Analysis of gas networks using mathematical models in the form of differential algebraic equations.

Part II

Maria Filipkovska

Friedrich-Alexander-Universität Erlangen-Nürnberg

Department of Data Science

Chair for Dynamics, Control and Numerics

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine

Department of Mathematical Physics (Mathematical Division)

Consider **implicit ordinary differential equations** (ODEs) of the form

$$\frac{d}{dt}[A(t)x(t)] + B(t)x(t) = f(t,x(t)), \qquad t \in [t_+,\infty), \tag{1}$$

$$A(t)\frac{d}{dt}x(t) + B(t)x(t) = f(t,x(t)), \tag{2}$$

where $t_+ \geq 0$, $A(t), B(t)$ $(t \in [t_+,\infty))$ are closed linear operators from X to Y with the domains $D_{A(t)}$, $D_{B(t)}$, $D = D_{A(t)} \cap D_{B(t)} \neq \{0\}$, X, Y are Banach spaces, $f\colon [t_+,\infty) \times X \to Y$.

The time-varying operators $A(t)$, $B(t)$ can be degenerate.

The differential equations (DEs) (1) and (2) with a degenerate (for some t) operator $A(t)$ are called **time-varying (nonautonomous) degenerate DEs** or **time-varying differential-algebraic equations (DAEs)**. In the terminology of DAEs, equations of the form (1), (2) are commonly referred to as **semilinear**.

We study the initial value problem (the Cauchy problem) for the DAEs (1), (2) with the initial condition

$$x(t_0) = x_0. \tag{3}$$

Fields of application of the theory of DAEs are control theory, radioelectronics, cybernetics, mechanics, robotics technology, economics, ecology and chemical kinetics.

In particular, semilinear DAEs are used in modelling

- transient processes in electrical circuits
- gas flow in networks
- dynamics of neural networks
- dynamics of complex mechanical and technical systems (e.g., robots)
- multi-sectoral economic models
- kinetics of chemical reactions

Notice that any type of a PDE can be represented as a DAE in infinite-dimensional spaces (an abstract DAE) and, possibly, a complementary boundary condition.

Model of a gas flow for a single pipe

We consider the mathematical model of a gas pipeline which consists of the **isothermal Euler equations** of the form

$$\rho_t = -\varphi_x, \tag{4}$$
$$\varphi_t = -p_x - g\rho h_x - 0.5\lambda D^{-1}\varphi|\varphi|\rho^{-1} \tag{5}$$

and the **equation of state for a real gas** in the form

$$p = RT_0\rho z(p), \tag{6}$$

- $x \in [0,L]$, $t \in [0,t_1) \subseteq [0,\infty]$, where $[t_0,t_1)$ is the time interval, $L < \infty$ is the pipe length and T_0 is the temperature
- $\rho = \rho(t,x)$, $\varphi = \varphi(t,x)$ ($\varphi := \rho v$, v is the velocity) and $p = p(t,x)$ are respectively the density, flow rate and pressure
- g is the gravitational constant, and R is the specific gas constant
- λ is the pipe friction coefficient, and D is the pipe diameter
- $h = h(x)$ is the height profile of the pipe over ground
- $z = z(p)$ is the compressibility factor

[P. Benner, S. Grundel, C. Himpe, C. Huck, T. Streubel, C. Tischendorf. *Gas Network Benchmark Models*, 2018]

Denote $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -\frac{d}{dx} & 0 \\ -g\,h_x & 0 & -\frac{d}{dx} \\ 0 & 0 & -1 \end{pmatrix}$, $f(u) = \begin{pmatrix} 0 \\ -\frac{\lambda}{2D}\frac{\varphi|\varphi|}{\rho} \\ RT_0\rho z(p) \end{pmatrix}$ and $u = (\rho, \varphi, p)^T$. Then we can write the system (4)–(6) as:

$$A\frac{d}{dt}u(t) + Bu(t) = f(u(t)), \tag{7}$$

where $u = u(t)(x) = (\rho(t,x), \varphi(t,x), p(t,x))^T$, $x \in [0,L]$, $t \in [0,T] \subset [0,t_1)$. The initial condition has the form:

$$u(0) = u_0, \qquad u_0 = u_0(x) = (\rho(0,x), \varphi(0,x), p(0,x))^T, \quad x \in [0,L], \tag{8}$$

where $p(0,x)$ is chosen so as to satisfy the equation (6) for $t = 0$, $x \in [0,L]$. We will assume that $u(t,x)$ satisfies suitable boundary conditions, for example,

$$\varphi(t,0) = \varphi_l(t), \quad p(t,0) = p_l(t), \quad t \in [0,T], \tag{9}$$

i.e., $u(t)(0) = u_l(t) = (\rho(t,0), \varphi_l(t), p_l(t))^T$, where $\varphi_l(t)$ and $p_l(t)$ are given.

Then we consider the IVP (1), (3), where $X = Y = L_2$ and $A(t), B(t)\colon H_0^1[0,L] = \{u(x) \in H^1[0,L] \mid u(t)(0) = u_l(t)\} \to L_2$ for each $t \in [0,t_1)$.

[P. Benner, S. Grundel, C. Himpe, C. Huck, T. Streubel, C. Tischendorf. *Gas Network Benchmark Models*, 2018]

[Azevedo-Perdicoulis, T.-P., Jank, G. Modelling aspects of describing a gas network through a DAE system. 2007]

The gas network is considered to be described by a connected finite graph $G = (V, E, \psi)$, where V denotes a set of vertices with $|V| = n$, and E denotes a set of edges with $|E| = m$. The mapping $\psi \colon E \to V \times V$ is called the incidence map, where $\psi_1(e) = v_1$ is the initial vertex and $\psi_2(e) = v_2$ is the final vertex.

We define a flow $\varphi \colon E \to \mathbb{R}$ and a pressure-drop $p \colon E \to \mathbb{R}$ on every edge, and a nodal-pressure $\Phi \colon V \to \mathbb{R}$ and a nodal-flow $F \colon V \to \mathbb{R}$ on every vertex.

The Kirchhoff First Law (KFL) says that the flow rate vanishes at any vertex of the graph:*The Kirchhoff Second Law* (KSL) says that the pressure drop vanishes at every fundamental circuit of the graph.

Notice that the network may comprise valves, reservoirs, compressor stations, supplying sources, and regulators.

The isothermal Euler equations are linearised around the operation levels (p_*, φ_*), whence we set $p = p_* + \Delta p$ and $\varphi = \varphi_* + \Delta\varphi$, with Δp and $\Delta\varphi$ as the deviations from the pressure-drop and flow, respectively, from the reference values p_* and φ_*.

Discretizing the gas net with respect to the space variable x, we obtain the gas network model in the form of a DAE of a type

$$\frac{\partial}{\partial t}(Au) = Bu + f(t,u), \tag{10}$$

where $u = (\Delta\varphi, \Phi, u_q, u_p)^T$, u_q denotes a control device controlling the flow in some edges by a "flow", u_p denotes a control device controlling the flow in some edges by a "pressure" and A is a certain degenerate matrix. The gas network model also may include the parameters denoting control devices controlling the pressure in the edges or nodes either through flow or pressure.

Assume that the characteristic operator pencil $\lambda A(t)+B(t)$ ($\lambda \in \mathbb{C}$ is a parameter), associated with the linear part of the DAE (1) or (2), is a **regular pencil of index not higher than 1**: for each $t \geq t_+$ the pencil $\lambda A(t)+B(t)$ be regular and there exist functions $C_1 : [t_+,\infty) \to (0,\infty)$, $C_2 : [t_+,\infty) \to (0,\infty)$ such that for every $t \in [t_+,\infty)$ the pencil resolvent $R(\lambda,t) = (\lambda A(t)+B(t))^{-1}$ satisfies the constraint

$$\|R(\lambda,t)\| \leq C_1(t), \quad |\lambda| \geq C_2(t). \tag{11}$$

Then for each $t \in [t_+,\infty)$, there exist the two pairs of mutually complementary projectors

