

# Eigenfunctions for Levy Fokker-Planck equations

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# Summary

$$Lu = (-\Delta)^s u - \frac{x}{2s} \cdot \nabla u - \frac{n}{2}u \quad \text{in } \mathbb{R}^n$$

## Comments:

- ▶  $s \in (0, 1)$ .
- ▶ May reduce to radial solutions  $u(r)$ ,  $r = |x|$

## Objectives:

- ▶ Eigenvalue problem  $Lu = \nu u$
- ▶ Regularity estimates for  $Lu = f$

Joint work with H. Chan, M. Fontelos, J. Wei

# The fractional Laplacian $(-\Delta)^s$

- ▶ Infinitesimal generator of a Levy process
- ▶ Pseudo-differential operator, principal symbol  $|\xi|^{2s}$ .

$$\widehat{(-\Delta)^s f} = |\xi|^{2s} \hat{f}$$

- ▶ Singular integral formulation

$$(-\Delta)^s f(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{f(x) - f(\xi)}{|x - \xi|^{n+2s}} d\xi$$

- ▶ **Nonlocal** operator
- ▶ Extension viewpoint

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# Fractional heat equation

$$\partial_\tau \varphi + (-\Delta_y)^s \varphi = 0, \quad \tau > 0, y \in \mathbb{R}^n$$

- **Self-similar variables:**  $\tau = \log t, \quad x = \frac{y}{t^{\frac{1}{2s}}}$

$$\partial_t u + (-\Delta_x)^s - \frac{1}{2s} x \cdot \nabla u = 0, \quad x \in \mathbb{R}^n, t > 0$$

- **Finite time blow-up:**  $t = -\log(T - \tau), \quad x = \frac{y}{(T-t)^{\frac{1}{2s}}}$

$$\partial_t u + (-\Delta_x)^s + \frac{1}{2s} x \cdot \nabla u = 0, \quad x \in \mathbb{R}^n, t > 0$$

- Note:  $L^*$  is the adjoint of  $L$ .

# Theorem 1

## Theorem\*

$L$  has eigenvalues  $\nu = k$ ,  $k \in \mathbb{N}$ , with eigenfunctions

$$w_k(r) = e^{-\frac{r^{2s}}{4}} L_k^{\left(\frac{n-2s}{2s}\right)}\left(\frac{r^{2s}}{4}\right)$$

The eigenfunctions form a complete orthonormal basis.

\* Up to a change of basis

Notes:

- ▶ Generalized Laguerre polynomials

$$L_k^{(\alpha)}(\rho) = \frac{1}{k!} \rho^{-\alpha} e^{\rho} \frac{d^k}{d\rho^k} (\rho^{\alpha+k} e^{-\rho}),$$

- ▶ Function spaces:

$$\|w\|_{L_w^2}^2 = \int_0^\infty w^2(r) e^{\frac{r^{2s}}{4}} r^{n-1} dr$$

## Local case $s = 1 \rightarrow$ Fourier transform

$$-\Delta u - \frac{x}{2} \cdot \nabla u - \frac{n}{2}u = \frac{\nu}{2}u$$

- ▶ Fourier transform in  $\mathbb{R}^n$ :  $\zeta = |\xi|$ ,  $\hat{u} = \phi(\zeta)$

$$\zeta^2 \phi + \frac{1}{2} \zeta \phi_\zeta = \frac{\nu}{2} \phi$$

- ▶ Solution of the ODE:  $\phi(\zeta) = \zeta^\nu e^{-\zeta^2}$
- ▶ Exponential decay for  $u$  as  $r \rightarrow \infty \Leftrightarrow$  smoothness at the origin  $\zeta = 0$  for  $\phi$
- ▶  $\nu = 2k$ ,  $k \in \mathbb{Z}$
- ▶ Inverse Fourier (dim=1)

$$u(r) = D_\ell(e^{-\frac{r^2}{4}}) = e^{-\frac{r^2}{4}} H_\ell(r)$$

- ▶ Rodrigues formula

# Mellin transform of $u = u(r)$

► **Definition**

$$\mathcal{M}u(z) := \int_0^\infty r^z u(r) \frac{dr}{r}.$$

► Usually,  $z = \lambda i$ ,  $\lambda \in \mathbb{R}$

► Equivalent to **Fourier** transform in the variable  $t = -\log r$ :

$$\mathcal{M}u(\lambda i) = \int_0^\infty r^{\lambda i} u(r) \frac{dr}{r} = \int_{\mathbb{R}} e^{-t\lambda i} u(e^{-t}) dt$$

