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Exact controllability to the trajectories of the one-phase Stefan problem.

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Joint work with Diego A. Souza and Enrique Fernández Cara.

Talk based on the paper: https://arxiv.org/pdf/2204.04750.pdf.

The controllability problem.

Statement of the controllability problem

We study the following Stefan problem:

$$\begin{cases} u_t - u_{xx} = 0 & \text{in} \quad Q_\ell, \\ u(0,t) = v(t) & \text{in} \quad (0,T), \\ u(\ell(t),t) = 0 & \text{in} \quad (0,T), \\ \beta \ell_t(t) = -u_x(\ell(t),t) & \text{in} \quad (0,T), \\ \ell(0) = \ell_0, & \\ u(\cdot\,,0) = u_0 & \text{in} \quad (0,\ell_0). \end{cases}$$

Here $Q_\ell := \{(t,x) : t \in (0,T), x \in (0,\ell(t))\}$, v is the control and $\beta > 0$. Our objective is to control exactly to trajectories with a positivity constraint.

Phenomena modelled by Stefan equation

- Liquid-solid interfaces.
- Tumour growth.
- Information diffusion in online social networks.

State of the art

- Controllability results of the Stefan's problem by E. Fernández-Cara, D. A. Souza and their collaborators regarding null controllability.

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- Controllability results to constant trajectories of the viscous Burgers equation with one moving endpoint equation by B. Geshkovski and E. Zuazua.
- Controllability of fluid-structure systems, for instance by Fernández-Cara, Takahashi, Tucsnak, etc.

Contribution of our paper

- From our knowledge, our result is the first one concerning the exact control to non-constant trajectories in the context of a free boundary parabolic system.
- We work with a positivity constraint.
- We prove a Carleman inequality for a system which has a nonlocallity on the boundary condition. This is a novelty in the literature.

First steps on the proof of the exact controllability to trajectories.

Main strategy to solve the controllability problem

- With a change of variables we obtain an equation in a cylindrical domain instead of a free boundary domain.
- Reduce the problem to a distributed control problem.
- Linearize the equation in a neighbourhood of a trajectory.
- Obtain the controllability of the linealized equation with the help of a Carleman inequality.
- Prove the exact controllability to trajectories with Liusternik-Graves' Inverse Theorem.

The system after changing of variable

Let us consider $p(y,t) = u(y\ell(t),t)$ and $q(t) = \ell(t)^2$. Then, the system is equivalent to:

$$\begin{cases} qp_t - p_{yy} + \dfrac{y}{\beta} p_y(1,\cdot)p_y = 0 & \text{in} \quad (0,T)\times(0,1), \\ p(0,\cdot) = v & \text{in} \quad (0,T), \\ p(1,\cdot) = 0 & \text{in} \quad (0,T), \\ p(\cdot\,,0) = p_0 & \text{in} \quad (0,1), \\ \beta q_t + 2p_y(1,\cdot) = 0 & \text{in} \quad (0,T), \\ q(0) = q_0, \end{cases} \tag{1}$$

The control system in a neighbourhood of a trajectory

Let $z = p - \overline{p}$ and $h = \frac{\beta}{2}(q - \overline{q})$. Then, the control problem in the trajectory (z, h) is given by:

$$\begin{cases} \bar{q}z_t - z_{xx} + \dfrac{x}{\beta}\bar{p}_x(1,\cdot)z_x + \dfrac{x}{\beta}\bar{p}_x z_x(1,\cdot) + \dfrac{2}{\beta}\bar{p}_t h + \dfrac{2}{\beta}hz_t + \dfrac{x}{\beta}z_x(1,\cdot)z_x = 0 & \text{in} \quad Q_1, \\ z(0,\cdot) = \widehat{v} & \text{in} \quad (0,T), \\ z(1,\cdot) = 0 & \text{in} \quad (0,T), \\ z(\cdot\,,0) = z_0 & \text{in} \quad (0,1), \\ h_t + z_x(1,\cdot) = 0 & \text{in} \quad (0,T), \\ h(0) = h_0, & \end{cases}$$

We can prove the existence and uniqueness of solutions of such system with Galerkin's method. Here $Q_1 := (0,T) \times (0,1)$.

