Maximal distance minimizers

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Problem

For a given compact set $M \subset \mathbb{R}^n$ and a given number $r > 0$ find a closed connected Σ, such that

$$\begin{cases} M \subset \overline{B}_r(\Sigma) \\ \mathcal{H}^1(\Sigma) \text{ is minimal.} \end{cases}$$

The problem was stated at 2003 and was actively reseached in works by Miranda, Paolini, Butazzo and Stepanov (in $\mathbb{R}^n$). They proved that a minimizer Σ exists and that a minimizer can not contain a loop.

There are two main types of questions concerning maximal distance minimizers:

- regularity: what is local behaviour of minimizers;
- explicit examples: what it the minimizer for the concrete set M. Even for a circle and a rectangle the question is non-trivial.

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Problem (Statement 1)

For a given compact set $M \subset \mathbb{R}^n$ and a given number $r > 0$ to find a closed connected Σ, such that

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Problem (Statement 2)

For a given compact set $M \subset \mathbb{R}^n$ and a given number $r > 0$ find a closed connected Σ, such that

$$\begin{cases} F_M(\Sigma) := \max_{y \in M} \mathit{dist}(y, \Sigma) \le r \\ \mathcal{H}^1(\Sigma) \text{ is minimal.} \end{cases}$$

Problem (Dual statement)

For a given compact set $M \subset \mathbb{R}^n$ and a given number $l > 0$ find a closed connected Σ, such that

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The simplest examples

Example for two points at a distance $R > 2r$ apart:

One example for three points:

Each tripod Σ is a minimizer for some three points and $r > 0$. But not vice versa.

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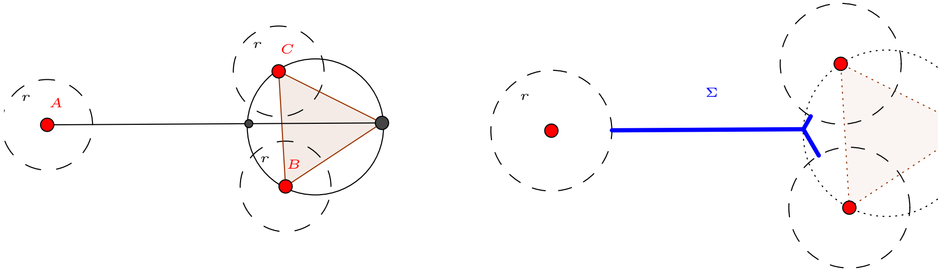

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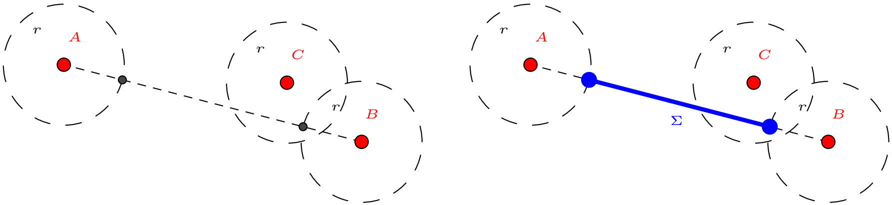
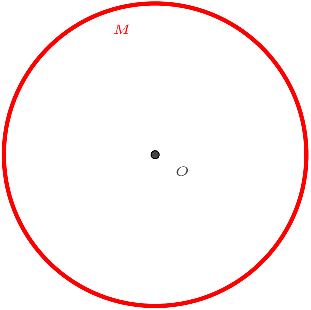
Another example for 3 points.



A segment Σ is minimizer for the border (or closure) of its r-neighbourhood.

Every maximal distance minimizer Σ for a set M and number $r > 0$ is also a minimizer for r-neighbourhood of Σ. Uniqueness is an open question here.

Let $M := \partial B_R(O)$, $R > 4.98r$. Then Σ is a horseshoe.



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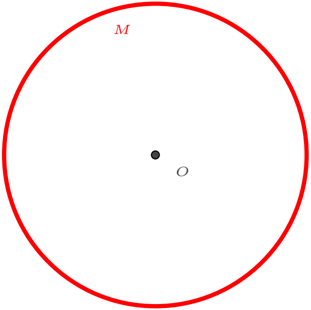

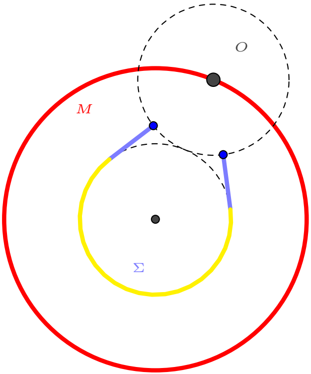

Solution for a concrete M. A curve with a great curvature radius

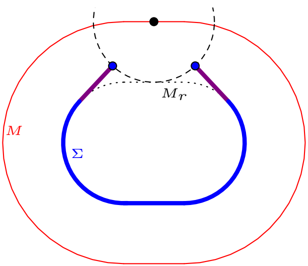

Figure: The solution for the set M with big radius of curvature

Theorem (Cherkashin, T., 2016)

For every closed convex curve M with minimal radius of curvature R and for every $r < R/5$ the set of minimizers contains only horseshoes. For the circumference $M = \partial B_R(O)$ the claim is true for $r < R/4.98$.

