# Constructive exact controls for semi-linear wave equations

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# Semi-linear wave equation

- Let $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, and $\omega \subset \Omega$. For $T > 0$, we set $Q_T := \Omega \times (0,T)$, $\Sigma_T := \partial\Omega \times (0,T)$ and $q_T := \omega \times (0,T)$. We denote $\boldsymbol{V} := H_0^1(\Omega) \times L^2(\Omega)$.
- We consider the semi-linear wave equation with distributed control

$$\Box := \partial_{tt} - \Delta \qquad \begin{cases} \Box y + g(y) = u \mathbb{1}_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y, \partial_t y)(\cdot, 0) = (y^0, y^1) & \text{in } \Omega. \end{cases} \tag{SL}$$

- $y = y(\boldsymbol{x}, t)$ is the state, $u = u(\boldsymbol{x}, t)$ is a control acting in $\omega$.

**Definition (global exact controllability)**

System (SL) is said *exactly controllable* in time $T$ iff

$$\forall (y^0, y^1), (z^0, z^1) \in \boldsymbol{V},\ \exists u \in L^2(q_T) \text{ s.t. } (y, \partial_t y)(\cdot, T) = (z^0, z^1),\ y \text{ sol. of (SL)}.$$

For $(y^0, y^1)$ and $(z^0, z^1)$ fixed, we call state-control pair (SC-pair) of (SL) any $(y, u)$ with $u$ a control and $y$ the associated controlled solution.

**Goal**

Find a sequence $(y_k, u_k)_{k\in\mathbb{N}}$ such that $(y_k, u_k) \underset{k\to\infty}{\longrightarrow} (y, u)$ a SC-pair of (SL).

- Assume $g \in C^1(\mathbb{R};\mathbb{R})$ such that $|g(r)| \le C(1+|r|)\ln(2+|r|),\ \forall r \in \mathbb{R}$.
- For $(y^0, y^1) \in \boldsymbol{V}$ and $u \in L^2(q_T)$, system

$$\Box := \partial_{tt} - \Delta \qquad \begin{cases} \Box y + g(y) = u 1_\omega & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y, \partial_t y)(\cdot, 0) = (y^0, y^1) & \text{in } \Omega, \end{cases} \tag{SL}$$

has a unique weak solution $y$ with $(y, \partial_t y) \in C([0,T]; \boldsymbol{V})$.

**Theorem [1] (global exact controllability)**

For $\boldsymbol{x}_0 \in \mathbb{R} \setminus \overline{\Omega}$, we set $\Gamma_+ := \{\boldsymbol{x} \in \partial\Omega;\ (\boldsymbol{x} - \boldsymbol{x}_0)\cdot \boldsymbol{\nu}(\boldsymbol{x}) > 0\}$ and denote $\mathcal{O}_\delta(\Gamma_+)$ its $\delta$-neighbourhood. Assume $T > 2\max_{\boldsymbol{x}\in\overline{\Omega}} |\boldsymbol{x} - \boldsymbol{x}_0|$ and $\mathcal{O}_\delta(\Gamma_+) \cap \Omega \subseteq \omega$ for some $\delta > 0$.

Moreover, if $g$ satisfies

$$\limsup_{|r|\to\infty} \frac{|g(r)|}{|r| \ln^{1/2}|r|} = 0,$$

then system (SL) is exactly controllable in time $T$.

1. Fu, Yong, Zhang – Exact controllability for multidimensional semilinear hyperbolic equations, SIAM J. Control Optim. 2007

- We define operator $K : L^\infty(0,T;L^d(\Omega)) \to L^\infty(0,T;L^d(\Omega))$ by $K(\xi) := y$, where $y$ is the solution of the linearized wave equation

$$\begin{cases} \Box y + \widehat{g}(\xi) y = u 1_\omega - g(0) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y, \partial_t y)(\cdot, 0) = (y^0, y^1) & \text{in } \Omega, \end{cases} \qquad \widehat{g}(r) := \begin{cases} \dfrac{g(r) - g(0)}{r} & \text{si } r \neq 0, \\ g'(0) & \text{si } r = 0, \end{cases}$$

associated with the control $u$ of minimal $L^2(q_T)$-norm such that $(y, \partial_t y)(\cdot, T) = (z^0, z^1)$ in $\Omega$.

