Controllability from the exterior of fractional heat equation

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Plan of the Talk

Plan of the Talk

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Fractional Laplace operator

- The *fractional Laplacian* $(-\partial_x)^s$, with $0 < s < 1$, is defined by the following singular integral

$$(-\partial_x)^s u(x) := C_{1,s}\, \text{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x-y|^{1+2s}}\, dy, \quad x \in \mathbb{R},$$

where $C_{1,s}$ is a normalization constant.

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where $C_{1,s}$ is a normalization constant.

- The N-dimensional fractional heat equation

$$\partial_t u + (-\Delta)^s u = 0$$

naturally arises from a probabilistic process in which a particle moves randomly in the space subject to a probability that allows long jumps with a polynomial tail.

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Applications

- G.M. Viswanathan, V. Afanasyev, S.V. Buldyrev, E.J. Murphy, P.A. Prince, and H. Eugene Stanley. Lévy flight search patterns of wandering albatrosses. Nature, 381(6581): 413–415, 1996.
- Nicolas E. Humphries et al. Environmental context explains Lévy and Brownian movement patterns of marine predators. Nature, 465(7301):1066–1069, 2010.

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Fractional heat equation

- We are interested in the controllability from the exterior of the fractional heat equation in the interval $(-1,1)$. That is,

$$\begin{cases} \partial_t u + (-\partial_x^2)^s u = 0 & \text{in } (-1,1)\times(0,T), \\ u = g\chi_{\mathcal{O}\times(0,T)} & \text{in } (\mathbb{R}\setminus(-1,1))\times(0,T), \\ u(\cdot,0) = u_0 & \text{in } (-1,1). \end{cases}$$

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Exterior condition?

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T), \\ u = g & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases}$$

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- Then, the EXTERIOR CONDITION is the right notion that replaces the classical nonhomogeneous boundary problems associated with local operators.

Main results

- Null controllability without constraints with $g \in L^2(0, T; H^s((-1, 1)^c))$.

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- Null controllability with L^∞-controls.
- Controllability to the trajectories under positive L^∞-controls.

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Similar results

Fractional heat equation

- U. Biccari and V. Santa-María (IMA J. Math. Control Inform. 2019): null controllability with interior control.

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- D. Pighin and E. Zuazua (Math. Control. Relat. Fields 2018): controllability under positive constraints for semilinear problem.

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Joint work with M. Warma

We are interested in the null controllability of the fractional heat equation in the interval $(-1,1)$. That is,

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More precisely, given u_0, find g such that the solution of (1) satisfies:

$$u(\cdot,T) = 0, \text{ in } (-1,1).$$

Here $\mathcal{O} \subset (\mathbb{R}\setminus(-1,1))$.

Basic properties of fractional Laplace operator

- Let $(-\Delta)^s_D$ be the selfadjoint operator in $L^2(\Omega)$ with domain

$$D((-\Delta)^s_D) := \left\{u \in H^s_0(\overline{\Omega}),\ (-\Delta)^s u \in L^2(\Omega)\right\}, \quad (-\Delta)^s_D u := (-\Delta)^s u.$$

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- $(-\Delta)^s_D$ has a compact resolvent and its eigenvalues form a non-decreasing sequence of real numbers $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ satisfying $\lim_{n\to\infty} \lambda_n = \infty$. In addition, the eigenvalues are of finite multiplicity.

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- Let $(\varphi_n)_{n\in\mathbb{N}}$ be the orthonormal basis of eigenfunctions associated with $(\lambda_n)_{n\in\mathbb{N}}$. Then $\varphi_n \in D((-\Delta)^s_D)$ for every $n \in \mathbb{N}$, $(\varphi_n)_{n\in\mathbb{N}}$ is total in $L^2(\Omega)$ and satisfies

$$\begin{cases} (-\Delta)^s \varphi_n = \lambda_n \varphi_n & \text{in } \Omega, \\ \varphi_n = 0 & \text{in } \Omega^c. \end{cases}$$

Theorem

Let $T > 0$ be any real number. Then for every $u_0 \in L^2(-1,1)$ and $g \in L^2((0,T); H^s((-1,1)^c))$, the system (1) has a unique weak solution $u \in C([0,T]; L^2(-1,1))$ given by

$$u(x,t) = \sum_{n=1}^{\infty} u_{0,n} e^{-\lambda_n t} \varphi_n(x) + \sum_{n=1}^{\infty} \left(\int_0^t (g(\cdot,\tau), \mathcal{N}_s \varphi_n)_{L^2((-1,1)^c)} e^{-\lambda_n((t-\tau)} d\tau \right) \varphi_n(x),$$