The linearized control system with a control that acts on the interior

When we linearized the previous system and prolong the domain to obtain the boundary control we obtain the following system:

$$\begin{cases} \bar{q}z_t - z_{xx} + \dfrac{x}{\beta}\bar{p}_x(1,\cdot)z_x + \dfrac{x}{\beta}\bar{p}_x z_x(1,\cdot) + \dfrac{2}{\beta}\bar{p}_t h = f_1 + w1_\omega & \text{in} \quad (-1,1)\times(0,T), \\ z(-1,\cdot) = 0 & \text{in} \quad (0,T), \\ z(1,\cdot) = 0 & \text{in} \quad (0,T), \\ z(\cdot,0) = z_0 & \text{in} \quad (-1,1), \\ h_t + z_x(1,\cdot) = f_2 & \text{in} \quad (0,T), \\ h(0) = h_0, \end{cases}$$

where f_1 and f_2 belong to appropriate spaces of functions that decay exponentially as $t \to T^-$ and will be made precise below.

The adjoint system

The adjoint system is given by the following equations:

$$\begin{cases} -\bar{q}\varphi_t - \varphi_{xx} - \dfrac{x}{\beta}\bar{p}_x(1,\cdot)\varphi_x + \dfrac{1}{\beta}\bar{p}_x(1,\cdot)\varphi = g_1 & \text{in} \quad (0,T)\times(-1,1), \\ \varphi(-1,\cdot) = 0 & \text{in} \quad (0,T), \\ \varphi(1,\cdot) = \gamma + \displaystyle\int_{-1}^{1} \frac{x}{\beta}\bar{p}_x(x,\cdot)\varphi(x,\cdot)\,dx & \text{in} \quad (0,T), \\ \varphi(\cdot,T) = \varphi_T & \text{in} \quad (-1,1), \\ \gamma_t = \displaystyle\int_{-1}^{1} \frac{2}{\beta}\bar{p}_t(x,\cdot)\varphi(x,\cdot)\,dx + g_2 & \text{in} \quad (0,T), \\ \gamma(T) = \gamma_T. & \end{cases}$$

The proof of existence relies on Leray-Schauder Fixed Point Principle, and the uniqueness on regularity estimates.

An important property of the adjoint system

When we multiply the solutions of such system by a cut-off function $\kappa(t)$ just depending on the time variable, we get a similar system, so regularity estimates apply. This implies that we can estimate the L^2-norm of the system at a time $t = 0$ with the L^2-norm of $(T/4, 3T/4)$.

The Carleman inequality.

Auxiliary weights

Let ω_0 be a non-empty open set, with $\omega_0 \subset\subset \omega$ and let be a function η in $C^2([-1,1])$ satisfying

$$\eta > 0 \text{ in } [-1,1], \quad \min_{x \in [-1,1] \setminus \omega_0} |\eta_x(x)| > 0, \quad \eta(-1) = \eta(1) = \min_{x \in [-1,1]} \eta(x).$$

Let us introduce the following associated weights:

$$\begin{aligned}
\alpha(x,t) &:= \frac{e^{2\lambda m \|\eta\|_\infty} - e^{\lambda(m\|\eta\|_\infty + \eta(x))}}{t(T-t)} && \forall (x,t) \in (0,T) \times (-1,1), \\
\xi(x,t) &:= \frac{e^{\lambda(m\|\eta\|_\infty + \eta(x))}}{t(T-t)} && \forall (t,x) \in (0,T) \times (-1,1), \\
\widehat{\alpha}(t) &:= \max_{x \in [-1,1]} \alpha(x,t) = \alpha(1,t) = \alpha(-1,t) && \forall t \in (0,T), \\
\widehat{\xi}(t) &:= \min_{x \in [-1,1]} \xi(x,t) = \xi(1,t) = \xi(-1,t) && \forall t \in (0,T),
\end{aligned}$$

where $\lambda > 0$ is a sufficiently large constant (to be chosen later) and $m > 1$.

