IX Partial Differential Equations, Optimal Design and Numerics

CENTRO DE CIENCIAS DE BENASQUE PEDRO PASCUAL

1D Nonlinear Hyperbolic Systems: Modeling, Control Theory and Applications

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Chair Dynamics, Control and Numerics FAU Alexander von Humboldt Stiftung/Foundation

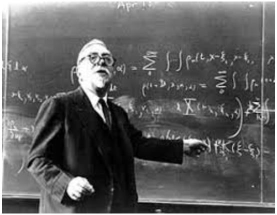
DFG Deutsche Forschungsgemeinschaft German Research Foundation

ConFlex

EMERGING TALENTS INITIATIVE

Origins: control and control theory

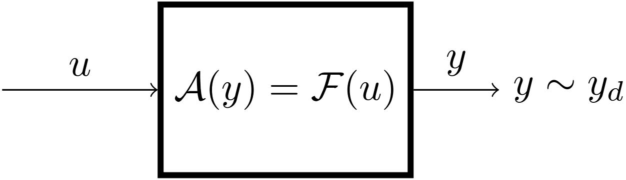
To control means to act, to put things in order to guarantee that the system behaves as desired.



In 1948, Norbert Wiener defined Cybernetics (or Control Theory) as **the science of control and communication in animals and machines.**

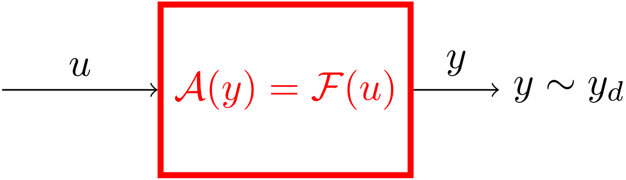
"...In a desirable future, engines would obey and imitate human beings.."
Cybernetics by N. Wiener (1894-1964)

Mathematical Formulation of Control Problem



- y is the state to be controlled.
- u is the control. It belongs to the set of admissible controls.

Mathematical Formulation of Control Problem

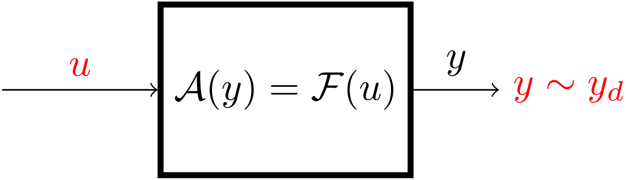


- Modeling:
  1. Model-driven: Meaningful Physical Model.
  2. Data-driven: Machine Learning.

Mathematical Formulation of Control Problem

$$\xrightarrow{\;\;u\;\;} \boxed{\mathcal{A}(y) = \mathcal{F}(u)} \xrightarrow{\;\;y\;\;} y \sim y_d$$

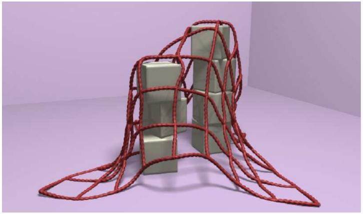
- Modeling
- Analysis: Existence and uniqueness of solution.



- Modeling
- Analysis: Existence (and uniqueness) of solution.
- Control Theory:
  1. Feasibility $\longrightarrow$ Controllability (To find at least one way to reach the target, e.g $y(x,T) = y_d(x)$, or $y(N_i,t) = y_d(t)$);
  2. Optimality $\longrightarrow$ Optimal control (To find the best way, in some sense, to reach the target.)

Nonlinear hyperbolic systems

- We mainly focus on **networked nonlinear hyperbolic systems** in which solutions can be tracked as propagating waves.
- **Key issue:** Developing and applying mathematical methods to **model**, **understand** and **control** the dynamics of PDEs arising in real world applications.



Network of large deflection strings
(Nonlinear vibrating strings)

NASA Flexible flight device
(Geometrically exact beams)

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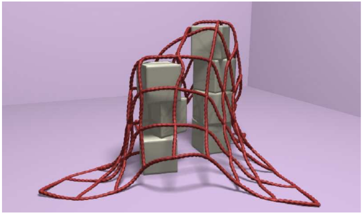

Gas transport networks
(Isothermal Euler equation)

Open canal
(Saint-Venant system)

1-D quasilinear hyperbolic system

We consider the following first-order 1-D quasilinear hyperbolic system

$$\boxed{\frac{\partial y}{\partial t} + A(y)\frac{\partial y}{\partial x} = F(y), \qquad (t,x) \in [0,T] \times [0,L],} \tag{1}$$

where

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- $A(y)$ is an $n \times n$ matrix with smooth entries $a_{ij}(y)(i, j = 1, ..., n)$, $F(y) = (f_1(y), \cdots, f_n(y))^{\mathrm{T}}$ is a given smooth vector function of y with $F(0) = 0$. Thus, $y = 0$ is an equilibrium of (1).

