Hautus-Yamamoto criteria for approximate and exact controllability of difference delay equations

Y. Chitour, S. Fueyo, G. Mazanti and M. Sigalotti

August 31, 2022

Talk IX Partial differential equations, optimal design and numerics 2022

Difference delay systems

- We consider difference delay systems of the form:

$$x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) + Bu(t), \; x \in \mathbb{R}^d, \; u \in \mathbb{R}^m, \; t \geq 0. \quad (1)$$

- d, m, N are positive integers, $A_j \in \mathbb{R}^{d \times d}$ for $1 \leq j \leq N$, $B \in \mathbb{R}^{d \times m}$, $0 < \Lambda_1 < \cdots < \Lambda_N$.

Interest $\Longrightarrow$ L^q controllability criteria in the frequency domain, $q \in [1, +\infty)$.

1-D hyperbolic PDE of conservation laws

$$\begin{cases} \partial_t R(t,x) + \Lambda \partial_x R(t,x) = 0, (t,x) \in \Omega_{\mathrm{hyp}}, \\ R(t,0) = KR(t,1) + Bu(t), \qquad t \geq 0, \end{cases}$$

with

- Diagonal matrix Λ:

$$\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_d\},\ \lambda_i > 0 \text{ for all } i = 1, \ldots, d. \tag{2}$$

- Domain :

$$\Omega_{\mathrm{hyp}} \ = \ \{(t,x) \in \mathbb{R}^2,\ 0 < x < 1 \text{ and } 0 < t < +\infty\}; \tag{3}$$

Characteristic method implies:

$$\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = K \begin{pmatrix} y_1(t-\Lambda_1) \\ \vdots \\ y_n(t-\Lambda_n) \end{pmatrix} + Bu(t),\ \text{for } t \geq 0,\ \Lambda_i = 1/\lambda_i. \tag{4}$$

Hautus criteria for ODE

Consider the ordinary differential equation in $\mathbb{R}^d$, $d \in \mathbb{N}$:

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \quad x(0) \in \mathbb{R}^d, \quad B \in \mathbb{R}^{d\times m}. \tag{5}$$

Definition

The ODE is controllable if for all $T > 0$ and $x(0), x_f \in \mathbb{R}^d$, there exists a function u defined on $[0, T]$ such that $x(T) = x_f$.

Theorem (Kalman)

The ODE system is controllable if and only if

$$\mathrm{Rank}\,[B,\, AB,\, \ldots,\, A^{d-1}B] = d. \tag{6}$$

Theorem (Hautus-Fattorini)

The ODE system is controllable if and only if

$$\mathrm{Rank}\,[pI_d - A, B] = d, \quad p \in \mathbb{C}. \tag{7}$$

Hautus Criteria for Difference delay system?

Taking the Laplace transform in the ODE leads to:

$$(pI_d - A)\,\hat{x}(p) = B\,\hat{u}(p), \quad p \in \mathbb{C}, \tag{8}$$

where $\hat{x}(p) = \int_{-\infty}^{+\infty} x(t)e^{-pt}dt$ and $\hat{u}(p) = \int_{-\infty}^{+\infty} u(t)e^{-pt}dt$.

We now do the same in the difference delay system where we take the Laplace transform in the system, with $x(t) = u(t) = 0$, $t < 0$:

$$\widehat{Q}_0(p)\widehat{x}(p) = B\widehat{u}(p), \quad p \in \mathbb{C}, \tag{9}$$

where

$$\widehat{Q}_0(p) := I_d - \sum_{j=1}^{N} e^{-p\Lambda_j} A_j, \qquad p \in \mathbb{C}.$$

We want L^q controllability criteria in terms of $\widehat{Q}_0(p)$ and B

Realization theory

Input u belongs to the subspace of $L^q(\mathbb{R}, \mathbb{R}^m)$ with compact support included in $\mathbb{R}_-$:

$$\begin{cases} x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) + Bu(t), & \text{for } t \geq \inf \operatorname{supp}(u), \\ x(t) = 0, & \text{for } t < \inf \operatorname{supp}(u), \\ y(t) = x(t - \Lambda_N), & \text{for } t \in [0, +\infty), \end{cases} \tag{10}$$