► **Mellin inversion formula**

$$u(r) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} r^{-z} \mathcal{M}u(z) dz$$

► **Plancherel identity**

$$\int_{\mathbb{R}} |\mathcal{M}u(\frac{n}{2} + \lambda i)|^2 d\lambda = \int_0^\infty |u(r)|^2 r^{n-1} dr$$

► **Function spaces**

$$\|u\|_{H^\beta}^2 = \int_{\mathbb{R}} (1 + |\lambda|^2)^\beta |\mathcal{M}u(\frac{n}{2} + \lambda i)|^2 d\lambda$$



# Mellin symbol of the fractional Laplacian

## Proposition

$$\mathcal{M}\{(-\Delta)^s u\}(z) = \Theta_s(z)\mathcal{M}u(z - 2s)$$

for the symbol

$$\Theta_s(z) = 2^{2s} \frac{\Gamma(\frac{z}{2})\Gamma(\frac{n+2s-z}{2})}{\Gamma(\frac{n-z}{2})\Gamma(\frac{z-2s}{2})}$$

▶ New proof: Hankel+Mellin

▶ **Cylindrical metric**

$$g = \frac{dr^2 + r^2 d\theta^2}{r^2} = dt^2 + d\theta^2 \quad (\text{change } r = e^{-t})$$

▶ Conformal fractional Laplacian on the cylinder

$$\mathcal{F}P_s v(z) = \Theta_s(z)\mathcal{F}v(z)$$

▶ Conformal law

$$P_s v = r^{\frac{n+2s}{2}} (-\Delta)^s \left( r^{-\frac{n-2s}{2}} v \right)$$

## Back to the eigenvalue problem

$$(-\Delta)^s u - \frac{r}{2s} \partial_r u - \frac{n}{2} u = \nu u$$

- ▶ Take Mellin transform

$$\Theta_s(z) \mathcal{M}u(z - 2s) = \left(\nu + \frac{n-z}{2s}\right) \mathcal{M}u(z)$$

- ▶ Change  $V(z) = 2^{-z} \frac{\Gamma(\frac{n-z}{2})}{\Gamma(\frac{z}{2})\Gamma(\nu + \frac{n-z}{2s})} \mathcal{M}u(z)$

$$V(z - 2s) = V(z)$$

- ▶ Solution is  $V(z) = e^{\frac{\pi i}{s} z \ell}$ ,  $\ell \in \mathbb{Z}$
- ▶ The “eigenfunction” is

$$\mathcal{M}u_\nu(z) = 2^z \frac{\Gamma(\frac{z}{2})\Gamma(\nu + \frac{n-z}{2s})}{\Gamma(\frac{n-z}{2})} \quad \rightarrow \quad u_\nu(r)$$

### Proposition

Under Fourier transform,  $\mathcal{F}u_\nu(\zeta) = \zeta^\nu e^{-\zeta^{2s}}$

## Choosing the function space

The “eigenfunction” is

$$\mathcal{M}u_\nu(z) = 2^z \frac{\Gamma(\frac{z}{2})\Gamma(\nu + \frac{n-z}{2s})}{\Gamma(\frac{n-z}{2})} =: \Lambda_s(z)\mathcal{W}_\nu(z),$$

where

$$\Lambda_s(z) := 2^z \frac{\Gamma(\frac{z}{2})}{\Gamma(\frac{n-z}{2})} 2^{-\frac{z}{s}} \frac{\Gamma(\frac{z}{2s})}{\Gamma(\frac{n-z}{2s})} \rightarrow \text{change of basis}$$

and

$$\mathcal{W}_\nu(z) := 4^{\frac{z}{2s}} \frac{\Gamma(\nu + \frac{n-z}{2s})\Gamma(\frac{z}{2s})}{\Gamma(\frac{n-z}{2s})} \rightarrow w_k(r) = e^{-\frac{r^{2s}}{4}} L_k^{(\frac{n-2s}{2s})}(\frac{r^{2s}}{4})$$