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where the *non-local normal derivative* $\mathcal{N}_s$ is given by

$$\mathcal{N}_s\varphi_n(x) := C_{1,s} \int_\Omega \frac{\varphi_n(x) - \varphi_n(y)}{|x-y|^{1+2s}} \, dy, \quad x \in (-1,1)^c.$$

Adjoint problem

Using the integration by parts formula, we have that the following backward system

$$
\begin{cases}
-\partial_t \psi + (-\partial_x^2)^s \psi = 0 & \text{in } (-1,1)\times(0,T),\\
\psi = 0 & \text{in } (\mathbb{R}\setminus(-1,1))\times(0,T),\\
\psi(\cdot,T) = \psi_0 & \text{in } (-1,1),
\end{cases}
\tag{2}
$$

can be viewed as the dual system associated with (1).

Theorem

Let $T > 0$ be a real number and $\psi_0 \in L^2(-1,1)$. Then the system (2) has a unique weak solution $\psi \in C([0,T]; L^2(-1,1))$ which is given by

$$\psi(x,t) = \sum_{n=1}^{\infty} \psi_{0,n} e^{-\lambda_n (T-t)} \varphi_n(x).$$

In addition, there is a constant $C > 0$ such that for all $t \in [0,T]$,

$$\|\psi(\cdot,t)\|_{L^2(-1,1)} \leq C \|\psi_0\|_{L^2(-1,1)}.$$

Controllability Identity

Lemma

The system (1) is null controllable at time $T > 0$ if and only if for each initial datum $u_0 \in L^2(-1,1)$, there exists a control function $g \in L^2((0,T); H^s((-1,1)^c))$ such that the solution ψ of the dual system (2) satisfies

$$\int_{-1}^{1} u_0(x)\psi(x,0)\, dx = \int_0^T \int_{\mathcal{O}} g(x,t)\mathcal{N}_s\psi(x,t)\, dxdt, \tag{3}$$

for each $\psi_0 \in L^2(-1,1)$,

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for each $\psi_0 \in L^2(-1,1)$, where the *non-local normal derivative* $\mathcal{N}_s$ is given by

$$\begin{aligned}\mathcal{N}_s\psi(x,t) &:= C_{1,s}\int_\Omega \frac{\psi(x,t)-\psi(y,t)}{|x-y|^{1+2s}}\,dy, \quad x \in (-1,1)^c \\ &= \sum_{n=1}^{\infty} \psi_{0,n} e^{-\lambda_n(T-t)}\mathcal{N}_s\varphi_n(x).\end{aligned}$$

Main result

Theorem 1 (M. Warma and S.Z.)

Let $0 < s < 1$ and let $\mathcal{O} \subset (\mathbb{R} \setminus (-1, 1))$ be an arbitrary nonempty open set. Then the following assertions hold.

(a) If $\frac{1}{2} < s < 1$, then the system (1) is null controllable at any time $T > 0$.

(b) If $0 < s \leq \frac{1}{2}$, then the system (1) is not null controllable at time $T > 0$.

(c) If $\frac{1}{2} < s < 1$, then the system (1) is exactly controllable to the trajectories at any time $T > 0$.

Proof of Theorem

The system (1) is null controllable if and only if the following observability inequality holds for the dual system: there exists a constant $C > 0$ such that

$$\|\psi(\cdot,0)\|^2_{L^2(-1,1)} \leq C \int_0^T \int_{\mathcal{O}} |\mathcal{N}_s \psi(x,t)|^2 \, dxdt. \tag{4}$$

Proof of Theorem

The system (1) is null controllable if and only if the following observability inequality holds for the dual system: there exists a constant $C > 0$ such that

$$(4) \qquad \|\psi(\cdot,0)\|^2_{L^2(-1,1)} \leq C \int_0^T \int_{\mathcal{O}} |\mathcal{N}_s \psi(x,t)|^2 \, dxdt.$$

Using the representations of ψ and $\mathcal{N}_s\psi$, and employing the orthonormality of the eigenfunctions in $L^2(-1,1)$, then the observability inequality (4) becomes

$$(5) \qquad \sum_{n=1}^{\infty} |\psi_{0,n}|^2 e^{-2\lambda_n T} \leq C \int_0^T \int_{\mathcal{O}} \left| \sum_{n=1}^{\infty} \psi_{0,n} e^{-\lambda_n (T-t)} \mathcal{N}_s \varphi_n(x) \right|^2 dxdt.$$

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Proof of Theorem...