The Carleman inequality

Let us assume that $R \in L^{\infty}(0, T; L^2(-1,1))$, $N \in W^{1,\infty}(0, T; L^2(-1,1))$ and $d \in C^1([0, T])$ with $d(t) > d_* > 0$ for all $t \in [0, T]$. There exist constants $\lambda_0 \geq 1$, $s_0 \geq 1$ and $C_0 > 0$ such that, for any $\lambda \geq \lambda_0$, any $s \geq s_0(T + T^2)$, any $(\psi_T, \gamma_T) \in H^1(-1,1) \times \mathbb{R}$ satisfying $\varphi_T(-1) = 0$ and $\varphi_T(1) = \gamma_T + (N(\cdot, T), \varphi_T)_2$ and any source terms $f \in L^2(Q)$ and $g \in L^2(0, T)$, the strong solution to:

$$\begin{cases} \psi_t + d(t)\psi_{xx} = f & \text{in} \quad (0, T) \times (-1,1), \\ \psi(-1, \cdot) = 0 & \text{in} \quad (0, T), \\ \psi(1, t) = \gamma(t) + (N(\cdot\,, t), \psi(\cdot\,, t))_2 & \text{in} \quad (0, T), \\ \psi(\cdot\,, T) = \psi_T & \text{in} \quad (-1,1) \\ \gamma_t(t) - (R(\cdot\,, t), \psi(\cdot, t))_2 = g & \text{in} \quad (0, T), \\ \gamma(T) = \gamma_T \end{cases}$$

satisfies:

$$\begin{aligned} &\iint_Q \left[(s\xi)^{-1}(|\psi_{xx}|^2 + |\psi_t|^2) + \lambda^2(s\xi)|\psi_x|^2 + \lambda^4(s\xi)^3|\psi|^2\right] e^{-2s\alpha}\, dx\, dt \\ &+ \int_0^T \left[\lambda^3(s\widehat{\xi})^3(|\psi(1,t)|^2 + |\gamma|^2) + \lambda(s\widehat{\xi})(|\psi_x(-1,t)|^2 + |\psi_x(1,t)|^2)\right] e^{-2s\widehat{\alpha}}\, dt \\ &\leq C_0 \left(s^3\lambda^4 \iint_{(0,T)\times\omega} \xi^3|\psi|^2 e^{-2s\alpha}\, dx\, dt + \iint_Q |f|^2 e^{-2s\alpha}\, dx\, dt + \int_0^T |g|^2 e^{-2s\widehat{\alpha}}\, dt \right). \end{aligned}$$

Steps of the proof of the Carleman inequality

- Start the proof from scratch; that is, from the equation satisfied by $w = e^{-s\alpha}\psi$.
- Replicate the steps of the heat equation for the terms in the interior and for the boundary terms of w and w_x.
- When considering the boundary terms there is a term given by $2\int_0^T dw_t w_x$. For that, we must consider that:

$$w_t = -s\alpha_t w + e^{-s\alpha}\psi_t,$$

  so with Cauchy-Schwarz inequality we leave a boundary term with ψ_t on the right-hand side.
- We perform the Carleman estimate as usual, though leaving ψ_t on the right-hand side. We also revert the change of variables.
- $\psi_t(1,t)$ can be written in terms of ψ and on the integral of ψ_t, and with that the boundary term is absorbed.
- We add γ as $\gamma = \psi(1,t) - (N(\cdot,t), \psi(\cdot,t))_2$.

Step 1: starting from scratch.