Still unknown: what is minimizer for a circle with $R > r > R/4.98$? (it conjectured for a circle by Paolini, Miranda and Stepanov that the answer still is a horseshoe)

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Still unknown:

1. What if $R > r > R/4.98$? (it conjectured for a circle by Paolini, Miranda and Stepanov that the answer still is a horseshoe)
2. What if M is a narrow stadium? (it is not a horseshoe!)

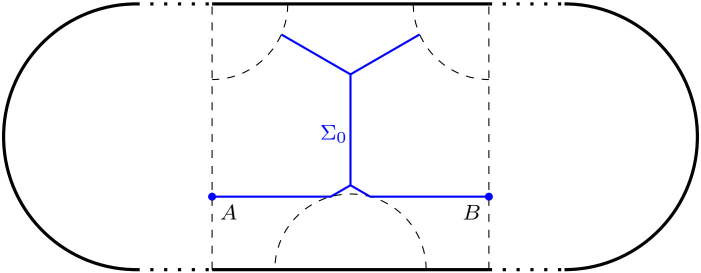

Figure: Horseshoe is not a minimizer for long enough stadium with $R < 1.75r$.

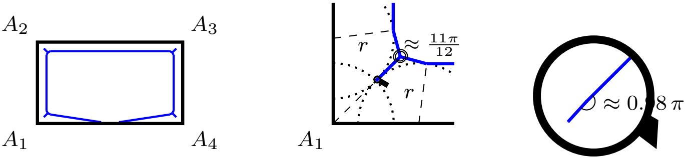

Figure: The minimizer for a rectangle M with $r < r_0(M)$.

When M is a rectangle, we described the topology of maximal distance minimizers (see our preprint arXiv:2106.00809).

Theorem (Cherkashin–Gordeev–Strukov–T,2021)

Let $M = A_1A_2A_3A_4$ be a rectangle, $r > 0$ be chosen small enough depending on M. Then any maximal distance minimizer has the topology depicted in the left part of Fig. 3. The middle part of the picture contains enlarged fragment of the minimizer near A_1; the labeled angles are equal to $\frac{2\pi}{3}$. The rightmost part contains much more enlarged fragment of minimizer near A_1. A minimizer consists of 21 segments; an approximation of the length of a minimizer is $Per - 8.473981r$, where Per is the perimeter of the rectangle.

Definition

We say that the ray $(ax]$ is a *tangent ray* of the set Σ at the point $x \in \Sigma$ if there exists a non stabilized sequence of points $x_k \in \Sigma$ such that $x_k \to x$ and $\angle x_k x a \to 0$.

Theorem (Gordeev, T., 2022)

Let Σ be a maximal distance minimizer for a compact set $M \subset \mathbb{R}^n$ and an $r > 0$ be fixed. Then

(i) *the angle between each pair of tangent rays at every point of Σ is at least $2\pi/3$. The number of tangent rays at every point of Σ is not greater than 3.*

(ii) *In planar case Σ is a union of a finite number of injective images of the segment $[0, 1]$ with non-intersecting interiors;*

Corollary

In planar case the number of triple points is finite.

Remark. It is not true for a Steiner tree, i. e. there exists an indecomposable Steiner tree with infinite number of triple points.

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- Find the minimizers for a circumference of radius $r < R < 4.98r$. Find the minimizers for a ball.
- Find the explicit estimate for the curvature radius at the horseshoe theorem
- Find the minimizers for a given stadium.
- Can maximal distance minimizer in Euclisean space have infinite many branching points?
- If Σ is a minimizer for some M then Σ is a minimizer for $\overline{B_r(\Sigma)}$. Is Σ the unique minimizer for $\overline{B_r(\Sigma)}$?
- Let Σ be a fixed planar compact set. It is interesting to determine whether Σ is a minimizer for M being a set of n points and some positive r. As an obvious necessary condition Σ should be a Steiner tree for some set of n points, but this condition is not sufficient. It turns out that a Steiner tree for the vertices of a square is not a maximal distance minimizer for every set of four points.

Thank you for your attention!

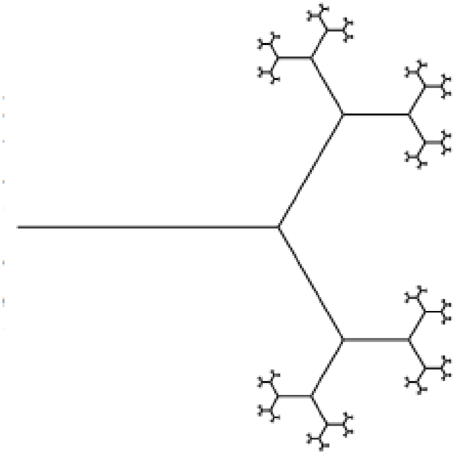

Figure: Indecomposable Steiner tree with infinite number of triple points