- The operator $K$ has a fixed point.
- With each fixed point $y$ of $K$, we associate the SC-pair $(y, u)$ of (SL).

**Fixed-point algorithm**

To build a SC-pair of (SL), we can consider the Picard iteration

$$\begin{cases} y_0 \in L^\infty(0,T;L^d(\Omega)) \text{ given}, \\ y_{k+1} = K(y_k), \quad \forall k \in \mathbb{N}. \end{cases}$$

However, $K$ is not necessarily contracting, such a strategy fails in general.

We consider the Hilbert space

$$\mathcal{H} := \Big\{(y,u) \in L^2(Q_T) \times L^2(q_T);\ (y, \partial_t y) \in C([0,T]; \boldsymbol{V}),\ \Box y \in L^2(Q_T)\Big\}$$

and the closed subspaces

$$\mathcal{A} := \Big\{(y,u) \in \mathcal{H};\ (y,\partial_t y)(\cdot,0) = (y^0, y^1),\ (y,\partial_t y)(\cdot,T) = (z^0,z^1)\Big\},$$
$$\mathcal{A}_0 := \Big\{(y,u) \in \mathcal{H};\ (y,\partial_t y)(\cdot,0) = (0,0),\ (y,\partial_t y)(\cdot,T) = (0,0)\Big\}.$$

## Least-squares functional

We define the functional $E : \mathcal{A} \to \mathbb{R}$ by

$$E(y,u) := \frac{1}{2}\big\|\Box y + g(y) - u 1_\omega\big\|^2_{L^2(Q_T)}$$

and consider the (non-convex) minimization problem

$$\min_{(y,u)\in\mathcal{A}} E(y,u).$$

Since $(y,u) \in \mathcal{A}$ is a SC-pair of (SL) iff it is a zero of $E$, we build a minimizing sequence converging to a zero of $E$.

# A first property of $E$

The functional $E$ is differentiable and we denote $E'(y,u)\cdot(Y,U)$ the derivative of $E$ at point $(y,u)\in\mathcal{A}$ in the direction $(Y,U)\in\mathcal{A}_0$.

## Proposition

$$\forall (y,u)\in\mathcal{A},\quad \sqrt{E(y,u)}\leq \frac{C}{\sqrt{2}}\Big(1+\|g'(y)\|_{L^\infty(L^3)}\Big)e^{C\|g'(y)\|^2_{L^\infty(L^d)}}\|E'(y,u)\|_{\mathcal{A}_0'}.$$

## Consequences

- Any critical point $(y,u)\in\mathcal{A}$ of $E$ (i.e. $E'(y,u)=0$) is a zero of $E$.
- If $(y_k,u_k)_{k\in\mathbb{N}}\subset\mathcal{A}$ satisfies $\|E'(y_k,u_k)\|_{\mathcal{A}_0'}\underset{k\to\infty}{\longrightarrow}0$ and $\|g'(y_k)\|_{L^\infty(L^3)}$ is uniformly bounded w.r.t. $k$, then $E(y_k,u_k)\underset{k\to\infty}{\longrightarrow}0$.
- Thus, a minimising sequence cannot stop in a local minimum, even if $E$ is not convex.
- We build a minimizing sequence $(y_k,u_k)_{k\in\mathbb{N}}$ such that $\|g'(y_k)\|_{L^\infty(L^3)}$ stays uniformly bounded w.r.t. $k$.