Want to write System (10) as a convolution operator with a kernel in the space of Radon measures $M(\mathbb{R})$ with support in $\mathbb{R}_+$, i.e. we want to find $A \in M_+(\mathbb{R})$ such that the input-output system (10) can be represented as

$$y(t) = \int_{-\infty}^{+\infty} A(t-\tau)u(\tau)d\tau = (A * u)(t), \qquad t \in \mathbb{R}^+, \tag{11}$$

where $*$ is the convolution product.

Realization theory

We have

$$y(t) = (A * u)(t), \qquad t \in \mathbb{R}^+ \quad \text{and} \quad A = Q^{-1} * P, \qquad (12)$$

where

$$Q := \delta_{-\Lambda_N} I_d - \sum_{j=1}^{N} \delta_{-\Lambda_N + \Lambda_j} A_j, , \quad P := B\delta_0,$$

and Q^{-1} is invertible over the Radon measure space $M(\mathbb{R})$.

Remark

We have $\widehat{Q}(p) = \widehat{Q}_0(p) e^{p\Lambda_N}$.

Realization theory

We define $\pi : \phi \to \phi|_{\mathbb{R}_+}$ the truncation operator on $L^q(\mathbb{R}, \mathbb{R}^d)$.
We can rewrite the system as:

$$y = \pi(A * u). \tag{13}$$

We define the state space of System (10) in terms of the distribution Q as

$$\left(L^q\right)^Q := \left\{ y \in L^q\left(\mathbb{R}_+, \mathbb{R}^d\right) \mid \pi(Q * y) = 0 \right\}. \tag{14}$$

State space in Radon measures:

$$(R)^Q := \left\{ \pi\phi \mid \ \phi \in (M(\mathbb{R}_+))^d \text{ and } \pi(Q * \pi\phi) = 0 \right\}. \tag{15}$$

State space in distributions:

$$(D)^Q := \left\{ \pi\phi \mid \ \phi \in \left(D'(\mathbb{R}_+)\right)^d \text{ and } \pi(Q * \pi\phi) = 0 \right\}. \tag{16}$$

Definition approximate controllability

The difference delay system is said to be:

1) *L^q approximately controllable (from the origin)* if for every $\phi \in L^q([-\Lambda_N, 0], \mathbb{R}^d)$, there exist $n \in \mathbb{N}$, $T_n > 0$ and $u_n \in L^q([0, T_n], \mathbb{R})$ such that

$$\lim_{n\to+\infty} \|y(T_n + \cdot) - \phi(\cdot)\|_{L^q([-\Lambda_N,0],\mathbb{R}^d)} = 0;$$

2) *R(adon) approximately controllable (from the origin)* if for every $\phi \in M([-\Lambda_N, 0], \mathbb{R}^d)$, there exist $n \in \mathbb{N}$, $T_n > 0$ and $u_n \in M([0, T_n], \mathbb{R})$ such that

$$y(T_n + \cdot) - \phi(\cdot) \underset{n\to+\infty}{\longrightarrow} 0, \quad \text{in distributional sense.}$$

3) *D(istributional) approximately controllable (from the origin)* if for every $\phi \in \mathcal{D}'([-\Lambda_N, 0], \mathbb{R}^d)$, there exist $n \in \mathbb{N}$, $T_n > 0$ and $u_n \in \mathcal{D}'([0, T_n], \mathbb{R})$ such that

$$y(T_n + \cdot) - \phi(\cdot) \underset{n\to+\infty}{\longrightarrow} 0, \quad \text{in distributional sense.}$$

Definition exact controllability

The difference delay system is said to be:

1) *L^q exactly controllable (from the origin)* if for every $\phi \in L^q([-\Lambda_N, 0], \mathbb{R}^d)$, there exist $T > 0$ and $u \in L^q([0, T], \mathbb{R})$ such that

$$y(T+\theta) = \phi(\theta), \quad \theta \in [-\Lambda_N, 0];$$

2) *R(adon) exactly controllable (from the origin)* if for every $\phi \in M([-\Lambda_N, 0], \mathbb{R}^d)$, there exist $T > 0$ and $u \in M([0, T], \mathbb{R})$ such that

$$y(T+\theta) = \phi(\theta) \quad \text{for } \theta \in [-\Lambda_N, 0] \text{ in distributional sense.}$$

3) *D(istributional) exactly controllable (from the origin)* if for every $\phi \in \mathcal{D}'([-\Lambda_N, 0], \mathbb{R}^d)$, there exist $T > 0$ and $u \in \mathcal{D}'([0, T], \mathbb{R})$ such that

$$y(T+\theta) = \phi(\theta) \quad \text{for } \theta \in [-\Lambda_N, 0] \text{ in distributional sense.}$$

Controllability in terms of realization theory

$X \in \{L^q, R, D\}$

- System (1) is
  1) X approximately controllable if and only if for every $\pi\phi \in (X)^Q$ there exists a sequence of inputs $(u_n)_{n\in\mathbb{N}}$ (in "X") such that:

$$\pi(A * u_n) \underset{n\to+\infty}{\longrightarrow} \pi\phi \quad \text{in} \quad X\left(\mathbb{R}_+, \mathbb{R}^d\right);$$

  2) X exactly controllable if and only if for every $\pi\phi \in (X)^Q$ there exists u (in "X") such that its associated output through System (10) satisfies

$$\pi(A * u) = \pi\phi.$$

Approximate controllability

Theorem (Yamamoto 88,89)

We have the following equivalences:

a) *L^q approximate controllability*

b) *Radon approximate controllability*

c) *Distributional approximate controllability*

d) *$\exists$ two sequences of distribution $(S_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$ compactly supported in $\mathbb{R}_-$ s.t.:*

$$Q * R_n + P * S_n \underset{n\to+\infty}{\longrightarrow} \delta_0 I_d, \quad \text{in distributional sense.} \tag{17}$$

e) *(Hautus-Yamamoto criteria) The two conditions hold true:*

   1) $\mathrm{rank}\left[\widehat{Q}(p), B\right] = d$ *for every* $p \in \mathbb{C}$,

   2) $\mathrm{rank}\,[A_N, B] = d$.

Fundamental ingredient: Approximate Bézout identity.

Sketch of Proof

Equivalence between a), b), c) follow from standard density arguments.
Equivalence of them with d) is essentially explained next slide for exact controllability.
d) implies e): Condition 1) follows as for the exact case explained later and Condition 2) follows from Paley-Wiener.
e) implies a): Harder.

- Consider the semigroup of right translation on $(L^q)^Q$. It has a generator A;
- The spectrum of A is composed of eigenvalues and they are the zero of $\det \widehat{Q}(.)$;
- Condition $\operatorname{rank}\left[\widehat{Q}(p), B\right] = d$ for every $p \in \mathbb{C}$ translate the exact controllability on each eigenspace;
- Condition $\operatorname{rank}[A_N, B] = d$ express that the closure of the span of all the eigenspaces is $(L^q)^Q$.

Exact controllability

Theorem (Yamamoto 2009)

The distributional exact controllability holds if and only if we have the existence of two distributions S and R compactly supported in $\mathbb{R}_-$ such that:

$$Q * R + P * S = \delta_0 I_d. \tag{18}$$

Theorem (Yamamoto 2008)

Let $d = 1$ and $Z := \{\lambda \in \mathbb{C} | \widehat{Q}(\lambda) = 0\}$. We have the equivalence:

1. *The distributional exact controllability holds.*
2. *Suppose that the algebraic multiplicity of each zero $\lambda \in \mathbb{C}$ is uniformly bounded. There exist $k \geq 0$ and $c > 0$ such that $|\lambda^k \widehat{P}(\lambda)| \geq c$ for all $\lambda \in Z$.*

Why the Bézout arises in the controllability?