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- $A(y)$ has n **distinct non-vanishing real** eigenvalues $\lambda_i(y)(i = 1, \cdots, n)$ and a complete set of left (resp. right) eigenvectors $\mathbf{l_i}(y) = (l_{i1}(y), \cdots, l_{in}(y))$:

$$\mathbf{l_i}(y)A(y) = \lambda_i(y)\mathbf{l_i}(y). \tag{2}$$

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- We may equivalently rewrite system into its characteristic form

$$\mathbf{l}_i(y)\left(\frac{\partial y_j}{\partial t} + \lambda_i(y)\frac{\partial y_j}{\partial x}\right) = \mathbf{l}_i(y)f_j(y) \qquad (i = 1, ..., n). \tag{3}$$

Remarks

- For simplicity: The eigenvalues of $A(0)$ are ordered, i.e.

$$\lambda_1(0) < \lambda_2(0) < \cdots < \lambda_m(0) < 0 < \lambda_{m+1}(0) < \cdots \lambda_n(0), \tag{4}$$

- In general, we call the following systems

$$\frac{\partial y}{\partial t} + \frac{\partial g(y)}{\partial x} = F(y) \tag{5}$$

  to be **hyperbolic balance laws**, where the **flux** $g := (g_1, ..., g_n)$ is a vector function of u. Obviously, system (5) can be written in the quasilinear form as (1) with the Jacobian matrix

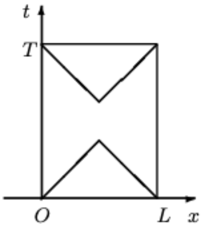
$$A(y) := D(g(y)). \tag{6}$$

Some Key Properties.

- Prescribe **suitable boundary conditions** for IBVP on a bounded domain
  - local & global
  - internal control & boundary control

Some Key Properties.

- Prescribe **suitable boundary conditions** for IBVP on a bounded domain
  - local & global
  - internal control & boundary control
- Controllability time $(T > 0)$
  - a finite speed of propagation of the hyperbolic wave



  - maximum determinate domains
  - $T(> 0)$ should be chosen as small as possible (optimal controllability time).

- **Nonlinearity.**
  - > **Weak solutions**. [of quasilinear hyperbolic systems $\rightarrow$ shock waves

Boundary Controllability. Difficulties may arise in...

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  - Coupling at the junction. Complexity and Nonlinearity in **interface conditions**.

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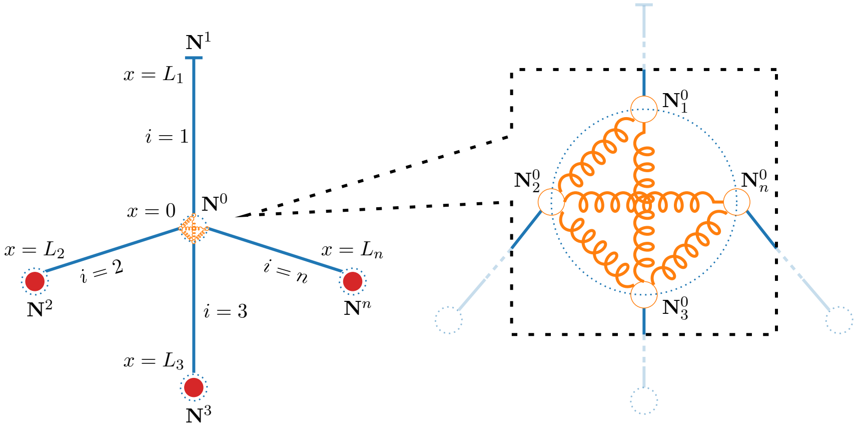
- **Networked Structure.**
  - Coupling at the junction. Complexity and Nonlinearity in **interface conditions**.
  - Structure of networks $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ may change the controllability results [Lagnese-Leugeing-Schmidt, '94]

Outline

- Motivation and Introduction
- Networks of Nonlinear Strings Coupled by Spring and Mass
  - 1d nonlinear wave equation
  - Dynamical boundary conditions
  - Description of coupling/interface conditions.
- Nonlinear Analysis & Control
  - Well-posedness of IBVP
  - Controllability
  - Control design: explicit constructive method with modular structure
- Extension & Perspectives

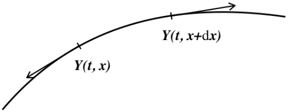
Networks of Nonlinear Strings Coupled by Spring and Mass



Modeling 1/3: 1d quasilinear wave equation

$y = y(t,x)$ the transversal displacements of the string, governed by 1d quasilinear wave equation

$$\boxed{y_{tt}(t,x) - (K(y_x(t,x))_x = 0, \quad 0 \le x \le L, t \ge 0}$$



- e.g. $K(y_x) = cy_x$ with constant $c > 0$.
- e.g. $K = T(|y_x|)\frac{y_x}{|y_x|}$, T is the tension being a function of the extension, following the tangential direction $\frac{y_x}{|y_x|}$.
- You can add $f(y, y_x, y_t)$ at the right side.

Modeling 2/3: boundary conditions

Typical boundary conditions with control $u(t)$ at $x = L$

$$
\begin{aligned}
&y = u(t) && \text{(Dirichlet type)},\\
&y_x = u(t) && \text{(Neumann type)},\\
&y_x + \theta y = u(t), && \text{(Robin type)},\\
&y_x + \gamma y_t = u(t), && \text{(Dissipative type)}.
\end{aligned}
$$

$\theta, \gamma > 0$.

Example (Dynamical Boundary Condition)

$$
\begin{aligned}
& m y_{tt}(0,t) + K(y_x(0,t)) \\
& \qquad + g(y_t(0,t)) + h(y(0,t)) = 0,
\end{aligned}
$$

where

- y_{tt}: Inertial force of the end point with mass m,
- $K(y_x)$: Stress,
- $g(y_t)$: Speed-dependent damping,
- $h(y)$: External force associated with displacement.

