frac{\widetilde{\phi} e_n}{|f'(x')|} dx' \right)^2 < \infty \right\}.$$

Note:

$$\left\|\widetilde{\phi}\right\|^2_{\widetilde{H}^{\frac{1}{2}}} = \sum\left(\int \frac{\widetilde{\phi}\lambda_n e_n}{|f'(x)|}\right)\left(\int \frac{\widetilde{\phi} e_n}{|f'(x)|}\right) = \int \widetilde{\phi}\,\mathcal{A}\widetilde{\phi} = \frac{\pi}{2}\sum_{n\geq 1} n a_n^2 \sim \left\|\widetilde{\phi}\right\|^2_{H^{\frac{1}{2}}_{w^{-1}}}.$$

Theorem 2. *The equation (62) is observable in time $T > T_0$ for some T_0 sufficiently large. That is, there exists a positive constant $C > 0$ such that for T sufficiently large*

$$\int_0^T \int_{-1}^1 \frac{\widetilde{\phi}^2}{|f'(x')|} dx'dt \geq CT\left(\left\|\widetilde{\phi}_1\right\|^2_{\widetilde{H}^{-\frac{1}{2}}} + \left\|\widetilde{\phi}_0\right\|^2_{L^2_{|f'|^{-1}}}\right).$$

We consider next the nonhomogeneous problem (as in (43))

$$\frac{\widetilde{\psi}_{tt}}{|f'|} + \mathcal{A}\widetilde{\psi} = \frac{\widetilde{h}}{|f'|}.$$

Theorem 3. *Given f a conformal mapping that satisfies condition (60), for any $\widetilde{h} \in L^2(0,T; L^2_{|f'|^{-1}})$ and $\left(\widetilde{\psi}_0, \widetilde{\psi}_1\right) \in \widetilde{H}^{\frac{1}{2}} \times L^2_{|f'|^{-1}}$ equation (67) has a unique weak solution*

$$(\widetilde{\psi}, \widetilde{\psi}') \in C([0,T]; H^{\frac{1}{2}}_{w^{-1},0} \times L^2_{|f'|^{-1}}).$$

Theorem 4. *The system (67) is exactly controllable in time T. That is, for any initial data $\left(\widetilde{\psi}_0, \widetilde{\psi}_1\right) \in \widetilde{H}^{\frac{1}{2}} \times L^2_{|f'|^{-1}}$, there exist $\widetilde{h} \in L^2(0,T; L^2_{|f'|^{-1}})$ and $T > 0$ such that*

$$\|\widetilde{\psi}\|_{H^{\frac{1}{2}}_{w^{-1},0}} = 0, \quad \textit{for } t > T.$$

Idea of proof:

$(\widetilde{\psi}_0, \widetilde{\psi}_1) = (\widetilde{\psi}(0), \widetilde{\psi}_t(0)) \in \widetilde{H}^{\frac{1}{2}} \times L^2_{|f'|^{-1}}$. So that $\widetilde{h}$ can be chosen as the minimizer of the functional

$$J[\widetilde{\phi}_0, \widetilde{\phi}_1] = \frac{1}{2}\int_0^T \int_{-1}^1 \frac{\widetilde{\phi}^2}{|f'|} dx' dt + \int_{-1}^1 \frac{\widetilde{\psi}_1 \widetilde{\phi}_0}{|f'|} dx' - \int_{-1}^1 \frac{\widetilde{\psi}_0 \widetilde{\phi}_1}{|f'|} dx'.$$

$$\left| \int_{-1}^1 \frac{\widetilde{\psi}_1 \widetilde{\phi}_0}{|f'|} dx' - \int_{-1}^1 \frac{\widetilde{\psi}_0 \widetilde{\phi}_1}{|f'|} dx' \right| \le \|\widetilde{\phi}_0\|_{L^2_{|f'|^{-1}}} \|\widetilde{\psi}_1\|_{L^2_{|f'|^{-1}}} + \|\widetilde{\phi}_1\|_{\widetilde{H}^{-\frac{1}{2}}} \|\widetilde{\psi}_0\|_{\widetilde{H}^{\frac{1}{2}}}.$$

Conclusions:

- Method to solve the sloshing problem in 2D based on conformal mapping and study of an integrodifferential operator.

- Observability and interior controllability.

- Efficient algorithms based on this approach.