- Assume system distributional exactly controllable. We have $\pi Q^{-1} \in (D)^Q$ so that there exists S a distribution such that $\pi(Q^{-1} * P * S) = \pi Q^{-1}$. Let the distribution $R = Q^{-1} * P * S - Q^{-1}$. Then its support is bounded on the left and $\pi R = 0$ then R is a compactly supported distribution in $\mathbb{R}_-$. Hence Bézout identity (after multiplying by Q on the left).
- Assume the Bézout identity holds true. Takes any $\pi\phi$ in $(D)^Q$.
  - Multiply Bézout on the left by Q^{-1} and on the right by $Q * \phi$:

$$\begin{aligned} \phi &= Q^{-1} * (\delta_0 Id) * Q * \phi = Q^{-1} * (P * S + Q * R) * Q * \phi \\ &= Q^{-1} * P * S * Q * \phi + R * Q * \phi. \end{aligned}$$

  - Since (!!) $\pi(R * Q * \phi) = \pi(R * \pi(Q * (\pi\phi))) = 0$, we get:

$$\pi\phi = \pi\left(Q^{-1} * P * S * Q * \phi\right) = \pi\left(Q^{-1} * P * \omega\right).$$

  - Thus $\omega = S * Q * \phi$ is a control leading to the target $\pi\phi$.

Bézout implies Hautus-Yamamoto for exact controllability

Proposition

If the Bézout holds true then we have:

1) $\mathrm{rank}\left[\widehat{Q}(p), B\right] = d$ *for every* $p \in \mathbb{C}$,

2) $\mathrm{rank}\,[A_N, B] = d$.

- Let us prove that Condition 1) is necessary. Let $g^T \in \mathbb{C}^d$ a nonzero line vector such that $g^T \widehat{Q}(p) = 0$. From the Bézout identity, one obtains:

$$g^T B \widehat{S}(p) = g^T \tag{19}$$

  so that $g^T B \neq 0$.

- Conditions 2) is necessary for the approximate distributionnal controllability so it is a necessary condition for the exact distributionnal controllability.

??Hautus-Yamamoto implies Bézout??
A Corona problem

- Assume that the hautus-Yamamoto criteria holds since it is necessary for the exact distributional controllability.
- Laplace transform in the Bézout identity:

$$\widehat{Q}(p)\widehat{R}(p) + B\widehat{S}(p) = I_d, \quad p \in \mathbb{C}. \tag{20}$$

- (assume here $m = 1$) For $p \in \mathbb{C}$ zero of $\det \widehat{Q}(.)$, let $g^T \widehat{Q}(p) = 0$ with $\|g^T\| = 1$. Then we must have $\widehat{S}(p) = g^T/(g^T B)$ is imposed.
- Interpolation problem: interpolation of $\widehat{S}(.)$ on the zero of $\det \widehat{Q}(p)$.
- It is a Corona problem for some class of holomorphic map.
- It is a complicated problem solved by Carleson in the case of bounded holomorphic on the unit disk in 1962. But in our case, $\widehat{Q}(p)$ is not bounded.

Questions

1. Controllability from the origin
   - Controllability in finite time? Bound on this finite time? Minimal time of controllability?
2. For L^q or Radon exact controllability, tremendous work to do.
   - Characterization in terms of a Bézout?
   - Hautus-type criteria for the L^q controllability?
   - Conjecture: L^q exact controllability holds true IFF the following two properties hold true:
     1) $\mathrm{rank}\left[\widetilde{Q}, B\right] = d$ for every $\widetilde{Q} \in \overline{\widehat{Q}(\mathbb{C})}$,
     2) $\mathrm{rank}\,[A_N, B] = d$.
3. Hautus-type criteria for stabilization of such systems? (for $\dot{x} = Ax + Bu$, NSC $= \mathrm{Rank}\,[pI_d - A, B] = d$ for $\mathrm{Re}(p) \geq 0$.)
4. Very Curious!!! Equivalence with controllability to constants (i.e., constant functions)

Exact controllability result for one delay or in small dimension

cf. Chitour et al [2020].