Modeling 2/3: nonlinear dynamical boundary condition

General form of **nonlinear dynamical boundary conditions**:

$$\boxed{y_{tt} = G(t, y, y_x, y_t),}$$

where G is a C^1 function:

- contains higher order derivative y_{tt},
- follows the dynamic law of $F = ma$,
- characterizes the relationship between force acting on the end-mass and its motion.

Modeling 3/3: coupling/interface condition - elastic spring

$$\begin{cases} \quad\quad y^i_{tt} - K_i(y^i_x)_x = 0, \quad 0 \le x \le L, t > 0, \qquad i = 1,2, \\ x = 0 : y^1_{tt}(0,t) = K_1(y^1_x(0,t)) - \kappa(y^1(0,t) - y^2(0,t)), \\ \quad\quad y^2_{tt}(0,t) = K_2(y^2_x(0,t)) + \kappa(y^1(0,t) - y^2(0,t)), \\ x = L : y^i = u^i(t), \qquad i = 1,2. \end{cases}$$

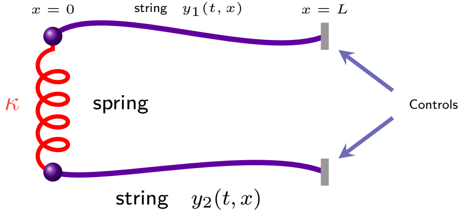


Figure: Two strings connected via masses and an elastic spring

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- If the spring stiffness tends to infinity, formally the system tends to the classical string-mass problem. [1]
- For spring-mass system it is known that the mass smoothens the waves while crossing the mass-point.[2]
- If the spring stiffness tends to zero, the strings become uncoupled.
- The spring coupling can be seen as a weakening of the classical transmission conditions at a multiple joint.[3]

[1] G. Leugering, 1998; F. Almusallams, 2015; Y.Wang, T.Li, 2018

[2] S. Hansen, E.Zuazua 1995

[3] G.Leugering,S.Micu, I.Roventa, Y.Wang, 2022

- **Kelvin Model**: a classical class of viscoelastic **solid** models.

$$
\begin{cases}
y^i_{tt} - K_i(y^i_x)_x = 0, \quad 0 \le x \le L, t > 0, \qquad i = 1,2, \\
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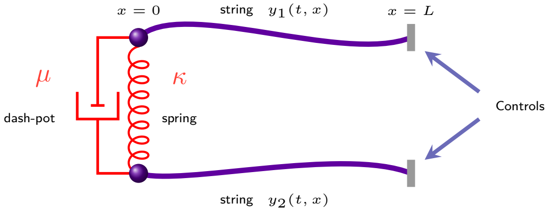
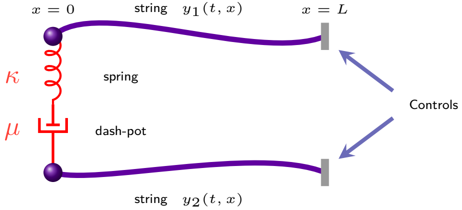


Figure: Networked strings and a Kelvin-type spring

Modeling 3/3: interface condition - viscoelastic spring

- **Maxwell Model**: a classical class of viscoelastic **fluid** models.

$$
\begin{cases}
y^i_{tt} - K_i(y^i_x)_x = 0, \quad 0 \le x \le L, \qquad i = 1,2, \\
x = 0 : y^1_{tt}(0,t) = K_1(y^1_x(t,0)) - \kappa(y^1(t,0) - y^2(t,0)) \\
\qquad + \dfrac{\kappa^2}{\mu} \displaystyle\int_0^t e^{-\frac{\kappa}{\mu}(t-\tau)}(y^1(\tau,0) - y^2(\tau,0))\mathrm{d}\tau \\
y^2_{tt}(0,t) = \cdots \\
x = L : y^i = u^i(t), \qquad i = 1,2.
\end{cases}
$$



Generalization

Consider the following **coupled** system of 1-D quasilinear wave equations $(i = 1, ..., n)$:

$$(\mathbf{E}) \begin{cases} y^i_{tt} - (K^i(y^i, y^i_x))_x = F(\mathbf{y}, \mathbf{y}_x, \mathbf{y}_t), \quad x \in [0, L_i], t \in [0, T] \\ y^i_{tt}(t, 0) = G^i(t, \mathbf{y}(t, 0), \mathbf{y}_x(t, 0), \mathbf{y}_t(t, 0)) \\ \qquad\qquad + \displaystyle\int_0^t H^i(t, s, \mathbf{y}(s, 0))\mathrm{d}s, \ t \in [0, T] \\ y^i(t, L_i) = u^i(t), \ t \in [0, T] \\ (y^i, y^i_t)(0, x) = (\phi^i(x), \psi^i(x)), \quad x \in [0, L_i]. \end{cases}$$

where

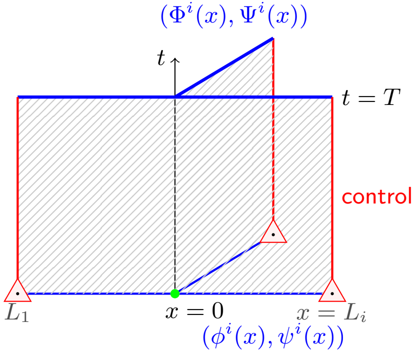
- $\mathbf{y} = (y^1, ..., y^n)^{\mathrm{T}}$ is an unknown vector function of (t, x),
- $K^i = K^i(y^i, y^i_x)$ are given C^2 functions of y^i and y^i_x,
- $\frac{\partial}{\partial y^i_x} K^i(y^i, y^i_x) > 0$,
- F^i, G^i, H^i are given C^1 functions of their arguments and 0 value at null state (i.e. 0 is an equiblium).

