An eigenvalue problem of Steklov type as $p \to +\infty$

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Chair DYNAMICS, CONTROL AND NUMERICS

FAU

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- $\Omega_0 \subset \mathbb{R}^n$ is an open bounded, connected and Lipschitz set
- B_{R_1} is the ball centered at the origin with radius $R_1 > 0$ and $B_{R_1} \Subset \Omega_0$;
- $\Omega := \Omega_0 \backslash \overline{B_{R_1}}$ and ν is the outer unit normal to $\partial\Omega_0$;
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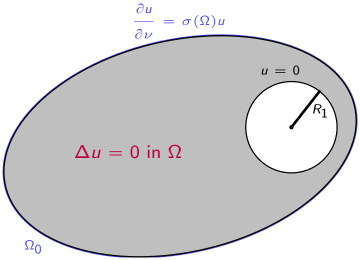

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$\sigma_1(\Omega) > 0$ if $R_1 \neq 0$.

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The following scaling invarian property holds, $\forall t > 0$:

$$\sigma_1(t\Omega) = t^{-1}\sigma_1(\Omega).$$

Aim

We want to study a shape optimization problem for $\sigma_1(\Omega)$ under volume and perimeter constraints;

Some references

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[HLS] - Hong, - Lim - Seo, ArXiv (2020)

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Motivation

The Steklov- Dirichlet and its related eigenvalue problems are of importance from both theoretical and applied perspectives.

- Partially free vibration modes of a thin planar membrane without mass on the interior and with mass on the boundary can be interpreted as Steklov-Dirichlet eigenfunctions;
- this problem has been studied in relation to hydrodynamics such as the sloshing problem (oscillations of fluid in a container).

The first Steklov-Dirichlet eigenvalue $\sigma_1(\Omega)$

$$\sigma_1(\Omega) = \inf_{\substack{v \in H^1_{\partial B_{R_1}}(\Omega) \\ v \not\equiv 0}} \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx}{\displaystyle\int_{\partial \Omega_0} v^2 \, d\mathcal{H}^{n-1}},$$

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- There exists a function $u \in H^1_{\partial B_{R_1}}(\Omega)$ which achieves the minimum, $\sigma_1(\Omega)$ is simple and the relative eigenfunctions have constant sign in Ω.
- When $\Omega_0 = B_{R_2}$, $R_2 > R_1$, i.e. $\Omega = A_{R_1,R_2}$, $u(x) = w(|x|) = w(r)$,

$$w(r) = \begin{cases} \ln r - \ln R_1 & n = 2 \\ \left(\dfrac{1}{R_1^{n-2}} - \dfrac{1}{r^{n-2}} \right) & n \geqslant 3 \end{cases}$$

$$\sigma_1(A_{R_1,R_2}) = \begin{cases} \dfrac{1}{R_2 \log\left(\frac{R_2}{R_1}\right)} & n = 2 \\ \dfrac{n-2}{R_2\left[\left(\frac{R_2}{R_1}\right)^{n-2} - 1\right]} & n \geqslant 3. \end{cases}$$

[A], [PPS], [SV], [F], [D], [HLS].

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$\sigma_1(A_{R_1,R_2}) \to 0$ if $R_1 \to 0$

Upper and lower bounds for $\sigma_1(\Omega)$

$$C\left(\frac{R_m}{R_M}\right)^{n-1}\sigma_1\left(A_{R_1,R_m}\right)\leqslant\sigma_1(\Omega)\leqslant\left(\frac{R_M}{R_m}\right)^{n-1}\sigma_1(A_{R_1,R_M}),$$

The equality case holds if and only if Ω is a spherical shell.

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where

- $\rho_0(x) = \sup\{\lambda \geqslant 0 \colon \lambda x \in \Omega_0\}$ with $x \in \mathbb{S}^{n-1}$, is the radial function of Ω_0. So, $\partial\Omega_0 = \{x\rho_0(x), x \in \mathbb{S}^{n-1}\}$;
- $R_m = \min_{\mathbb{S}^{n-1}} \rho_0$; minimal distance of $\partial\Omega_0$ from the origin
- $R_M = \max_{\mathbb{S}^{n-1}} \rho_0$; maximal distance of $\partial\Omega_0$ from the origin
- $C = \dfrac{1}{\max_{\mathbb{S}^{n-1}}\left(\sqrt{1+\frac{|\nabla_\tau \rho_0|^2}{\rho_0^2}}\right)}$

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- $R_m=\min_{\mathbb{S}^{n-1}}\rho_0$;
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- $\sigma_1(\Omega)>0$ being $R_1>0$ fixed.
- $\sigma_1(\Omega)\to 0$ as $R_1\to 0$.