It is a well known result for parabolic equations, that an inequality of the type (5) holds if and only if the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ and the eigenfunctions $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfy the following Müntz conditions

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty, \quad \text{and} \quad \|\mathcal{N}_s \varphi_n\|_{L^2(\mathcal{O})} \geq \eta > 0, \quad \forall n \in \mathbb{N}. \tag{6}$$

Proof of Theorem...

The eigenvalues $\{\lambda_n\}_{n\geq 1}$ satisfy[1]

$$(7) \qquad \lambda_n = \left(\frac{n\pi}{2} - \frac{(2-2s)\pi}{8}\right)^{2s} + O\left(\frac{1}{n}\right) \quad \text{as} \quad n \to \infty.$$

Therefore, we have the following two situations (assuming that the nonlocal normal derivative is uniformly bounded by below).

- If $0 < s \leq \frac{1}{2}$, then the series (6) will have the behavior of the harmonic series, which implies that it is divergent.
- On the other hand, if $\frac{1}{2} < s < 1$, hence, $2s > 1$, then using (7) we can deduce that the series (6) is convergent.

The proof of Parts (a) and (b) is complete.

[1] Mateusz Kwaśnicki . Eigenvalues of the fractional Laplace operator in the interval. J. Funct. Anal. **262** 2012

Theorem 2 (M. Warma and S.Z.)

Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be the orthogonal basis of normalized eigenfunctions associated with the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$. Then, for every nonempty open set $\mathcal{O} \subset (\mathbb{R} \setminus (-1,1))$, there exists a scalar $\eta > 0$ such that for every $k \in \mathbb{N}$, the function $\mathcal{N}_s\varphi_k$ is uniformly bounded from below by η in $L^2(\mathcal{O})$. Namely,

$$\exists\, \eta > 0,\ \forall\, k \in \mathbb{N},\ \|\mathcal{N}_s\varphi_k\|_{L^2(\mathcal{O})} \geq \eta.$$

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Proof: Main Ingredients

- Approximation of the eigenfunctions $\{\varphi_k\}_k$ of fractional Laplacian:

$$\|\varrho_k-\varphi_k\|_{L^2(-1,1)}\leq\begin{cases}\frac{C(1-s)}{k} & \text{when } \frac{1}{2}\leq s<1\\ \frac{C(1-s)}{k^{2s}} & \text{when } 0<s<\frac{1}{2}.\end{cases}$$

Continuation...

- Arguing by contradiction: there is a constant $C > 0$ independent of n such that for n large enough, we have

$$\|\mathcal{N}_s \varphi_{k_n}\|_{H^{-s}(\mathcal{O})} \leq \frac{C}{n}.$$

- Let the operator L be defined by

$$L : H_0^s(\overline{\Omega}) \to H^{-s}(\mathcal{O}),\ v \mapsto Lv := ((-\partial_x^2)^s v)|_{\mathcal{O}} = (\mathcal{N}_s v)|_{\mathcal{O}}.$$

It has been shown[2] that the operator L is compact, injective with dense range.

[2] T. Ghosh, A. Rüland, M. Salo, and G. Uhlmann. Uniqueness and reconstruction for the fractional Calderón problem with a single measurement. J. Funct. Anal. **279**(1) 2020

Plan of the Talk

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Joint work with H. Antil, U. Biccari, R. Ponce, M. Warma

In this part, we are concerned with the constrained controllability from the exterior of the one-dimensional non-local heat equation

$$\begin{cases} \partial_t u + (-\partial_x^2)^s u = 0 & \text{in } (-1,1)\times(0,T), \\ u = g\chi_{\mathcal{O}} & \text{in } (-1,1)^c\times(0,T), \\ u(\cdot,0) = u_0 & \text{in } (-1,1). \end{cases} \tag{8}$$

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Our principal goal is to analyze whether the parabolic equation (8) can be driven from any given initial datum $u_0 \in L^2(-1,1)$ to a desired final target by means of the L^∞-control action, but preserving some non-negativity constraints on the control and/or the state.

- Let $1/2 < s < 1$. System (8) is null controllable if and only if there exists a constant $C = C(T) > 0$ such that the following observability inequality

$$\|\psi(\cdot,0)\|^2_{L^2(-1,1)} \le C \left(\int_0^T \int_{\mathcal{O}} |\mathcal{N}_s \psi(x,t)| \, dxdt \right)^2 \tag{9}$$

holds for every $\psi_T \in L^2(-1,1)$, where ψ is the unique weak solution of the dual problem.