$$P_j(t) \colon D \to D_j(t) \text{ and } Q_j(t) \colon Y \to Y_j(t), \ j = 1,2,$$

which generate the direct decompositions

$$D = D_1(t) \dot{+} D_2(t), \quad Y = Y_1(t) \dot{+} Y_2(t) \quad \text{such that} \tag{12}$$

the pair of subspaces $X_1(t)$, $Y_1(t)$ and $X_2(t)$, $Y_2(t)$ are invariant under the operators $A(t)$, $B(t)$, and $A_j(t) = A(t)|_{D_j(t)}$, $B_j(t) = B(t)|_{D_j(t)} : D_j(t) \to Y_j(t)$, $j = 1,2$, are such that $A_2(t) = 0$, and there exist $A_1^{-1}(t)$ and $B_2^{-1}(t)$ if $D_1(t) \neq \{0\}$, $D_2(t) \neq \{0\}$ respectively ($D_2(t) = \operatorname{Ker} A(t) \cap D$, $Y_1(t) = A(t)D$)

$$A(t) = A_1(t) \dot{+} A_2(t), \ B(t) = B_1(t) \dot{+} B_2(t) \colon D_1(t) \dot{+} D_2(t) \to Y_1(t) \dot{+} Y_2(t) \tag{13}$$

[Rutkas A.G., Vlasenko L.A. Existence of solutions of degenerate nonlinear differential operator equations, *Nonlinear Oscillations*, 2001]

For each $t \in [t_+,\infty)$ the projectors can be determined by the formulas [Rutkas A.G., Vlasenko L.A. *Nonlinear Oscillations*, 2001]

$$\begin{aligned} P_1(t) &= \frac{1}{2\pi i} \oint\limits_{|\lambda|=C_2(t)} R(\lambda,t)\,A(t)\,d\lambda, \quad P_2(t) = I_X - P_1(t), \\ Q_1(t) &= \frac{1}{2\pi i} \oint\limits_{|\lambda|=C_2(t)} A(t)\,R(\lambda,t)\,d\lambda, \quad Q_2(t) = I_Y - Q_1(t). \end{aligned} \tag{14}$$

and the auxiliary operator $G(t) = A(t) + B(t)P_2(t)\colon D \to Y$ has the bounded inverse $G^{-1}(t) \in (Y,X)$.

Let $X = Y = D = \mathbb{R}^n$.

For each t any $x \in \mathbb{R}^n$ can be uniquely represented in the form

$$x = x_{p_1}(t) + x_{p_2}(t), \quad x_{p_i}(t) = P_i(t)x \in X_i(t).$$

The DAE (1) $[A(t)x(t)]' + B(t)x(t) = f(t,x(t))$ is reduced to the equivalent system

$$[P_1(t)x(t)]' = \left[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\right]P_1(t)x(t) + G^{-1}(t)Q_1(t)f(t,x(t)),$$
$$G^{-1}(t)Q_2(t)[f(t,x(t)) - A'(t)P_1(t)x(t)] - P_2(t)x(t) = 0 \qquad \text{or}$$

$$x_{p_1}'(t) = \left[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\right]x_{p_1}(t) + G^{-1}(t)Q_1(t)f(t,x), \quad (15)$$
$$G^{-1}(t)Q_2(t)[f(t,x_{p_1}(t) + x_{p_2}(t)) - A'(t)x_{p_1}(t)] - x_{p_2}(t) = 0. \quad (16)$$

Introduce the manifold

$$L_{t_+} = \{(t,x) \in [t_+,\infty) \times \mathbb{R}^n \mid Q_2(t)[B(t)x + A'(t)P_1(t)x - f(t,x)] = 0\}. \quad (17)$$

The consistency condition $(t_0,x_0) \in L_{t_+}$ for the initial point (t_0,x_0) is one of the necessary conditions for the existence of a solution of the initial value problem (1), (3).

$V'_{(15)}(t,x_{p_1}(t)) = \frac{\partial V}{\partial t}(t,x_{p_1}(t)) + \Big(\frac{\partial V}{\partial z}(t,x_{p_1}(t)), \big[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\big]x_{p_1}(t) + G^{-1}(t)Q_1(t)f(t,x_{p_1}(t) + x_{p_2}(t))\Big)$ is the derivative of the function $V(t,z)$ along the trajectories of the equation (15), where $V(t,z)$ is a continuously differentiable and positive definite scalar function.