Upper and lower bounds for $\sigma_1(\Omega)$

$$\sigma_1(\Omega) \leqslant \frac{2}{n\omega_n^{\frac{1}{n}}\left(\left(\frac{V(\Omega)}{2\omega_n}+R_1^n\right)^{\frac{1}{n}}-R_1\right)^2}(V(\Omega))^{\frac{1}{n}}$$

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An optimization problem for $\sigma_1(\Omega)$

Questions

- Does there exist an optimal set that maximizes $\sigma_1(\Omega)$ when R_1 and $V(\Omega_0)$ are fixed?
- If yes, what is the shape of the miximiser?
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- $\Omega = B_{R_2} \setminus \overline{B_{R_1}(x_0)}$, where B_{R_2} is a ball centered at the origin and $B_{R_1}(x_0)$ is a ball centered at $x_0 \in \mathbb{R}^n$ such that $B_{R_1}(x_0) \Subset B_{R_2}$.

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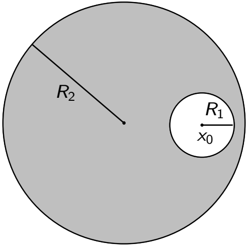

An optimization problem for $\sigma_1(\Omega)$: annular sets

Theorem

Among all doubly connected domains of $\mathbb{R}^n$, $n \geqslant 2$ of the form $B_{R_2} \setminus \overline{B_{R_1}(x_0)}$, with $B_{R_1}(x_0) \Subset B_{R_2}$ and R_1, R_2 fixed, $\sigma_1(\Omega)$ achieves its maximal value if and only if when the two balls are concentric.

[F] ($n \geqslant 2$), [SV]($n > 2$) (see also [S])

The optimization problem: Existence

Let $R_1 > 0$, $\kappa > n\omega_n R_1^{n-1}$ and $\omega > 0$ be fixed and let

$$\mathcal{A}_{R_1}(\omega) := \left\{\Omega = \Omega_0 \setminus \overline{B}_{R_1},\ \ \Omega_0 \subset \mathbb{R}^n \text{ open, convex } : \ B_{R_1} \Subset K : \ V(\Omega) = \omega\right\}.$$

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Theorem [GPPS]

There exists $E \in \mathcal{A}_{R_1}(\omega)$, such that $\sup_{\Omega \in \mathcal{A}_{R_1}(\omega)} \sigma_1(\Omega) = \sigma_1(E)$.

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There exists $\Omega \in \mathcal{B}_{R_1}(\kappa)$, such that $\displaystyle\sup_{D \in \mathcal{B}_{R_1}(\kappa)} \sigma_1(D) = \sigma_1(\Omega)$.

→ Let $K \subset \mathbb{R}^n$ be a convex body such that $K \subset \Omega_0$ and let $\Omega_K = \Omega_0 \setminus \overline{K}$. The existence result still holds in this case.

The optimization problem: Shape of the optimal set

Theorem [GPPS]

Let $R_1 > 0$, $\Omega_0 \subset \mathbb{R}^n$ be an open, bounded and convex set, $n \geqslant 2$, such that $B_{R_1} \subset \Omega_0 \subseteq B_{\bar{R}}$, where $B_{\bar{R}}$ is the ball centered at the origin with radius $\bar{R}$ given by

$$\bar{R} = \begin{cases} R_1 e^{\sqrt{2}} & \text{if } n = 2 \\ R_1 \left[\frac{(n-1)+(n-2)\sqrt{2(n-1)}}{n-1} \right]^{\frac{1}{n-2}} & \text{if } n \geqslant 3. \end{cases}$$

Then,

$$\sigma_1(\Omega) \leqslant \sigma_1(A_{R_1,R_2}),$$

where $\Omega = \Omega_0 \setminus \overline{B}_{R_1}$, A_{R_1,R_2} is the spherical shell of radii $R_1 < R_2$ having the same volume as Ω.