Let $w = e^{-s\alpha}\psi$. Then,

$$\begin{aligned} e^{-s\alpha} f - sd\alpha_{xx} w =& [dw_{xx} + (s\alpha_t + s^2 d\alpha_x^2)w] \\ &+ [w_t + 2sd\alpha_x w_x]. \end{aligned}$$

Step 2: integrating by parts

After integrating by parts before using the boundary conditions we get that:

$$
\begin{aligned}
2(dw_{xx} + (s\alpha_t + s^2 d\alpha_x^2)w, w_t + 2sd\alpha_x w_x) = & \iint_Q (-2sd^2\alpha_{xx} + d_t)|w_x|^2 \\
& + \iint_Q \left(-s\alpha_{tt} - s^2(d\alpha_x^2)_t - 2s^2 d(\alpha_x\alpha_t)_x - 6s^3 d^2\alpha_x^2\alpha_{xx}\right)|w|^2 \\
& + \int_0^T \left[2dw_t w_x + 2sd^2\alpha_x|w_x|^2 + 2s^2 d\alpha_x(\alpha_t + sd\alpha_x^2)|w|^2\right]_{x=-1}^{x=1}.
\end{aligned}
$$

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& + \int_0^T \left[2dw_t w_x + 2sd^2\alpha_x|w_x|^2 + 2s^2 d\alpha_x(\alpha_t + sd\alpha_x^2)|w|^2\right]_{x=-1}^{x=1}.
\end{aligned}
$$

On the boundary

$$\alpha_x \cdot n = -\partial_n \eta \xi > 0,$$

so the only boundary term which is not positive is $\int_0^T 2dw_t w_x$.

Step 3: dealing with the boundary term

First, because of the Dirichlet boundary condition on the left hand-side $w_t(-1,\cdot)=0$ on $(0,T)$. As for the boundary condition on $x=1$, considering that $w_t=-s\alpha_t w+e^{-s\alpha}\psi_t$:

$$\int_0^T 2dw_t w_x\Big|_{x=1} \geq 2\int_0^T d\psi_t w_x e^{-s\widehat{\alpha}}\Big|_{x=1} - Cs^3\int_0^T \widehat{\xi}^3|w|^2\Big|_{x=1} - Cs\int_0^T \widehat{\xi}|w_x|^2\Big|_{x=1}$$

The second and third term can be absorbed for λ large enough. As for the first term, we use a weighted Cauchy-Schwartz to absorb w_x and leave ψ_t for the moment.

Step 4: adding the higher order derivatives

After absorbing the terms, we may add the higher order derivatives as usual. In fact, we have:

$$\begin{aligned}
&\iint_Q s^{-1}\xi^{-1}(|w_t|^2+|w_{xx}|^2)+\iint_Q s\lambda^2\xi|w_x|^2+s^3\lambda^4\iint_Q \xi^3|w|^2 \\
&+s\lambda\int_0^T \widehat{\xi}|w_x|^2\Big|_{x=-1}+\int_0^T\left(s^3\lambda^3\widehat{\xi}^3|w|^2+s\lambda\widehat{\xi}|w_x|^2\right)\Big|_{x=1} \\
&\leq C\left(\|e^{-s\alpha}f\|^2_{L^2(Q)}+s^3\lambda^4\int_0^T\xi^3|w|^2+s^{-1}\lambda^{-1}\int_0^T \xi^{-1}e^{-2s\alpha}|\psi_t|^2\Big|_{x=1}\right).
\end{aligned}$$