For $(y,u) \in \mathcal{A}$, we denote $(Y,U) \in \mathcal{A}_0$ the SC-pair* of the linearized wave equation

$$\begin{cases} \Box Y + g'(y)Y = U\mathbb{1}_\omega + \left(\Box y + g(y) - u\mathbb{1}_\omega\right) & \text{in } Q_T, \\ Y = 0 & \text{on } \Sigma_T, \\ (Y, \partial_t Y)(\cdot, 0) = (0,0) & \text{in } \Omega, \\ (Y, \partial_t Y)(\cdot, T) = (0,0) & \text{in } \Omega. \end{cases} \tag{LZ}$$

* we choose the SC-pair associated with the control of minimal $L^2(q_T)$-norm

**Proposition**

For any $(y,u) \in \mathcal{A}$, the SC-pair $(Y,U) \in \mathcal{A}_0$ satisfies the *a priori* estimate

$$\|(Y, \partial_t Y)\|_{L^\infty(\boldsymbol{V})} + \|U\|_{L^2(q_T)} \leq C e^{C\|g'(y)\|^2_{L^\infty(L^d)}} \sqrt{E(y,u)}.$$

Besides, if $g'$ is uniformly $s$-Hölder on $\mathbb{R}$ with $s \in (0,1]$, we have

$$\boxed{E'(y,u) \cdot (Y,U) = 2E(y,u).}$$

This property implies that $-(Y,U)$ is a descent direction for $E$ at point $(y,u)$.

## Minimizing sequence

We define the sequence $(y_k, u_k)_{k\in\mathbb{N}}$ by

$$\begin{cases} (y_0, u_0) \in \mathcal{A} \text{ given}, \\ (y_{k+1}, u_{k+1}) = (y_k, u_k) - \lambda_k (Y_k, U_k), \quad \forall k \in \mathbb{N}, \\ \lambda_k = \arg\min_{\lambda\in[0,1]} E\big((y_k, u_k) - \lambda(Y_k, U_k)\big), \end{cases}$$

where $(Y_k, U_k) \in \mathcal{A}_0$ is the SC-pair of system (LZ) associated with $(y_k, u_k)$.

## Theorem [2] (strong convergence)

Assume $(\Omega, \omega, T)$ satisfies the standard geometric control condition. If $g'$ is uniformly $s$-Hölder on $\mathbb{R}$ with $s \in (0, 1]$ and satisfies

$$\exists\alpha \geq 0,\ \exists\beta \in [0, \beta^\star(s)), \quad |g'(r)| \leq \alpha + \beta \ln^{1/2}(1 + |r|), \quad \forall r \in \mathbb{R},$$

then, for any $(y_0, u_0) \in \mathcal{A}$, $E(y_k, u_k) \underset{k\to\infty}{\longrightarrow} 0$ and the sequence $(y_k, u_k)_{k\in\mathbb{N}}$ strongly converges to a SC-pair $(y^\star, u^\star) \in \mathcal{A}$ of (SL).

Moreover, the convergence is at least linear,
and at least of order $1 + s$ after a finite number of iterations.

2. B., Lemoine, Munch – Constructive exact controls for semi-linear wave equations, preprint

# Minimizing sequence and strong convergence

## Minimizing sequence

We define the sequence $(y_k, u_k)_{k\in\mathbb{N}}$ by

$$\begin{cases} (y_0, u_0) \in \mathcal{A} \text{ given}, \\ (y_{k+1}, u_{k+1}) = (y_k, u_k) - \lambda_k (Y_k, U_k), \quad \forall k \in \mathbb{N}, \\ \lambda_k = \underset{\lambda\in[0,1]}{\arg\min}\, E\big((y_k, u_k) - \lambda(Y_k, U_k)\big), \end{cases}$$

where $(Y_k, U_k) \in \mathcal{A}_0$ is the SC-pair of system (LZ) associated with $(y_k, u_k)$.

## Link with the Newton algorithm

We have $E(y,u) = \frac{1}{2}\|F(y,u)\|^2_{L^2(Q_T)}$, with $F(y,u) := \Box y + g(y) - u\mathbb{1}_\omega$.

• We check that the descent algorithm for $E$ coincide with the *damped* Newton algorithm for $F$, and coincide with the standard Newton algorithm when $\lambda_k \equiv 1$.

• The optimization of $\lambda_k$ ensures the global convergence of the descent algorithm.

• We also have $\lambda_k \to 1$ as $k \to \infty$, which explain the superlinear convergence of the descent algorithm after a finite number of iterations.

We illustrate the result of convergence of the least-squares algorithm and compare its performance with two fixed-point algorithms.