Problem Description: Exact boundary controllability

Exact Boundary controllability (EBC) problem

Let $T > 0$. For any given $(\phi^i, \psi^i)(x)$ and final state $(\Phi^i, \Psi^i)(x)$ in $C^2([0, L_i]; \mathbb{R}) \times C^1([0, L_i; \mathbb{R}])(i = 1, ..., n)$, do there exist n boundary controls $\{u^i(t)\}_{i=1}^n (0 \le t \le T)$ such that the solution of system (E) satisfies

$$(y^i, y^i_t)(T, x) = (\Phi^i(x), \Psi^i(x))?$$



- The wellpoesedness of IBVP: the **existence** and **uniqueness** of semi-global classical solution $y^i \in C^2([0, T] \times [0, L_i]; \mathbb{R})^3$ with small norm (Y.Wang, 2017).
- HUM method (J.Lions, 1980s) and duality method (E.Zuazua, 1990s) can not be applied on this case.
- Local controllability: (ϕ, ψ) and (Φ, Ψ) are close to the equilibrium point (Y.Wang, Li, Leugering, 2019).

Wellposedness

We introduce $\mathbf{w}^i = (w_1^i, w_2^i, w_3^i)^T := (y^i, y_x^i, y_t^i)^T$. Then we get

$$\boxed{\frac{\partial}{\partial t}\begin{pmatrix} w_1^i \\ w_2^i \\ w_3^i \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -K^i_{w_2^i} & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} w_1^i \\ w_2^i \\ w_3^i \end{pmatrix} = \begin{pmatrix} w_2^i \\ 0 \\ F^i(\mathbf{w}^i) + K^i_{w_1^i} w_2^i \end{pmatrix}}$$

with $(t, x) \in [0, T] \times [0, L_i]$. This, in turn, can be rewritten in the form of a quasilinear hyperbolic system

$$\boxed{\mathbf{w}_t^i + A^i(x, \mathbf{w}^i)\mathbf{w}_x^i = \tilde{F}(\mathbf{w}^i),}$$

where A^i has 3 distinct real eigenvalues:

$$\lambda_i^- = -\sqrt{K^i_{w_2^i}(w_1^i, w_2^i)}, \quad \lambda_i^0 = 0, \quad \lambda_i^+ = \sqrt{K^i_{w_2^i}(w_1^i, w_2^i)}.$$

We may integrate the boundary conditions w.r.t. time and obtain a kind of **non-local (of time)** boundary condition in the first order system (FOS):

$$
(\mathbf{FOS}) \begin{cases}
\mathbf{w}^i_t + A^i(x,\mathbf{w}^i)\mathbf{w}^i_x = \tilde{F}(\mathbf{w}^i), \quad x \in [0,L_i], t \in [0,T] \\
w^i_2(t,0) = \psi^i(0) + \int_0^t G^i(\tau,\mathbf{w}^i(\tau,0))\,\mathrm{d}\tau \\
\qquad\qquad + \int_0^t \int_0^\tau H^i(\tau,s,w^i_1(s,0))\,\mathrm{d}s\,\mathrm{d}\tau,\ \ t \in [0,T] \\
w^i_1(t,L_i) = u^i(t),\ \ t \in [0,T] \\
\mathbf{w}^i(0,x) = \mathbf{w}^{0,i}(x) = (\phi^i(x), \psi^i(x), \phi^{i\prime}(x)), \quad x \in [0,L_i].
\end{cases}
$$

- Local existence of C^1 solution to (FOS) (T.Li, '85): $\exists!$ C^1 solution on $\mathcal{R}(\delta) = \{(t,x) | 0 \le t \le \delta, 0 \le x \le L\}$, where δ depends on the initial and boundary data.
- For given $T > 0$, NO results on existence of semi-global classical solutions before.

Lemma: A uniform priori estimate of solution to (FOS) [Y.W.'19]

$$\|w(t,\cdot)\|_1 \triangleq \|w(t,\cdot)\| + \left\|\frac{\partial w}{\partial x}(t,\cdot)\right\| \leq C(T), \qquad 0 \leq t \leq T, \tag{7}$$

where $\|\cdot\|$ denotes C^0-norm.
Main Idea in the Proof: We apply

$$\mathbf{l}_i^- = (\mathbf{0}, \sqrt{K^i_{w_2^i}}, 1), \quad \mathbf{l}_i^0 = (1, \mathbf{0}, \mathbf{0}), \quad \mathbf{l}_i^+ = (\mathbf{0}, -\sqrt{K^i_{w_2^i}}, 1)$$

to (FOS) and define **Riemann variables** as

$$v_i = \mathbf{l_i}(w)w, \quad \bar{v}_i = \mathbf{l_i}(w)w_x.$$

They follow

$$\frac{Dv_i}{D_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(w)\bar{v}_j\bar{v}_k + \sum_{j=1}^{n} \tilde{\beta}_{ij}(w)\tilde{F}_j(w) \qquad (i = 1, ..., n),$$

$$\frac{D\bar{v}_i}{D_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(w)\bar{v}_j\bar{v}_k + \sum_{j=1}^{n} \tilde{\gamma}_{ij}(w)\bar{v}_j \qquad (i = 1, ..., n),$$

along the characteristic curves, where

$$\frac{D}{D_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x}.$$

Uniform Priori Estimate (ctd.)