Step 5: coming back to the variable

Let us consider $\psi = e^{-s\alpha} w$. With some easy absorptions, we obtain that:

$$I(s,\lambda,\psi) \ \le C\Big(\iint_Q e^{-2s\alpha}|f|^2 + s^3\lambda^4 \int_0^T e^{-2s\alpha}\xi^3|\psi|^2 + s^{-1}\lambda^{-1}\int_0^T \widehat{\xi}^{-1} e^{-2s\widehat{\alpha}}|\psi_t|^2\big|_{x=1}\Big),$$

where we have set:

$$I(s,\lambda,\psi) := \iint_Q e^{-2s\alpha}\left[(s\xi)^{-1}(|\psi_t|^2+|\psi_{xx}|^2) + s\lambda^2\xi|\psi_x|^2 + s^3\lambda^4\xi^3|\psi|^2\right] + s^3\lambda^3\int_0^T e^{-2s\widehat{\alpha}}\widehat{\xi}^3|\psi|^2\big|_{x=1} + s\lambda\int_0^T e^{-2s\widehat{\alpha}}\widehat{\xi}|\psi_x|^2\big|_{x=-1} + s\lambda\int_0^T e^{-2s\widehat{\alpha}}\widehat{\xi}|\psi_x|^2\big|_{x=1}.$$

Step 6: absorption of the boundary term

We must absorb the term:

$$s^{-1}\lambda^{-1}\int_0^T \xi^{-1}e^{-2s\alpha}|\psi_t|^2\big|_{x=1}$$

with

$$\iint_Q e^{-2s\alpha}\left[(s\xi)^{-1}|\psi_t|^2+s^3\lambda^4\xi^3|\psi|^2\right].$$

For that, it suffices to use:

$$\psi_t = g + (R(\cdot,t),\psi(\cdot,t))_2 + \partial_t(N(\cdot,t),\psi(\cdot,t))_2$$

and recall that the minimum of the weights is obtained on the boundary.

Step 7: addition of γ

Since $\gamma = \psi(1,t) - (N(\cdot,t),\psi(\cdot,t))_2$, we have that:

$$s^3\lambda^3 \int_0^T e^{-2s\widehat{\alpha}}\widehat{\xi}^3|\gamma|^2 \leq C\left(s^3\lambda^3 \int_0^T e^{-2s\widehat{\alpha}}\widehat{\xi}^3|\psi|^2\Big|_{x=1} + s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\psi|^2,\right)$$

so we can add that term on the left-hand side.

Exact controllability to trajectories of the non-linear system.

A Carleman inequality for our system

Corollary

Assume that $(\bar{p}, \bar{q})$ *belong to the space* $[W^{1,\infty}(0,T;H^1(-1,1)) \cap H_0^{1,2}(Q)] \times W^{1,\infty}(0,T)$ *with* $\overline{q}(t) \in (q_*, +\infty)$ *for all* $t \in [0,T]$*. There exist constants* $\lambda_0 \geq 1$, $s_0 \geq 1$ *and* $C_0 > 0$ *such that, for any* $\lambda \geq \lambda_0$, *any* $s \geq s_0(T+T^2)$, *any* $\varphi_T \in H^1(-1,1)$ *any* $\gamma_T \in \mathbb{R}$ *with*

$$\varphi_T(-1) = 0 \quad \text{and} \quad \varphi_T(1) = 2\gamma_T + \frac{1}{\beta}\int_{-1}^{1} \bar{p}_x(x,T) x \varphi_T(x)\, dx$$

and any right hand sides $g_1 \in L^2(Q)$ *and* $g_2 \in L^2(0,T)$*, the strong solution to the adjoint system satisfies:*

$$\begin{aligned}
&\iint_Q \left[(s\xi)^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \lambda^2(s\xi)|\varphi_x|^2 + \lambda^4(s\xi)^3|\varphi|^2\right] e^{-2s\alpha}\, dx\, dt \\
&\quad + \int_0^T \left[|\gamma_t|^2 + \lambda(s\widehat{\xi})\left(|\varphi_x(-1,t)|^2 + |\varphi_x(1,t)|^2\right) + \lambda^3(s\widehat{\xi})^3\left(|\varphi(1,t)|^2 + |\gamma|^2\right)\right] e^{-2s\widehat{\alpha}}\, dt \\
&\quad \leq C_0\left(\iint_Q |g_1|^2 e^{-2s\alpha}\, dx\, dt + \int_0^T |g_2|^2 e^{-2s\widehat{\alpha}}\, dt + s^3\lambda^4\int_{(0,T)\times\omega} \xi^3|\varphi|^2 e^{-2s\alpha}\, dx\, dt\right).
\end{aligned}$$

Sketch of the proof

Proof.