- Let $\Omega = (0,1)^2$, $\omega$ as displayed below and $T = 3$.
- We take $y^0(\boldsymbol{x}) = 100\sin(\pi x_1)\sin(\pi x_2)$, $y^1 = 0$ and $(z^0, z^1) = (0,0)$.
- We consider the semi-linearity $g(r) = -c_g\, r \ln^{1/2}(2 + |r|)$, with $c_g \in \mathbb{R}$.
- We initialize with the SC-pair $(y_0, u_0) \in \mathcal{A}$ of the linear wave equation (i.e. for $g \equiv 0$).
- The descent direction $(Y, U)$ is approximated with a standard method [2] based on the minimization of the conjugate functional.
- The trajectories of the wave equations involved are approximated using finite differences in time and finite elements in space.

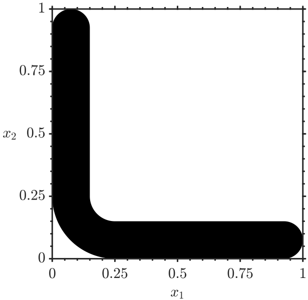


Fig. 1 – Control domain $\omega$.

2. Glowinski, Li, Lions – A numerical approach to the exact boundary controllability of the wave equation, Japan J. Appl. Math. 1990

| $\sharp$iterate $k$ | $\sqrt{2E(y_k, u_k)}$ | $\lambda_k$ | $\Vert y_k \Vert_{L^2(Q_T)}$ | $\Vert u_k \Vert_{L^2(q_T)}$ |
|---|---|---|---|---|
| 0 | $3.66 \times 10^{2}$ | 1 | 37.653 | 1339.39 |
| 1 | $4.87 \times 10^{1}$ | 0.996 | 31.353 | 1223.44 |
| 2 | $8.65 \times 10^{-1}$ | 1 | 32.101 | 1348.12 |
| 3 | $5.82 \times 10^{-5}$ | 1 | 32.104 | 1348.09 |
| 4 | $7.17 \times 10^{-14}$ | – | 32.104 | 1348.09 |

Tab. 1 – Convergence of the algorithm.

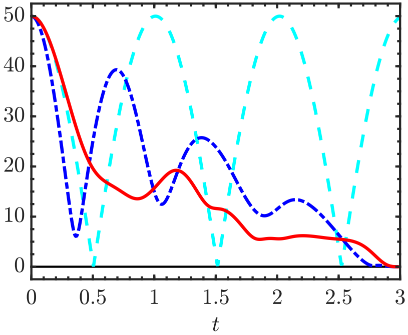


Fig. 2 – (red solid) $\Vert y^\star(\cdot,t)\Vert_{L^2(\Omega)}$ ; (blue dashed) $\Vert y_0(\cdot,t)\Vert_{L^2(\Omega)}$ ; (cyan dashed) $\Vert y(\cdot,t;0)\Vert_{L^2(\Omega)}$.

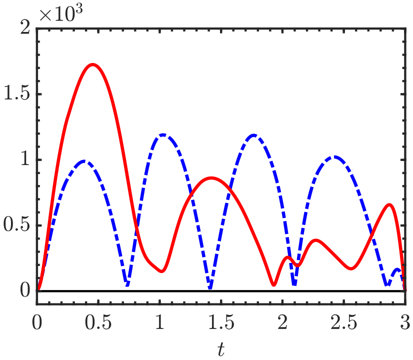


Fig. 3 – (red solid) $\Vert u^\star(\cdot,t)\Vert_{L^2(\omega)}$ ; (blue dashed) $\Vert u_0(\cdot,t)\Vert_{L^2(\omega)}$.

| $\sharp$iterate $k$ | $\sqrt{2E(y_k, u_k)}$ | $\lambda_k$ | $\|y_k\|_{L^2(Q_T)}$ | $\|u_k\|_{L^2(q_T)}$ |
|---|---|---|---|---|
| 0 | $7.32 \times 10^{2}$ | 1 | 37.653 | 1339.39 |
| 1 | $1.59 \times 10^{2}$ | 0.998 | 57.718 | 1164.11 |
| 2 | $2.43 \times 10^{0}$ | 1 | 59.934 | 1138.75 |
| 3 | $4.09 \times 10^{-3}$ | 1 | 59.892 | 1141.01 |
| 4 | $1.80 \times 10^{-9}$ | – | 59.891 | 1141.01 |

Tab. 2 – Convergence of the algorithm.