Let

$$T_1 = \min_{\substack{i=1,\dots,n;\\ \|w\|\leq \eta_0}} \frac{L}{|\lambda_i(w)|} > 0.$$

For $(t,x) \in \mathcal{R}(T_1)$, we estimate $|v_i(t,x)|$ by integrating (backward) along the characteristic curve (three cases, $\lambda_i <, =, > 0$). It will arrive at $(0,\alpha)$, or (t_*, L), or $(t_*, 0)$. In different cases, we could obtain

$$|v_i(t,x)| \leq \|v(0,\cdot)\| + C_1 \int_0^t v_i(\tau)\, d\tau,$$

or

$$|v_i(t,x)| \leq A\|v_i(0,\cdot)\| + \|u'\| + C_2 \int_0^t v_i(\tau)\, \mathrm{d}\tau, \qquad \forall t \in [0, T_1],$$

where $v(\tau) = \sup_{0\leq t\leq \tau} \|v(t,\cdot)\|$.
Using **Gronwall inequality** it follows that

$$|v(t,x)| \leq C \max\{\|u'\|, \|v(0,\cdot)\|\} \triangleq C\alpha_0, \qquad \forall t \in [0, T_1],$$

with $C > 1$.
Then **repeating** $N = \left[\frac{T}{T_1}\right] + 1$ times, we have

$$|v(t)| \leq C^N \alpha_0, \qquad \forall t \in [(N-1)T_1, T].$$

$$
(\mathbf{E})\begin{cases}
y^i_{tt}-(K^i(y^i,y^i_x))_x=F(\mathbf{y},\mathbf{y}_x,\mathbf{y}_t), \quad x\in[0,L_i], t\in[0,T]\\
y^i_{tt}(t,0)=G^i(t,\mathbf{y}(t,0),\mathbf{y}_x(t,0),\mathbf{y}_t(t,0))\\
\qquad\qquad +\displaystyle\int_0^t H^i(t,s,\mathbf{y}(s,0))\mathrm{d}s,\ t\in[0,T]\\
y^i(t,L_i)=u^i(t),\ t\in[0,T]\\
(y^i,y^i_t)(0,x)=(\phi^i(x),\psi^i(x)), \quad x\in[0,L_i].
\end{cases}
$$

The system (E) is locally exact controllable

- with n controls [G.Leugering, T.Li, Y.Wang, '18,'19].
- This result can be improved by **reducing the number of controls to** $n-1$, but the space of controlled initial data is **asymmetric** [G.Leugering, S.Micu, I.Robenta, Y.Wang, '22] [G.Leugering, C.Rodriguez, Y.Wang, '22].

Exact boundary Controllability

Theorem (Full Control)

In a neighbourhood of an equilibrium (around 0), the system (E) is locally exact boundary controllable when

- ***(Controllability time, sharp)***

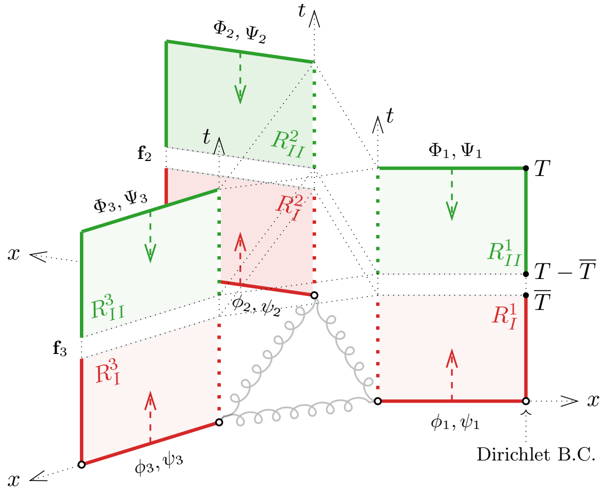
  $$T > 2\bar{T},$$

  where we define the maximal travelling time for the strings:

  $$\bar{T} = \max_{i=1,\ldots n} \frac{L_i}{\sqrt{K^i_{y_x}(0,0)}}.$$

- $$det\left(\frac{\partial G^i(t,0,0,0)}{\partial y^j_x}\right)_{n\times n} \neq 0, \forall t \in [0,T].$$

Proof of Theorem

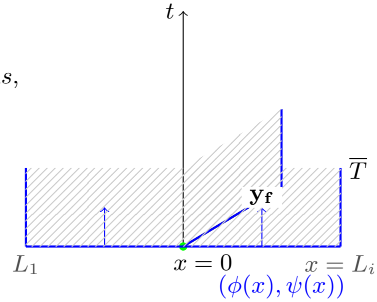


Proof. A constructive method with modular structure

Step 1: Solving the forward mixed initial-boundary value problem on domain $\mathcal{R}_f := [0,\bar{T}] \times \bar{\Omega}$, where $\bar{\Omega} := \Pi_{i=1,\dots,n}[0, L^i]$.

$$\begin{cases} y^i_{tt} - (K^i(y^i, y^i_x))_x = F^i(\mathbf{y}, \mathbf{y_x}, \mathbf{y_t}), \\ y^i_{tt}(t,0) = G_i(t, \mathbf{y}, \mathbf{y_t}, \mathbf{y_x}) + \int_0^t H^i(t,s,\mathbf{y})\mathrm{d}s, \\ y^i(t, L_i) = q^i(t), \\ (y^i, y^i_t)(0,x) = (\phi_i(x), \psi_i(x)). \end{cases}$$

where $q^i(t)(i = 1, \dots, n)$ are any given C^2 functions of t.