Main results:

- **Theorems on the existence and uniqueness of global solutions**

  Some advantages: the restrictions of the type of the global Lipschitz condition (including contractive mapping) are not used.
- **Theorem on the Lagrange stability of the DAE** (the boundedness of solutions)
- **Theorem on the Lagrange instability of the DAE** (solutions have finite escape time)
- **Theorem on the ultimate boundedness (dissipativity) of the DAE** (the ultimate boundedness of solutions)
- **Theorems on the Lyapunov stability and instability of the equilibrium state of the DAE**
- **Theorems on asymptotic stability and asymptotic stability in the large of the equilibrium state** (complete stability of the DAE)
- **Numerical methods**

The obtained theorems were used for the study of certain mathematical models of electrical circuits with nonlinear and time-varying elements.

The IVP (1), (3): $\frac{d}{dt}[A(t)x(t)] + B(t)x(t) = f(t,x(t)), \quad x(t_0) = x_0.$

Definitions

Let $f(t,0) \equiv 0$ and $f\colon [t_+,\infty) \times U_R^x(0) \to \mathbb{R}^n$, where

$$U_R^x(0) = \{x \in \mathbb{R}^n \mid \|x\| < R\}.$$

An *equilibrium position* (a *stationary solution*) $x_*(t) \equiv 0$ of the DAE (1) ($f(t,0) \equiv 0$) is called **Lyapunov stable**, or simply **stable**, if for each $\varepsilon > 0$ ($\varepsilon < R$) and each $t_0 \in [t_+,\infty)$ there exists a number $\delta = \delta(\varepsilon, t_0) > 0$ ($\delta \le \varepsilon$) such that for any consistent initial point (t_0,x_0) satisfying the condition $\|x_0\| < \delta$ there exists a global solution $x(t)$ of the IVP (1), (3) and this solution satisfies the inequality $\|x(t)\| < \varepsilon$ for all $t \in [t_0,\infty)$.
If, in addition, there exists $\widetilde{\delta} = \widetilde{\delta}(t_0) > 0$ ($\widetilde{\delta} \le \delta$) such that for each solution $x(t)$ with an initial point (t_0,x_0) satisfying the condition $\|x_0\| < \widetilde{\delta}$ the requirement $\lim\limits_{t\to\infty} x(t) = 0$ is fulfilled, then $x_*(t) \equiv 0$ is called **asymptotically Lyapunov stable**, or simply **asymptotically stable**.

If in the previous definition the number δ is independent of t_0, then the solution is called **uniformly Lyapunov stable** or **uniformly stable** (on $[t_+,\infty)$).

An *equilibrium position* $x_*(t) \equiv 0$ of the DAE (1) $(f(t,0) \equiv 0)$ is called **Lyapunov unstable**, or simply **unstable**, if for some $\varepsilon > 0$ $(\varepsilon < R)$, $t_0 \in [t_+,\infty)$ and any $\delta > 0$ there exist a solution $x_\delta(t)$ of the IVP (1), (3) and a time moment $t_1 > t_0$ such that $\|x_0\| < \delta$ and $\|x_\delta(t_1)\| \geq \varepsilon$.

Let $f(t,0) \equiv 0$ and $f\colon [t_+,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$.
If the equilibrium position $x_*(t) \equiv 0$ of the DAE (1) $(f(t,0) \equiv 0)$ is asymptotically stable and, moreover, for each point $(t_0,x_0) \in L_{t_+}$ there exists a global solution $x(t)$ of the IVP (1), (3) and $\lim\limits_{t\to\infty} x(t) = 0$, then $x_*(t) \equiv 0$ is called **asymptotically stable in the large**, and the DAE is called **completely stable** or **asymptotically stable**.

Remark. Since the Lagrange instability of a solution implies its Lyapunov instability, the theorems on the Lagrange instability of DAEs can also be treated as Lyapunov instability theorems.