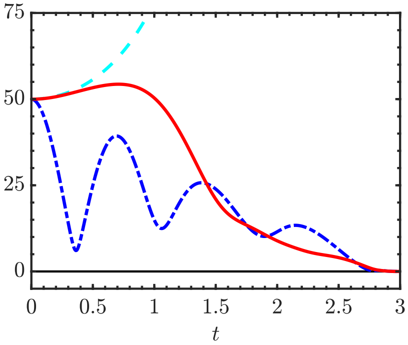


Fig. 4 – (red) $\|y^\star(\cdot, t)\|_{L^2(\Omega)}$ ; (blue dashed) $\|y_0(\cdot, t)\|_{L^2(\Omega)}$ ; (cyan dashed) $\|y(\cdot, t; 0)\|_{L^2(\Omega)}$.

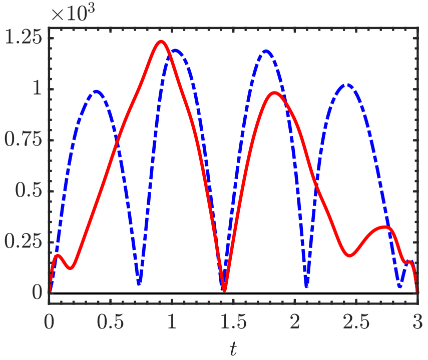


Fig. 5 – (red) $\|u^\star(\cdot, t)\|_{L^2(\omega)}$ ; (blue dashed) $\|u_0(\cdot, t)\|_{L^2(\omega)}$.

| $\sharp$iterate $k$ | $\sqrt{2E(y_k, u_k)}$ | $\lambda_k$ | $\Vert y_k \Vert_{L^2(Q_T)}$ | $\Vert u_k \Vert_{L^2(q_T)}$ |
|---|---|---|---|---|
| 0 | $1.46 \times 10^{3}$ | 1 | 37.653 | 1339.39 |
| 1 | $2.70 \times 10^{2}$ | 0.985 | 42.479 | 2601.16 |
| 2 | $1.55 \times 10^{1}$ | 1 | 44.309 | 2696.17 |
| 3 | $1.94 \times 10^{-2}$ | 1 | 44.34 | 2700.73 |
| 4 | $9.66 \times 10^{-9}$ | – | 44.34 | 2700.73 |

Tab. 3 – Convergence of the algorithm.

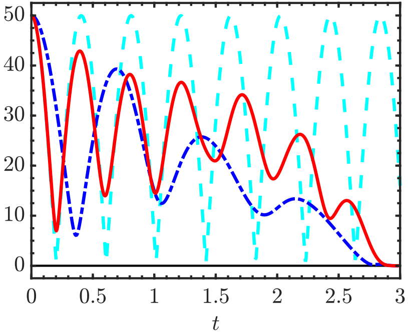


Fig. 6 – (red solid) $\Vert y^\star(\cdot,t)\Vert_{L^2(\Omega)}$ ; (blue dashed) $\Vert y_0(\cdot,t)\Vert_{L^2(\Omega)}$ ; (cyan dashed) $\Vert y(\cdot,t;0)\Vert_{L^2(\Omega)}$.

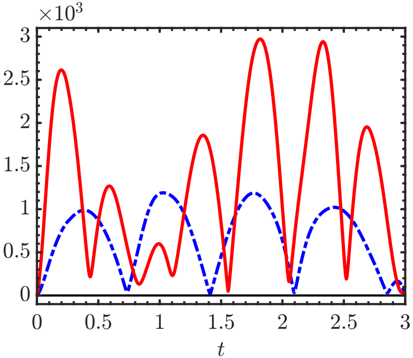


Fig. 7 – (red solid) $\Vert u^\star(\cdot,t)\Vert_{L^2(\omega)}$ ; (blue dashed) $\Vert u_0(\cdot,t)\Vert_{L^2(\omega)}$.