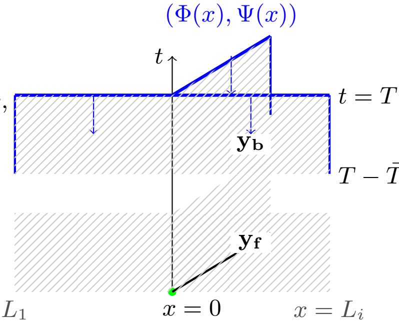


Proof. A constructive method with modular structure

Step 2: Solving the backward mixed initial-boundary value problem on domain $\mathcal{R}_b := [T-\bar{T}, T] \times \bar{\Omega}$.

$$\begin{cases} y^i_{tt} - (K^i(y^i, y^i_x))_x = F^i(\mathbf{y}, \mathbf{y_x}, \mathbf{y_t}), \\ y^i_{tt}(t,0) = G_i(t, \mathbf{y}, \mathbf{y_t}, \mathbf{y_x}) + \int_0^t H^i(t,s,\mathbf{y})\mathrm{d}s, \\ y^i(t, L_i) = \bar{q}^i(t), \\ (y^i, y^i_t)(T, x) = (\Phi_i(x), \Psi_i(x)). \end{cases}$$

where $\bar{q}^i(t)(i = 1, ..., n)$ are any given C^2 functions of t.



Proof. A constructive method with modular structure

Step 3 (Key): Observe and connect the trace (not unique).

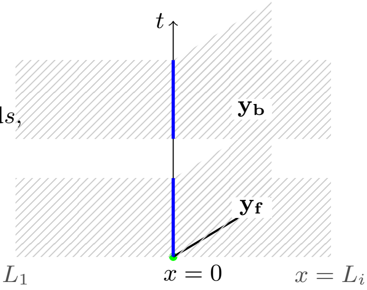
$$(c^i(t), \bar{c}^i(t)) = \begin{cases} (y_f^i(t,0), y_{fx}^i(t,0)), & 0 \le t \le T_1, \\ (y_b^i(t,0), y_{bx}^i(t,0)), & T - T_1 \le t \le T, \end{cases}$$

1. Since $\mathcal{R}_f$ and $\mathcal{R}_b$ do not overlap, we first construct $c^i(t) \in C^3[0,T]$.
2. Because all $(c^i, \bar{c}^i)$ should satisfy n interface conditions

$$y_{tt}^i(t,0) = G_i(t, \mathbf{y}, \mathbf{y_t}, \mathbf{y_x}) + \int_0^t H^i(t,s,\mathbf{y})\mathrm{d}s,$$

we can solve $(\bar{c}^1, ..., \bar{c}^2)(t)$ when

$$det\left(\frac{\partial G^i(t,0,0,0)}{\partial y_x^j}\right)_{n\times n} \neq 0, \forall t \in [0,T].$$

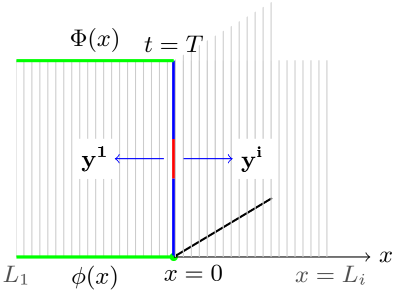


Proof. A constructive method with modular structure

Step 4: Change the role of t and x. We solve the side-wise problem on each string (here we omit index i in the systems).

$$\begin{cases} y_{xx} - K_{y_x}^{-1}(y, y_x) y_{tt} = -K_{y_x}^{-1}(y, y_x)\Big(K_y(y, y_x) y_x + F(\cdot)\Big), \\ (y, y_x)(t, 0) = (c(t), \bar{c}(t)), \quad \text{(new I.C.)}, \\ y(0, x) = \phi(x), y(T, x) = \Phi(x), \quad \text{(new B.C.)}. \end{cases}$$

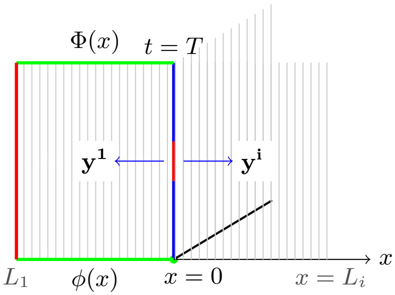
We get the solution $\mathbf{y} = (y^1, ..y^n)$ on the domain $\mathcal{R} = [0, T] \times \bar{\Omega}$.



Proof. A constructive method with modular structure

Step 5: Compute the controls by the trace!

$$u^i(t) := y^i(t, L^i), t \in [0, T].$$



Proof. A constructive method with modular structure

Step 6: Verify that $\mathbf{y} = \mathbf{y}(t,x)$ satisfies all data.