1. Case one delay: All controllability concepts are equivalent to a Kalman condition.
   Rationally dependent delays: reduce to the case of one delay.
2. Case $N = d = 2$ and $m = 1$. Let the geometric set :

$$\mathcal{S} := \left\{ \beta + |\alpha|^{\left(1-\frac{\Lambda_2}{\Lambda_1}\right)} e^{i\theta+2k\pi)\left(1-\frac{\lambda_2}{\Lambda_1}\right)} \middle| k \in \mathbb{R} \right\} \tag{21}$$

   Kalman criteria:
   - L^2 approximate controllability iff $0 \notin \mathcal{S}$.
   - L^2 exact controllability iff $0 \notin \overline{\mathcal{S}}$.
3. Approximate controllability to constants equivalent to L^2 Approximate controllability in all dimension. Same for exact, also true only for $N = d = 2$ and $m = 1$.

Exact controllability result for one delay or in small dimension

The result obtained for $N = d = 2$, $m = 1$, by Chitour et al are based on two things:

- A representation formula of the following type:

$$x_t = \Upsilon_2(t)x_0 + E_2(t)u. \tag{22}$$

- Computations via some subtle transforms can be performed for the dual operator of $E_2(t)$ so that $E_2(t)^*$ is bounded below implying the L^2 exact controllability.

Such arguments have no chance to work in greater dimensions
$\Longrightarrow$ Yamamoto's work is interesting

Some partial answers

It is the object of the talk and it is an outgoing work.

1. In fact, we can prove that the L^q exact controllability from the origin is in equivalent to the L^q controllability in finite time T for all $T \geq d\Lambda_N$.
2. We have that the L^1 controllability is equivalent to the solvability of a Bézout over a Radon measure algebra.
3. At the moment, we are trying to solve the Bézout identity.

Representation formula

Definition

The family of matrices $\Xi_n \in \mathcal{M}_{d,d}(\mathbb{R})$, $n \in \mathbb{Z}^N$, is defined

$$\Xi_n = \begin{cases} 0 & \text{if } n \in \mathbb{Z}^N \backslash \mathbb{N}^N, \\ I_d & \text{if } n = 0, \\ \sum_{k=1}^{N} A_k \Xi_{n-e_k} & \text{if } n \in \mathbb{N}^N \text{ and } |n| > 0, \end{cases} \tag{23}$$

where e_k denotes the k-th canonical vector of $\mathbb{N}^N$.

Representation formula

For $T \in [0, +\infty)$, we introduce the following two operators:

1) $\Upsilon_q(T) : L^q([-\Lambda_N, 0], \mathbb{R}^d) \longrightarrow L^q([-\Lambda_N, 0], \mathbb{R}^d)$

$$(\Upsilon_q(T)x_0)(s) = \sum_{\substack{(n,j)\in\mathbb{N}^N\times[\![1,N]\!] \\ -\Lambda_j \leq T+s-\Lambda\cdot n<0}} \Xi_{n-e_j} A_j x_0(T + s - \Lambda \cdot n),$$

2) $E_q(T) : L^q([0, T], \mathbb{R}^m) \longrightarrow L^q([-\Lambda_N, 0], \mathbb{R}^d)$

$$(E_q(T)u)(t) = \sum_{\substack{n\in\mathbb{N}^N \\ \Lambda\cdot n \leq T+t}} \Xi_n Bu(T + t - \Lambda \cdot n).$$

Theorem (Variation-of-constants formula)