Lemma

$\{w_k\}_k$  is a complete o.n. basis for the scalar product

$$\langle w, w' \rangle = \int_0^\infty w(r)w'(r)e^{-\frac{r^{2s}}{4}} r^{n-1} dr$$

Prove  $\nu = k$ ,  $k \in \mathbb{N}$

## Theorem

If  $\nu + \frac{n}{2s}$  is not a non-positive integer, then

$$w_\nu(r) = 2s \frac{\Gamma(\nu + \frac{n}{2s})}{\Gamma(\frac{n}{2s})} M\left(\nu + \frac{n}{2s}; \frac{n}{2s}; -\frac{r^{2s}}{4}\right) \\ + 2s \sum_{\ell=0}^{\lfloor -\nu - \frac{n}{2s} \rfloor} \frac{\sin(\pi(\nu + \ell))}{\pi} \Gamma(\nu + 1 + \ell) \Gamma(\nu + \frac{n}{2s} + \ell) \frac{(-1)^\ell}{\ell!} \left(\frac{r^{2s}}{4}\right)^{-\nu - \frac{n}{2s} - \ell}$$

Moreover,  $w_\nu \in L_w^2$  iff  $\nu \in \mathbb{N}$ . In fact, as  $r \rightarrow +\infty$ ,

$$w_k(r) \sim e^{-\frac{r^{2s}}{4}} \left(-\frac{r^{2s}}{4}\right)^k \quad \text{for } \nu = k \in \mathbb{N},$$

$$w_\nu(r) \sim \left(\frac{r^{2s}}{4}\right)^{-\nu - \frac{n}{2s}} \quad \text{for } \nu \notin \mathbb{N}.$$

# The change of basis

- ▶ Recall  $\Lambda_s(z) := 2^z \frac{\Gamma(\frac{z}{2})}{\Gamma(\frac{n-z}{2})} 2^{-\frac{z}{s}} \frac{\Gamma(\frac{z}{2s})}{\Gamma(\frac{n-z}{2s})} \rightarrow$  change of basis
- ▶ Define the convolution

$$\Phi_s w(r) := \varphi_s \star w(r) = \int_0^\infty \varphi_s\left(\frac{r}{r'}\right) w(r') \frac{dr'}{r'}$$

Lemma

$$\|\Phi_s w\|_{H^\beta} = 2^{\frac{n}{2}(1-\frac{1}{s})} \|w\|_{H^\beta}.$$

$$\boxed{(-\Delta)^s u + \frac{1}{2s} x \cdot \nabla u = f, \quad u = u(r)}$$

## Theorem 2

Given  $f$  satisfying the compatibility condition

$$\int_{\mathbb{R}^n} f(x) e_s(x) dx = 0,$$

then there exists a solution  $u$  satisfying

$$\|u\|_{H^{\beta+2s}} \leq C \|f\|_{H^\beta}$$

**Remark:**

$$e_s(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} e^{-|\xi|^{2s}} d\xi \sim \frac{1}{(1+|x|)^{n+2s}}$$

# Proof via Fourier

$$\boxed{(-\Delta)^s u + \frac{1}{2s} x \cdot \nabla u = f \quad \text{in } \mathbb{R}^n}$$

- ▶ Fourier transform

$$|\xi|^{2s} \widehat{u} - \frac{n}{2s} \widehat{u} - \frac{1}{2s} \xi \cdot \nabla \widehat{u} = \widehat{f}, \quad \xi \in \mathbf{R}^n$$