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holds for every $\psi_T \in L^2(-1,1)$, where ψ is the unique weak solution of the dual problem.

- We prove that there exists a constant $\eta > 0$ such that for every $k \in \mathbb{N}$, $\mathcal{N}_s \varphi_k$ is uniformly bounded from below by η in $L^1(\mathcal{O})$. Namely,

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$$\exists\, \eta > 0,\ \forall\, k \in \mathbb{N},\ \|\mathcal{N}_s\varphi_k\|_{L^1(\mathcal{O})} \geq \eta.$$

- The dual system is L^1-observable in time T.

Controllability to trajectories under positive constraints

Theorem

Let $\mathcal{O} \subset \overline{\mathcal{O}} \subset [-1,1]^c$ be an arbitrary nonempty bounded open set and s a real number such that $1/2 < s < 1$. Let $0 < \widehat{u}_0 \in L^2(-1,1)$ and an exterior control $\widehat{g} \in L^\infty(\mathcal{O} \times (0,T))$ for which there is a positive constant α such that $\widehat{g} \geq \alpha$ a.e. in $\mathcal{O} \times (0,T)$. Consider a positive trajectory $\widehat{u}$ of (8) with initial datum $\widehat{u}_0$ and exterior condition $\widehat{g}$.

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Then, for every $u_0 \in L^2(-1,1)$ there exist $T > 0$ large enough and a non-negative control $g \in L^\infty(\mathcal{O} \times (0,T))$ such that the corresponding very weak solution u of (8) satisfies $u(\cdot,T) = \widehat{u}(\cdot,T)$ a.e. in $(-1,1)$. In addition, if $u_0 \geq 0$ a.e. in $(-1,1)$, then $u \geq 0$ a.e. in $(-1,1) \times (0,T)$.

According to previous Theorem the constrained controllability to trajectories holds true if the time horizon is large enough, let us define the minimal controllability time $T_{\min}$ by

(10)

$$T_{\min} := \inf \left\{ T > 0 : \exists\, 0 \leq g \in L^{\infty}((0,T); L^{\infty}(\mathcal{O})) \text{ such that } \; u(\cdot, T) = \widehat{u}(\cdot, T) \right\}.$$

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$$T_{\min} := \inf\Big\{ T > 0 : \exists\, 0 \leq g \in L^\infty((0,T); L^\infty(\mathcal{O})) \text{ such that } \; u(\cdot, T) = \widehat{u}(\cdot, T) \Big\}.$$

Theorem

Let $T := T_{\min}$ be the minimal controllability time given by (10). Then, there exists a non-negative control $g \in \mathcal{M}(\mathcal{O} \times (0,T))$ such that the corresponding solution u of (8) satisfies $u(\cdot, T) = \widehat{u}(\cdot, T)$ a.e. in $(-1, 1)$.

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Numerical simulations

- **Main idea:**
  Approximation of exterior (Dirichlet) control problem by parabolic Robin problem:
  Let $n \in \mathbb{N}$ and y^n be the solution of the following Robin problem[3]

$$\begin{cases} \partial_t y^n + (-\partial_x^2)^s y^n = 0 & \text{in } (-1,1)\times(0,T), \\ \mathcal{N}_s y^n + n y^n = ng & \text{in } (\mathbb{R}\setminus(-1,1))\times(0,T), \\ y^n(\cdot,0) = y_0 & \text{in } (-1,1), \end{cases} \tag{11}$$

[3] H. Antil, D. Verma and M. Warma, External optimal control of nonlocal PDEs. Inverse Problems, **35**(8):084003, 35, 2020

- For the target trajectory, we consider

$$\widehat{u}(x, T) := \frac{\Gamma\left(\frac{1}{2}\right) 2^{-2s} e^{T}}{\Gamma(1+s)\Gamma\left(\frac{1}{2}+s\right)} \left(1-|x|^2\right)_+^s,$$

which is known to be the exact solution to the Dirichet problem evaluated at the final time T, i.e., $\widehat{u}$ satisfies

$$\begin{cases} \partial_t \widehat{u} + (-\partial_x^2)^s \widehat{u} = z_{exact} + e^t & \text{in } (-1,1)\times(0,1), \\ \widehat{u} = z_{exact} & \text{in } ((-2,2)\setminus(-1,1))\times(0,1), \\ \widehat{u}(\cdot,0) = z_{exact}(\cdot,0) & \text{in } (-1,1), \end{cases}$$

where

$$z_{exact}(x,t) := \frac{\Gamma\left(\frac{1}{2}\right) 2^{-2s} e^{t}}{\Gamma(1+s)\Gamma\left(\frac{1}{2}+s\right)} \left(1-|x|^2\right)_+^s.$$