Asymptotic stability in the large for explicit ODEs was considered in [Krasovsky N.N. *Some problems of the theory of stability of motion*, 1959].

$$B_{r_1}(0) = \{z \in \mathbb{R}^n \mid \|z\| \le r_1\},$$
$$B_{r_1,r_2}^{x_{p_1},x_{p_2}}(0) = \{x \in \mathbb{R}^n \mid \|x_{p_i}(t)\| \le r_i, x_{p_i}(t) = P_i(t)x, i = 1,2\}. \tag{18}$$

Theorem 6 (Lyapunov stability and asymptotic stability of equilibrium position of the DAE). Let $f \in C([t_+,\infty) \times U_R^x(0),\mathbb{R}^n)$, $f(t,0) \equiv 0$, $\partial f/\partial x \in C([t_+,\infty) \times U_R^x(0),L(\mathbb{R}^n))$, $A,B \in C^1([t_+,\infty),L(\mathbb{R}^n))$, and the pencil $\lambda A(t) + B(t)$ satisfy (11), where $C_2 \in C^1([t_+,\infty),(0,\infty))$. Assume that for each $t_* \in [t_+,\infty)$ and $x_{p_1}^*(t_*) = 0$, $x_{p_2}^*(t_*) = 0$ the operator
$\Phi_{t_*,x_{p_1}^*(t_*),x_{p_2}^*(t_*)} \colon X_2(t_*) \to Y_2(t_*)$,
$\Phi_{t_*,x_{p_1}^*(t_*),x_{p_2}^*(t_*)} = \left[\frac{\partial}{\partial x}\left[Q_2(t_*)f(t_*,x_{p_1}^*(t_*) + x_{p_2}^*(t_*))\right] - B(t_*)\right]P_2(t_*)$, is invertible. Then the following statements are true:

1. Let there exist numbers $r_1,r_2 > 0$, $r_1 + r_2 < R$, and a positive definite function $V \in C^1([t_+,\infty) \times B_{r_1}(0),\mathbb{R})$ such that for all $t \in [t_+,\infty)$, $x \in B_{r_1,r_2}^{x_{p_1},x_{p_2}}(0)$ the following inequality holds:

$$V'_{(15)}(t,x_{p_1}(t)) \le 0. \tag{19}$$

Then the equilibrium position $x_*(t) \equiv 0$ of the DAE (1) is Lyapunov stable.

2. Let there exist numbers $r_1, r_2 > 0$, $r_1 + r_2 < R$, and positive definite functions $V \in C^1([t_+, \infty) \times B_{r_1}(0), \mathbb{R})$, $W \in C(B_{r_1}(0), \mathbb{R})$, and $U \in C(B_{r_1}(0), \mathbb{R})$ such that $V(t,z) \leq W(z)$ for all $t \in [t_+, \infty)$, $z \in B_{r_1}(0)$, and

$$V'_{(15)}(t, x_{p_1}(t)) \leq -U\big(x_{p_1}(t)\big) \tag{20}$$

for all $t \in [t_+, \infty)$, $x \in B_{r_1, r_2}^{x_{p_1}, x_{p_2}}(0)$, $x_{p_1}(t) \neq 0$;
also, let

$$G^{-1}(t)Q_2(t)[f(t, P_1(t)x + P_2(t)x) - A'(t)P_1(t)x] \to 0$$
$$\text{as } x \to 0 \text{ uniformly in } t \text{ on } [T, \infty) \text{ for some } T > t_+. \tag{21}$$

Then the equilibrium position $x_*(t) \equiv 0$ of the DAE (1) is asymptotically Lyapunov stable.