Idea of the proof

$$\sigma_1(\Omega) = \inf_{\substack{u \in H^1_{\partial B_{R_1}}(\Omega) \\ u \not\equiv 0}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\displaystyle\int_{\partial\Omega_0} u^2 \, d\mathcal{H}^{n-1}}, \leqslant \frac{\displaystyle\int_\Omega |\nabla w|^2 \, dx}{\displaystyle\int_{\partial\Omega_0} w^2 \, d\mathcal{H}^{n-1}}$$

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Test function: $w(r) = \begin{cases} \ln r - \ln R_1 & n = 2 \\ \left(\frac{1}{R_1^{n-2}} - \frac{1}{r^{n-2}} \right) & n \geqslant 3 \end{cases}$

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- $\displaystyle\int_\Omega |\nabla w|^2 \, dx \leqslant \int_{A_{R_1,R_2}} |\nabla w|^2 \, dx.$

$|\nabla w|^2$ is a non-negative radially symmetric decreasing function for any $n \geqslant 2$. Then it follows by the Hardy-Littlewood inequality

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- $\displaystyle\int_\Omega |\nabla w|^2 \, dx \leqslant \int_{A_{R_1,R_2}} |\nabla w|^2 \, dx.$
- $\displaystyle\int_{\partial\Omega_0} w^2 \, d\mathcal{H}^{n-1} \geqslant \int_{\partial B_{R_2}} w^2 \, d\mathcal{H}^{n-1}.$

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- $\displaystyle\int_\Omega |\nabla w|^2 \, dx \leqslant \int_{A_{R_1,R_2}} |\nabla w|^2 \, dx.$

- $\displaystyle\int_{\partial\Omega_0} w^2 \, d\mathcal{H}^{n-1} \geqslant \int_{\partial B_{R_2}} w^2 \, d\mathcal{H}^{n-1}.$

  - The radial representation $\partial\Omega_0 = \{x\,\rho_0(x), x \in \mathbb{S}^{n-1}\}$,
  - Jensen's inequality (The restrictions on Ω allow us to use the convexity of a one-dimensional function).

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- $\displaystyle\int_\Omega |\nabla w|^2 \, dx \leqslant \int_{A_{R_1,R_2}} |\nabla w|^2 \, dx.$
- $\displaystyle\int_{\partial\Omega_0} w^2 \, d\mathcal{H}^{n-1} \geqslant \int_{\partial B_{R_2}} w^2 \, d\mathcal{H}^{n-1}.$

The second claim cannot holds under perimeter constraint!

A numerical counterexample

- Let $R_2 = 1$ and $R_1 = 10^{-5}$.
- Let Ω_0 be an ellipse with the same perimeter as $A_{R_1,1}$ and let a and b its semi-axes.

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In order to compute the integral over the ellipse, we used the formula $P(\Omega_0) = 2\pi\sqrt{\frac{a^2+b^2}{2}}$, which is an approximation by excess for the perimeter of the ellipse.

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- Let $R_2 = 1$ and $R_1 = 10^{-5}$.
- Let Ω_0 be an ellipse with the same perimeter as $A_{R_1,1}$ and let a and b its semi-axes.

In order to compute the integral over the ellipse, we used the formula $P(\Omega_0) = 2\pi\sqrt{\frac{a^2+b^2}{2}}$, which is an approximation by excess for the perimeter of the ellipse. Chosen $b = 1.1$ we obtain

$$D(A_{R_1,1}) \approx 832,820208 > 828,919156 \approx D(\Omega_0),$$

where $D(\Omega_0) = \int_{\partial\Omega_0} w^2 ds$ and w is the eigenfunction of $\sigma_1(A_{R_1,1})$.

A numerical counterexample

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This means that we cannot study separately the numerator and denominator to obtain inequality under perimeter constraint.

Works in progress

- Remove the restriction on Ω_0 in the isoperimetric result by using a new technique.
- Prove the isoperimetric result also with the perimeter constraint.

Thank you for your attention!