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- To obtain $T_{\min}$, we consider the following constrained optimization problem:

$$\text{minimize} \quad T \tag{12}$$

subject to

$$\begin{cases} T > 0, & \\ \partial_t u^n + (-\partial_x^2)^s u^n = 0 & \text{in } (-1,1)\times(0,T), \\ \mathcal{N}_s u^n + n u^n = n g \chi_{\mathcal{O}\times(0,T)} & \text{in } ((-2,2)\setminus(-1,1))\times(0,T), \\ u^n(\cdot,0) = u_0 \geq 0 & \text{in } (-1,1), \\ g \geq 0 & \text{in } \mathcal{O}\times(0,T), \end{cases} \tag{13}$$

which we solve using `CasADi` open-source tool for nonlinear optimization and algorithmic differentiation.

- Set the initial datum to be

$$u_0(x) := \frac{1}{2}\cos\left(\frac{\pi}{2}x\right).$$

In this case, we have that $u_0 < \widehat{u}(\cdot, T)$ in $(-1,1)$

By solving (12) we obtain that $T_{\min} = 0.4739$.

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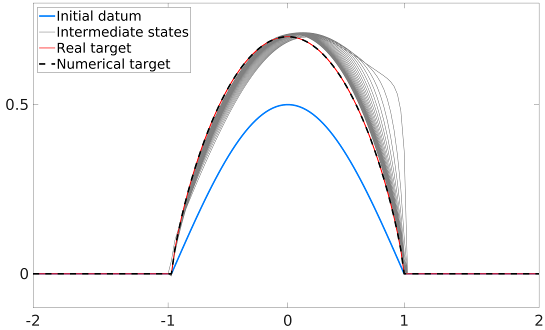

Figure: Evolution of the solution to (11) in the time interval $(0, T_{\min})$ with $s = 0.8$. The blue line is the initial configuration u_0. The red line is the target $\widehat{u}(\cdot, T)$ $(T = T_{\min})$ configuration. The black dashed line is the numerical solution at $T = T_{\min}$

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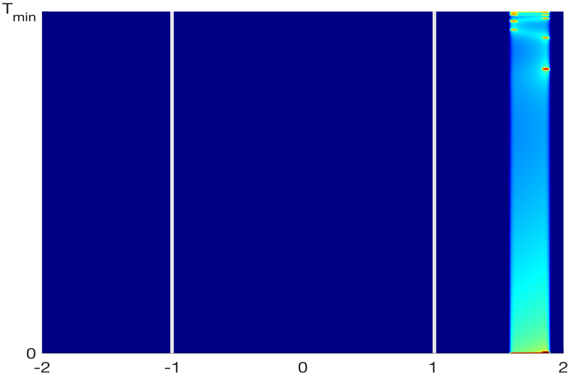

Figure: Minimal-time control: space-time distribution of the control. The white lines delimit the dynamics region $(-1, 1)$.

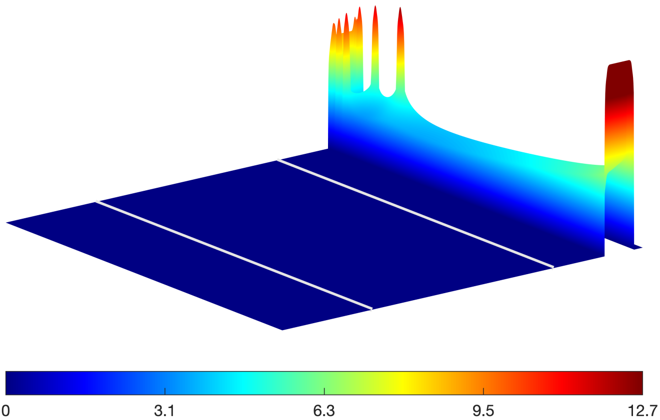

Figure: Minimal-time control: intensity of the impulses in logarithmic scale. In the (x, t) plane in blue the time t varies from $t = 0$ (bottom) to $t = T_{\min}$ (top).

Future work

- For $s \in (0, \frac{1}{2})$, null controllability with moving control?

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- For $s \in (0, \frac{1}{2})$, null controllability with moving control?
- Existence of minimal time T_{min}.
- Numerical analysis without the approximation of exterior Dirichlet control problem by parabolic Robin problem.

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