Theorem 7 (Lyapunov instability of equilibrium position of the DAE).
Let $f \in C([t_+,\infty) \times U_R^x(0),\mathbb{R}^n)$, $f(t,0) \equiv 0$, $\partial f/\partial x \in C([t_+,\infty) \times U_R^x(0),L(\mathbb{R}^n))$, $A,B \in C^1([t_+,\infty),L(\mathbb{R}^n))$, and the pencil $\lambda A(t)+B(t)$ satisfy (11), where $C_2 \in C^1([t_+,\infty),(0,\infty))$. Assume that for each $t_* \in [t_+,\infty)$ and $x_{p_1}^*(t_*)=0$, $x_{p_2}^*(t_*)=0$ the operator $\Phi_{t_*,x_{p_1}^*(t_*),x_{p_2}^*(t_*)}\colon X_2(t_*) \to Y_2(t_*)$,
$\Phi_{t_*,x_{p_1}^*(t_*),x_{p_2}^*(t_*)} = \left[\frac{\partial}{\partial x}\left[Q_2(t_*)f(t_*,x_{p_1}^*(t_*)+x_{p_2}^*(t_*))\right]-B(t_*)\right]P_2(t_*)$, is invertible. Let there exist numbers $T \ge t_+$ and $r_1,r_2>0$, $r_1+r_2<R$, and a function $V \in C^1([T,\infty) \times B_{r_1}(0),\mathbb{R})$ such that

1. $V(t,z) \to 0$ uniformly in t on $[T,\infty)$ as $\|z\| \to 0$;
2. there exists a positive function $U \in C(B_{r_1}(0),[0,\infty))$ such that

$$V'_{(15)}(t,x_{p_1}(t)) \ge U\big(x_{p_1}(t)\big) > 0 \quad \text{or} \quad V'_{(15)}(t,x_{p_1}(t)) \le -U\big(x_{p_1}(t)\big) < 0 \qquad (22)$$

for all $t \in [T,\infty)$, $x \in B_{r_1,r_2}^{x_{p_1},x_{p_2}}(0)$, $x_{p_1}(t) \ne 0$;

3. for each $\Delta_1>0$ and each $\Delta_2>0$, $\Delta_i \le r_i$, there exist $x_{p_1}(T) \ne 0$ and $x_{p_2}(T)$ such that $\|x_{p_i}(T)\| < \Delta_i$, $i=1,2$, and $V(T,x_{p_1}(T))\,V'_{(15)}(T,x_{p_1}(T)) > 0$ (i.e., the sign of the function V coincides with the sign of the derivative $V'_{(15)}$ at $(T,x_{p_1}(T))$).

Then the equilibrium position $x_*(t) \equiv 0$ of the DAE (1) is Lyapunov unstable.

Theorem 8 (asymptotic stability in the large or complete stability of the DAE). Let $f \in C([t_+,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t,0) \equiv 0$, $\partial f/\partial x \in C([t_+,\infty) \times \mathbb{R}^n, L(\mathbb{R}^n))$, $A, B \in C^1([t_+,\infty), L(\mathbb{R}^n))$, the pencil $\lambda A(t) + B(t)$ satisfy (11), where $C_2 \in C^1([t_+,\infty),(0,\infty))$. Let the conditions 1), 2) of Theorem 1 or 1), 2) of Theorem 2 (on the global solvability), as well as condition (21), be satisfied. Assume that there exist positive definite functions $V \in C^1([t_+,\infty) \times \mathbb{R}^n, \mathbb{R})$, $W \in C(\mathbb{R}^n, \mathbb{R})$ and $U \in C(\mathbb{R}^n, \mathbb{R})$ such that

1. $V(t,z) \le W(z)$ for all $t \in [t_+,\infty)$, $z \in \mathbb{R}^n$;
2. $V(t,z) \to \infty$ uniformly in t on $[t_+,\infty)$ as $\|z\| \to \infty$;
3. for all $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $x_{p_1}(t) \ne 0$ $\big(x_{p_i}(t) = P_i(t)x,\ i = 1,2\big)$, the inequality (20) holds.

Then the equilibrium position $x_*(t) \equiv 0$ of the DAE (1) is asymptotically stable in the large (the DAE (1) is completely stable).