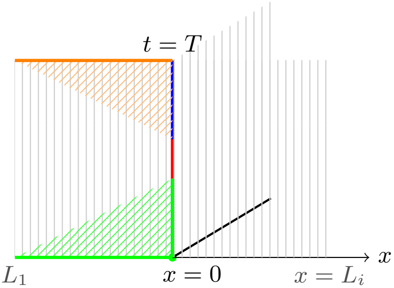
$$y^i(t,x) \equiv y^i_f(t,x)$$

on the domain $\left\{(t,x)\middle|\, 0 \le t \le \frac{T_1}{L_1}(L-x), 0 \le x \le L_1\right\}$ (Uniqueness of C^2 solution on the one-sided mixed initial boundary value problem, T.Li 2010).

- Note that

$$T_1 = \sup_{|w_1^i|+|w_2^i|\le\varepsilon}\left(\frac{L_1}{\sqrt{K^i_{w_2^i}(w_1^i,w_2^i)}}\right).$$

- $y_t^1(0,x) = \psi^1(x)(0 \le x \le L_1)$.
- $y_t^1(T,x) = \Psi^1(x)(0 \le x \le L_1)$.



Outline

- Motivation
- Modeling
  - 1d nonlinear wave equation
  - Dynamical boundary conditions
  - Coupling/interface conditions.
- Analysis & Control
  - Well-posedness of IBVP
  - Controllability
  - Control design: explicit constructive method with modular structure
- Extension & Perspectives

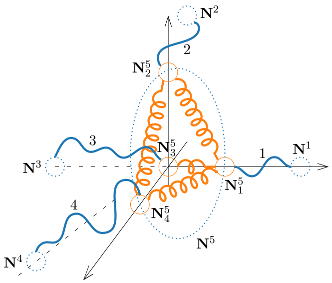
Extension

From the modeling perspective, one can consider..

- Extension to nonlinear mechanical systems: elastic multi-body structures, flexible structures, robotic systems.
- Extension to pipe-flow: gas network, water networks, flow-structure interaction.

Extension 1/4: spatial vibration of strings

- The result can be extended to spatial vibration of strings (Joint work with G. Leugering, C. Rodriguez, to appear)



$$\rho_i \mathbf{R}^i_{tt}(x,t) = [\mathbf{G}^i(\mathbf{R}^i_x(x,t))]_x - \rho_i g \mathbf{e}, \text{ in } (0, L_i) \times (0, T), i \in \mathcal{I}$$

The multiple node condition:

$$\epsilon_{ij} \mathbf{G}^i(\mathbf{R}^i_x(x_{ij}, t)) + m^j_i \mathbf{R}^i_{tt}(x_{ij}, t) \\ + \kappa_j \left[\left(\sum_{k \in \mathcal{I}^j} a^j_{ik} \right) \mathbf{R}^i(x_{ij}, t) - \sum_{k \in \mathcal{I}^j} a^j_{ik} \mathbf{R}^k(x_{kj}, t) \right] = 0, \quad t \in (0,T),\ j \in \mathcal{J}^M,\ i \in \mathcal{I}^j$$

Extension 2/4: geometrically exact beams

- Project Conflex. Control of flexible structures and fluid-structure interactions (Funding: Marie Sklodowska-Curie grant No.765579).
- GEB: any magnitude of displacement and rotation. [G. Leugering, C. Rodriguez, Y.Wang, JMPA, '21]

Geometrically exact beam (GEB)

FAU

States: position of centerline $\mathbf{p}(x,t) \in \mathbb{R}^3$, rotation of cross sections $\mathbf{R}(x,t) \in \mathbb{R}^{3\times 3}$, where $x \in [0, L]$.

$$X = \begin{bmatrix} x \\ X_2 \\ X_3 \end{bmatrix} \Rightarrow \text{at time } t \geq 0, \;\; \bar{\mathbf{p}}(X,t) = \mathbf{p}(x,t) + \mathbf{R}(x,t)(X_2 e_2^* + X_3 e_3^*).$$

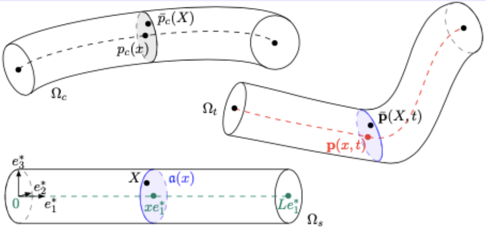


Figure: Ω_s, Ω_c and Ω_t: beam in straight-reference, curved-reference and current configuration respectively.

Extension 2/4: geometrically exact beams

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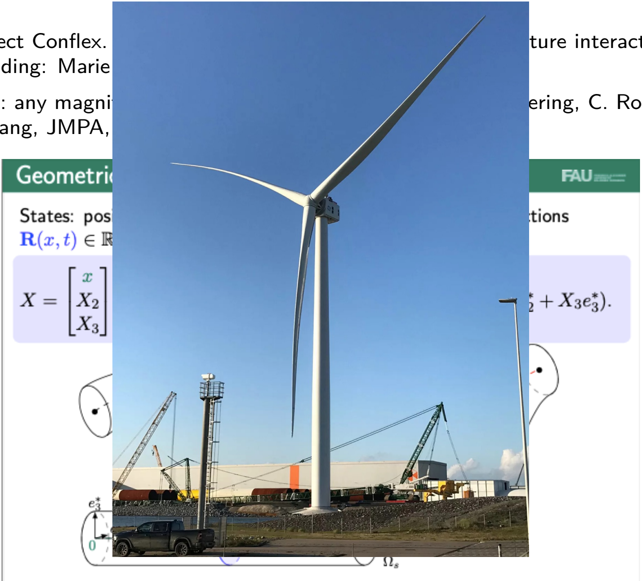
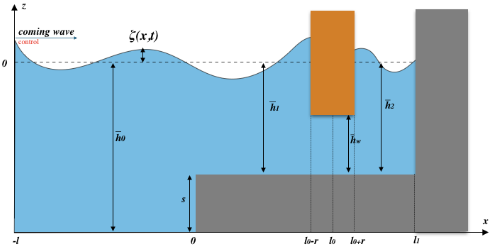


Figure: Ω_s, Ω_c and Ω_t: beam in straight-reference, curved-reference and current configuration respectively.