Let us apply the proven Carleman with the following data:

$$d = \frac{1}{\bar{q}}, \quad f = -\frac{1}{\bar{q}}\left[g_1 + \frac{\bar{p}_x(1,\cdot)}{\beta}(x\varphi_x - \varphi)\right],$$

$$N(x,t) = \frac{x}{\beta}\bar{p}_x(x,t), \quad R = \frac{2}{\beta}\bar{p}_t \text{ and } g = g_2.$$

□

Some additional weights

Let us define some additional weights:let the function $r = r(t)$ be given by

$$r(t) = \begin{cases} T^2/4 & \text{in } [0, T/2], \\ t(T-t) & \text{in } [T/2, T] \end{cases}$$

and let us set $D_1 = (-1,1) \times (0, T/2)$, $D_2 = (-1,1) \times (T/2, T)$,

$$\zeta(x,t) := \frac{e^{2\lambda m \|\eta\|_\infty} - e^{\lambda(m\|\eta\|_\infty + \eta(x))}}{r(t)} \quad \text{and} \quad \mu(x,t) := \frac{e^{\lambda(m\|\eta\|_\infty + \eta(x))}}{r(t)} \quad \forall (x,t) \in Q,$$

for η defined previously. Let us also introduce the notation:

$$\widehat{\zeta}(t) := \max_{x\in[-1,1]} \zeta(x,t),\ \widehat{\mu}(t) := \min_{x\in[-1,1]} \mu(x,t), \quad \forall t \in (0,T),$$

$$\zeta^*(t) := \min_{x\in[-1,1]} \zeta(x,t),\ \mu^*(t) := \max_{x\in[-1,1]} \mu(x,t), \quad \forall t \in (0,T).$$

Estimate of the time $t = 0$

Proposition

Under the conditions in Corollary 1, the unique strong solution to the adjoint system satisfies:

$$+\int_0^T \left[|\gamma_t|^2 + \widehat{\mu}\left(|\varphi_x(-1,t)|^2 + |\varphi_x(1,t)|^2\right) + \widehat{\mu}^3\left(|\gamma|^2 + |\varphi(1,t)|^2\right)\right] e^{-2s\widehat{\zeta}}\,dt$$
$$+\iint_Q \left[\mu^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \mu|\varphi_x|^2 + \mu^3|\varphi|^2\right] e^{-2s\zeta}\,dx\,dt + \|\varphi(\cdot\,,0)\|^2_{H^1(-1,1)} + |\gamma(0)|^2$$
$$\leq C_2\left(\iint_Q |g_1|^2 e^{-2s\zeta^*}\,dx\,dt + \int_0^T |g_2|^2 e^{-2s\widehat{\zeta}}\,dt + \iint_{(0,T)\times\omega} (\mu^*)^3|\varphi|^2 e^{-2s\zeta^*}\,dx\,dt\right),$$

for a positive constant C_2 depending on T, s and λ, with s and λ as in Corollary 1.