The conditions 1), 2) of Theorem 1 (on the global solvability):

1) for each $t \in [t_+,\infty)$ and each $x_{p_1}(t) \in X_1(t)$ there exists a unique $x_{p_2}(t) \in X_2(t)$ such that $(t,x_{p_1}(t)+x_{p_2}(t)) \in L_{t_+}$;

2) for each $t_* \in [t_+,\infty)$, $x^*_{p_i}(t_*) \in X_i(t_*)$, $i=1,2$, such that $(t_*,x^*_{p_1}(t_*)+x^*_{p_2}(t_*)) \in L_{t_+}$ the operator $\Phi_{t_*,x^*_{p_1}(t_*),x^*_{p_2}(t_*)}\colon X_2(t_*) \to Y_2(t_*)$, $\Phi_{t_*,x^*_{p_1}(t_*),x^*_{p_2}(t_*)} = \left[\frac{\partial}{\partial x}\left[Q_2(t_*)f(t_*,x^*_{p_1}(t_*)+x^*_{p_2}(t_*))\right] - B(t_*)\right] P_2(t_*)$, is invertible;

The conditions 1), 2) of Theorem 2 (on the global solvability):

1) for each $t \in [t_+,\infty)$, $x_{p_1}(t) \in X_1(t)$ there exists $x_{p_2}(t) \in X_2(t)$ such that $(t,x_{p_1}(t)+x_{p_2}(t)) \in L_{t_+}$;

2) for each $t_* \in [t_+,\infty)$, $x^*_{p_1}(t_*) \in X_1(t_*)$, $x^i_{p_2}(t_*) \in X_2(t_*)$ such that $(t_*,x^*_{p_1}(t_*)+x^i_{p_2}(t_*)) \in L_{t_+}$, $i=1,2$, the operator function $\Phi_{t_*,x^*_{p_1}(t_*)}(x_{p_2}(t_*))$ defined by $\Phi_{t_*,x^*_{p_1}(t_*)}\colon X_2(t_*) \to L(X_2(t_*),Y_2(t_*))$, $\Phi_{t_*,x^*_{p_1}(t_*)}(x_{p_2}(t_*)) = \left[\frac{\partial}{\partial x}\left[Q_2(t_*)f(t_*,x^*_{p_1}(t_*)+x_{p_2}(t_*))\right] - B(t_*)\right] P_2(t_*)$, is basis invertible on $[x^1_{p_2}(t_*),x^2_{p_2}(t_*)]$.

Outlooks

1. It is planned to extend the obtained results to the case when X, Y are Banach spaces, $f\colon [t_+,\infty)\times X \to Y$ and $A(t), B(t)\colon X \to Y$ $(t \in [t_+,\infty))$ are closed linear operators with the domains $D_{A(t)}$, $D_{B(t)}$, $D = D_{A(t)} \cap D_{B(t)} \neq \{0\}$.

   In this case we will require that $A(t)$, $B(t)$ be strongly continuously differentiable on $[t_+,\infty)$ (i.e., for each $d \in D$ the functions $A(t)d$, $B(t)d$ be continuously differentiable on $[t_+,\infty)$).

2. It is planned to consider semilinear time-varying DAEs of index higher than 1.

   A regular pencil $\lambda A(t)+B(t)$ is a regular pencil of **index** ν $(\nu \in \mathbb{N})$ if there exist functions $C_1, C_2\colon [t_+,\infty) \to (0,\infty)$ such that for every $t \in [t_+,\infty)$

$$\|R(\lambda,t)\| \le C_1(t)|\lambda|^{\nu-1}, \quad |\lambda| \ge C_2(t). \tag{23}$$

   [Rutkas A.G., Vlasenko L.A. Existence of solutions of degenerate nonlinear differential operator equations, *Nonlinear Oscillations*, 2001]

   Then for each $t \in [t_+,\infty)$ there exist the two pairs of mutually complementary projectors $P_j(t)$, $Q_j(t)$, $j = 1,2$, (14) which generate the direct decompositions of D and Y (12) such that the operators $A(t)$, $B(t)$ have the block representations (13), where $A_1^{-1}(t)$ and $B_2^{-1}(t)$ exist.

   In general, the order of pole of the resolvent $(A(t)+\mu B(t))^{-1}$ at the point $\mu = 0$ is called the *index* of the regular pencil $\lambda A(t)+B(t)$.

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Thank you for your attention!