# Comparison with 2 fixed-point algorithms

We compare the least-squares algorithm with the algorithms

$$\begin{cases} (y_0, u_0) \in \mathcal{A} \text{ given}, \\ \Box y_{k+1} + \widehat{g}(y_k) y_{k+1} = u_{k+1} \mathbb{1}_\omega - g(0), \quad \forall k \in \mathbb{N}, \end{cases} \tag{$\widehat{\text{FP}}$}$$

$$\begin{cases} (y_0, u_0) \in \mathcal{A} \text{ given}, \\ \Box y_{k+1} = u_{k+1} \mathbb{1}_\omega - g(y_k), \quad \forall k \in \mathbb{N}. \end{cases} \tag{FP}$$

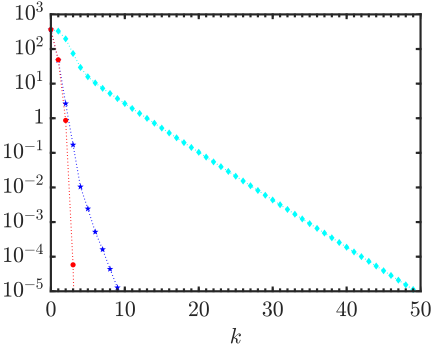


Fig. 8 – $c_g = 5$ – Evolution of $\sqrt{2E(y_k, u_k)}$,
(●) algo. (LS) ; (⋆) algo. ($\widehat{\text{FP}}$) ; (♦) algo. (FP).

• For large values of $c_g$, we expect a different dynamic, with the first values of the optimal step $\lambda_k$ being far from 1.

• Unfortunately, with the numerical scheme used to approximate the descent direction, large values of $c_g$ lead to numerical instabilities and overflow in the computation, due to the exponential growth of the uncontrolled solution.

• To conclude, we illustrate this dynamic in 1D, using a more robust numerical scheme. Namely, we use a space-time mixed formulation [3] to approximate the descent direction.

• Let $\Omega = (0, 1)$, $\omega = (0.2, 0.4)$ and $T = 3$.

• We take $y^0(x) = 100 \sin(\pi x)$, $y^1 = 0$ and $(z^0, z^1) = (0, 0)$.

• We consider the semi-linearity $g(r) = -c_g \, r \ln^{1/2}(2 + |r|)$, with $c_g \in \mathbb{R}$.

3. Cindea, Munch – A mixed formulation for the direct approximation of the control of minimal L2-norm, for linear type wave equations, Calcolo 2015

# Simulation for $c_g = 50$

$g(r) = -c_g \, r \ln^{1/2}(2 + |r|)$

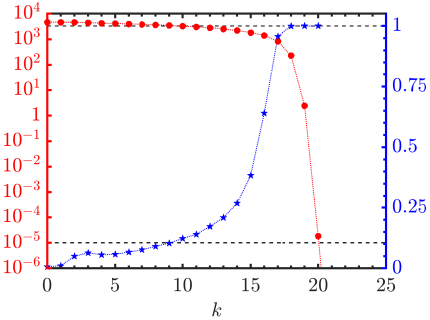


Fig. 9 – (●, left axis) $\sqrt{2E(y_k, f_k)}$ vs $k$ ;
(★, right axis) $\lambda_k$ vs $k$.

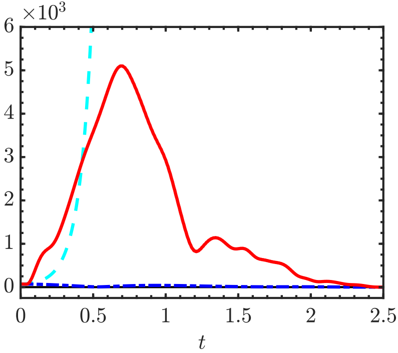


Fig. 10 – (red solid line) $\|y^\star(\cdot, t)\|_{L^2(\Omega)}$ ;
(blue dashed line) $\|y_0(\cdot, t)\|_{L^2(\Omega)}$ ; (cyan dashed line) $\|y(\cdot, t; 0)\|_{L^2(\Omega)}$.

# Thank you for your attention

A. B., J. Lemoine, A. Münch
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