For $T \in [0, +\infty)$, $u \in L^q([0, T], \mathbb{R}^m)$, $x_0 \in L^q([-\Lambda_N, 0], \mathbb{R}^d)$, and $t \in [0, T]$, we have

$$x_t = \Upsilon_q(t)x_0 + E_q(t)u. \tag{24}$$

Controllability in finite time

Definition (Exact control. in finite time)

System (1) is said to be L^q *exactly controllable in time* $T > 0$ if for every $x_0, \phi \in L^q([-\Lambda_N, 0], \mathbb{R}^d)$, there exists $u \in L^q([0, T], \mathbb{R}^m)$ such that

$$x(T + \theta) = \phi(\theta), \quad \theta \in [-\Lambda_N, 0].$$

Proposition

1. *System* (1) *is exactly controllable in time* $T > 0$ *if and only if* $\operatorname{Ran} E_q(T) = L^q([0, T], \mathbb{R}^m)$.
2. *System* (1) *is exactly controllable from the origin if and only if*

$$\bigcup_{T \geq 0} \operatorname{Ran} E_q(T) = L^q([-\Lambda_N, 0], \mathbb{R}^d). \tag{25}$$

Controllability in finite time

Lemma ("Cayley-Hamilton")

There exist real coefficients α_k, *for* $k \in \mathbb{N}^N$ *with* $0 < |k| \leq d$, *such that, for every* $n \in \mathbb{N}^N$ *with* $|n| \geq d$,

$$\Xi_n = -\sum_{\substack{k \in \mathbb{N}^N \\ 0<|k|\leq d}} \alpha_k \Xi_{n-k}. \tag{26}$$

Theorem

For all $T \in [d\Lambda_N, +\infty)$ *and* $q \in [1, +\infty)$, *we have*

$$\mathrm{Ran}\, E_q(T) = \mathrm{Ran}\, E_q(d\Lambda_N). \tag{27}$$

Theorem

Let $q \in [1, +\infty)$. *System* (1) *is* L^q *exactly controllable from the origin if and only if it is* L^q *exactly controllable in time* $T = d\Lambda_N$.

Radon Bézout's identity characterization

Theorem

System (10) *is* L^1 *exactly controllable from the origin if and only if there exist two matrices* R *and* S *with entries in* $M(\mathbb{R}_-)$ *such that*

$$Q * R + P * S = \delta_0 I_d. \tag{28}$$

Sketch of proof:

$\Longrightarrow$ If L^1 exactly controllable, $\exists S_n \in L^1(\mathbb{R}, \mathbb{R}^{m\times d})$ compactly supported in $[-d\Lambda_N, 0]$ such that

$$\pi(Q^{-1} * P * S_n) \underset{n\to+\infty}{\longrightarrow} \pi Q^{-1}, \quad \text{in distribution sense.} \tag{29}$$

Open mapping theorem implies $\|S_n\| \leq C$ for $C > 0$ independent of n. Weak compactness Radon measure there exists S a Radon measure compactly supported in $\mathbb{R}_-$ so that $S_n \underset{n\to+\infty}{\longrightarrow} S$.

Radon Bézout's identity characterization

We have

$$\pi(Q^{-1} * P * S) = \pi Q^{-1}. \tag{30}$$

$\Longleftarrow$ Similar to the characterization of the exact distrib. control.

Theorem

System (10) is L^1 exactly controllable in time $T \geq d\Lambda_N$ if and only if there exist two matrices R and S with entries in $M(\mathbb{R}_-)$ such that

$$Q * R + P * S = \delta_0 I_d. \tag{31}$$

A necessary L^1 exact controllability condition

L^1 Bézout characterization implies the following corollary.

Corollary

L^1 exact controllability implies:

1) $\mathrm{rank}\left[\widetilde{Q}, B\right] = d$ *for every* $\widetilde{Q} \in \overline{\widehat{Q}(\mathbb{C})}$,

2) $\mathrm{rank}\,[A_N, B] = d$.

To obtain a necessary and sufficient L^1 exactly controllability, we need to solve a Corona problem over $M(\mathbb{R}_-)$.

Thank You!