Extension 3/4: shallow water systems with a partially immersed obstacle

- Project Conflex. [G.Vergara-Hermosilla, G.Leugering, Y.Wang, COCV '21].
- One dimensional nonlinear shallow water system, describing the free surface flow of water as well as the flow under a fixed gate structure.



$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(q^2/h\right) + gh\partial_x \zeta = 0, \end{cases}$$

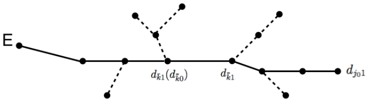
where $\zeta(t,x)$ is the free surface elevation, $h(t,x)$ is the fluid height, $q(t,x)$ is the horizontal discharge.

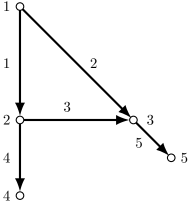
$$\zeta_0(t,0) = \zeta_1(t,0), \quad q_0(t,0) = q_1(t,0),$$

$$\begin{cases} q_2(t, l_0 + r) = q_1(t, l_0 - r) = q_w(t), \\ \left[\dfrac{q_i^2}{2h_i^2} + g\zeta_i\right]_{i=1,x=l_0-r}^{i=2,x=l_0+r} = -\alpha \dfrac{\mathrm{d}}{\mathrm{d}t} q_w(t), \end{cases}$$

where $\alpha = 2r/h_w$.

Extension 4/4: tree- or A-shaped network

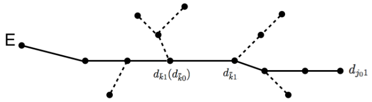


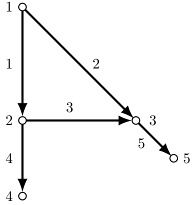


For trees with one clamped node (simple node E), the sharp estimate of the contollability time is determined the "longest" chain-like subnetwork.

$$T_{i_0 j_0} = \max_{i,j \in \tilde{\mathcal{S}}, i \neq j} \sum_{k \in \mathcal{D}_{ij}} \frac{L_k}{\sqrt{K^k_{v_k}(0,0)}}.$$

Extension 4/4: tree- or A-shaped network





For A-shaped network:

- No exact boundary controllability on a network with loops in general. [J.E. Lagnese, G. Leugering, E.J.P.G. Schmidt, '94].
- BUT it is possible to get the exact boundary controllability of nodal profile. See [K.Zhuang, G.Leugering, T.Li '19] [M.Guget etc. '11] [T.Li, Y.Wang '21] [C.Rodriguez, G.Leugering, Y. '21] for different nonlinear systems.

Perspectives 1/3

- If masses are present, at multiple nodes a complex smoothing pattern appears leading to **asymmetric control spaces** [Hansen and Zuazua SICON 1996], [Leugering, Micu, Roventa, Wang JEE 2022] [observability property on transport equation with interior mass, in discussion with E. Zuazua].

Perspectives 1/3

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- Controllability properties: nodal profile control (especially on networks with cycles), observability, sychronization, optimal control, turnpike, feedback control and stabilization, constrained controllability [of gas flow, in discussion with M. Gugat].

Perspective 2/3

- Realization and Numerics:
  1. Random batch method for hyperbolic system via the characteristic method [in discussion with D.Veldman, Benasque 2022].
  2. Non-overlapping domain decomposition on complex spatial structures [in discussion with G.Leugering, Benasque 2022].
  3. Numerical approximation of constructive method [M. Gugat, J. Habermann, M. Hintermüller, O. Huber, 2021].
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Perspective 2/3

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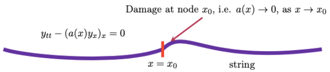
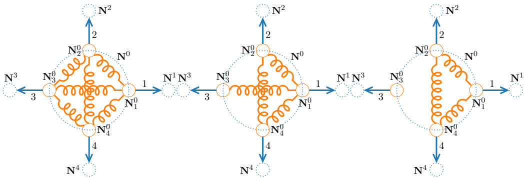
Perspective 3/3

- Interplay between degeneration and control. [Funded by DFG]
  1. Lack of one-sided exact controllability in $y_{tt} - (x^\alpha y_x)_x = 0$ when $\alpha \geq 2$.[F. Alabau-Boussouira, P. Cannarsa and G. Leugering, '17]
  2. Moving controllability? $y_{tt} - y_{xx} = f(t)\delta_{\gamma(t)}(x)$ [C. Castro '19].
  3. Some relaxed version of the damage problem?
     ◇ Damped wave equation, $y_{tt} - y_{xx} + u(x)y_t = 0$ [in discussion with J. Yong, D. Veldman, E. Zuazua];
     ◇ Missing springs in the coupling [in discussion with G. Leugering, C. Rodriguez].





Benasque 2022