Estimate of the time $t = 0$

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Under the conditions in Corollary 1, the unique strong solution to the adjoint system satisfies:

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\begin{aligned}
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&+\iint_Q \left[\mu^{-1}(|\varphi_t|^2 + |\varphi_{xx}|^2) + \mu|\varphi_x|^2 + \mu^3|\varphi|^2\right] e^{-2s\zeta}\,dx\,dt + \|\varphi(\cdot\,,0)\|^2_{H^1(-1,1)} + |\gamma(0)|^2 \\
&\le C_2\left(\iint_Q |g_1|^2 e^{-2s\zeta^*}\,dx\,dt + \int_0^T |g_2|^2 e^{-2s\widehat{\zeta}}\,dt + \iint_{(0,T)\times\omega} (\mu^*)^3|\varphi|^2 e^{-2s\zeta^*}\,dx\,dt\right),
\end{aligned}
$$

for a positive constant C_2 depending on T, s and λ, with s and λ as in Corollary 1.

Proof.

The proof relies on the previous Carleman inequality and regularity estimates for the adjoint system. □

Additional weights and operators

Let us consider the weights:

$$\rho_0(t) := e^{s\zeta^*(t)},\ \rho_1(t) := e^{s\widehat{\zeta}(t)},\ \rho_2(t) := \mu^{*-3/2}(t)e^{s\zeta^*(t)} \quad \forall t \in (0,T),$$

$$\rho_3(t) := e^{s\widehat{\zeta}(t)}\widehat{\mu}^{-3/2}(t),\ \rho_4(t) := \rho_3^{1/2}(t) \quad \forall t \in (0,T).$$

Let us introduce the linear operators

$$\mathcal{L}_1(z,h) := \bar{q}z_t - z_{xx} + \frac{x}{\beta}\bar{p}_x(1,\cdot)z_x + \frac{x}{\beta}\bar{p}_x z_x(1,\cdot) + \frac{2}{\beta}\bar{p}_t h \quad \text{and} \quad \mathcal{L}_2(z,h) := h_t + z_x(1,\cdot)$$

and the space E, given by

$$E := \big\{(z,h,w) \in L^2_{\rho_0}(Q) \times L^2_{\rho_1}(0,T) \times L^2_{\rho_2}(\omega \times (0,T)) :$$
$$\mathcal{L}_1(z,h) - w1_\omega \in L^2_{\rho_3}(Q), \mathcal{L}_2(z,h) \in L^2_{\rho_3}(0,T),\ h \in H^1_{\rho_4}(0,T) \text{ and } z \in H^{1,2}_{0,\rho_4}(Q)\big\}.$$

Null controllability of the linearized system around a trajectory

Proposition

Assume that $(f_1, f_2) \in L^2_{\rho_3}(Q) \times L^2_{\rho_3}(0, T)$ *and that* $(z_0, h_0) \in H^1_0(-1, 1) \times \mathbb{R}$*. Then, there exists a solution to the linearized system around a trajectory satisfying* $(z, h) \in E$*.*

Null controllability of the linearized system around a trajectory

Proposition

Assume that $(f_1, f_2) \in L^2_{\rho_3}(Q) \times L^2_{\rho_3}(0,T)$ *and that* $(z_0, h_0) \in H^1_0(-1,1) \times \mathbb{R}$*. Then, there exists a solution to the linearized system around a trajectory satisfying* $(z,h) \in E$*.*

The proof is based on duality and regularity estimates to ensure that the solutions belong to the the stated spaces.

Controllability of the non-linear system

We shall apply Liusternik-Graves' Inverse Function Theorem with $\mathcal{B}_1 = E$, $\mathcal{B}_2 = F_1 \times F_2$ and

$$\Lambda(z,h,w) = \Big(\mathcal{L}_1(z,h) - w1_\omega + \frac{2}{\beta} h z_t + \frac{x}{\beta} z_x(1,\cdot) z_x, \\ \mathcal{L}_2(z,h), z(\cdot,0),\ h(0)\Big) \tag{2}$$

for every $(z,h,w) \in E$. Here, we have introduced the Hilbert spaces $F_1 := L^2_{\rho_3}(Q) \times L^2_{\rho_3}(0,T)$ for the right hand sides and $F_2 := H^1_0(-1,1) \times \mathbb{R}$ for the initial conditions.

Thank you for your attention!
Is there any question?