- ▶ Radial variables  $v(\zeta) = \widehat{w}(\zeta)$ ,  $\zeta = |\xi|$

$$(2s\zeta^{2s} - n)v - \zeta v' = 2sh, \quad \text{for } \zeta \in (0, \infty)$$

- ▶ ODE

$$(\zeta^n e^{-\zeta^{2s}} v)' = -2sh\zeta^{n-1} e^{-\zeta^{2s}}$$

- ▶ Explicit solution

$$\begin{aligned} v(\zeta) &= -2s\zeta^{-n} e^{\zeta^{2s}} \int_{\zeta}^{\infty} h(\varrho) \varrho^{n-1} e^{-\varrho^{2s}} d\varrho \\ &= 2s\zeta^{-n} e^{\zeta^{2s}} \int_0^{\zeta} h(\varrho) \varrho^{n-1} e^{-\varrho^{2s}} d\varrho. \end{aligned}$$

- ▶ Estimate  $\|u\|_{H^{\beta+2s-\epsilon}} \leq C\|f\|_{H^{\beta}}$

- ▶ Mellin transform
- ▶ Magic change
- ▶ Riemann-Hilbert problem

$$V(\lambda) - V(\lambda - 2si) = F(\lambda)$$

- ▶ Conformal mapping  $\eta = e^{\pi\lambda/s}$

$$W(\eta + 0i) - W(\eta - 0i) = G(\eta)$$

- ▶ Cauchy formula

$$W(\eta) = -\frac{1}{2\pi i} \int_0^\infty \frac{G(\eta')}{\eta - \eta'} d\eta'$$

- ▶ Fredholm condition

$$\frac{1}{\eta - \eta'} = -\frac{1}{\eta'} - \frac{\eta}{(\eta - \eta')\eta'}$$



# Regularity via Mellin

$$\begin{aligned} \int_0^\infty \left| \eta^\delta \int_0^\infty \frac{\phi(\eta') d\eta'}{\eta - \eta'} \right|^2 d\eta &= \int_{-\infty}^\infty d\lambda \left| \int_0^\infty \eta^{i\lambda - \frac{1}{2} + \delta} \left( \int_0^\infty \frac{\phi(\eta') d\eta'}{\eta - \eta'} \right) d\eta \right|^2 \\ &= \int_{-\infty}^\infty d\lambda \left| \int_0^\infty \phi(\eta') \left( \int_0^\infty \frac{\eta^{i\lambda - \frac{1}{2} + \delta}}{\eta - \eta'} d\eta \right) d\eta' \right|^2 \\ &= \int_{-\infty}^\infty d\lambda \left| \int_0^\infty \eta'^{i\lambda + \delta - \frac{1}{2}} \phi(\eta') \left( \int_0^\infty \frac{\eta^{i\lambda + \delta - \frac{1}{2}}}{\eta - 1} d\eta \right) d\eta' \right|^2 \\ &\leq \pi^2 \int_{-\infty}^\infty \left| \cot \left[ \left( i\lambda + \delta - \frac{1}{2} \right) \pi \right] \right|^2 \left| \int_0^\infty \eta'^{i\lambda + \delta - \frac{1}{2}} \phi(\eta') d\eta' \right|^2 d\lambda \\ &\leq c \int_{-\infty}^\infty \left| \int_0^\infty \eta'^{i\lambda + \delta - \frac{1}{2}} \phi(\eta') d\eta' \right|^2 d\lambda = c \int_0^\infty |\phi(\eta')|^2 (\eta')^{2\delta} d\eta'. \end{aligned}$$

- ▶ Add a critical Hardy potential

$$L_{\kappa}u = (-\Delta)^s u - \frac{x}{2s} \cdot \nabla u - \frac{n}{2}u + \frac{\kappa}{|x|^{2s}} \quad \text{in } \mathbb{R}^n$$

- ▶ To construct type II singularities for the fractional heat equation (Herrero-Velázquez)

Thank you!!