Stabilization and asymptotic behavior of the system related to water waves problem

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Outline

1. Stability of a class of infinite dimensional systems
2. Boundary control problem in rectangle
   - 2.1 Governing equations for gravity waves;
   - 2.2. Well-posedness and stabilization (linearized equation);
   - 2.3. Asymptotic analysis;
3. Perspectives

Transport equation with boundary control:

$$(*)\ \begin{cases} \dfrac{\partial z}{\partial t}(t,x) = -\dfrac{\partial z}{\partial x}(t,x) & \forall\, t, x \geqslant 0, \\ z(t,0) = u(t). \end{cases}$$

Take $U = \mathbb{C}$, $X = L^2[0,\infty)$ and $Z = \mathcal{H}^1[0,\infty)$
Define

$$Az = -\frac{\mathrm{d}z}{\mathrm{d}x} \quad \text{with} \quad \mathcal{D}(A) = \mathcal{H}_0^1[0,\infty);$$

$$B = \delta_0 \in \mathcal{L}(\mathbb{C}, \mathcal{H}^{-1}[0,\infty))$$

$$(*) \iff \dot{z}(t) = Az(t) + Bu(t)$$

A generates the unilateral right shift seimigroup $\mathbb{T}_t$ on X,

$$(\mathbb{T}_t z)(x) = z(x-t) \qquad \forall\, x \in [0,\infty).$$

B is an admissible control operator for $\mathbb{T}$.
$\Longrightarrow$ $(*)$ is a well-posed boundary control system !

Consider the linear control system:

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \\ z(0) = z_0. \end{cases} \tag{1}$$

with the state space X (Hilbert) and the input space U (Hilbert).
$A : \mathcal{D}(A) \to X$ is skew-adjoint with compact resolvents;
$u \in L^2_{\text{loc}}([0,\infty); U)$ and $B \in \mathcal{L}(U, X)$.

- Energy estimate:

$$\|z_0\|^2 - \|z(t)\|^2 = -2\int_0^t \langle Az, z\rangle \mathrm{d}\tau - 2\int_0^t \langle u, B^* z\rangle_U \mathrm{d}\tau,$$

$$\Longrightarrow \quad \|z_0\|^2 - \|z(t)\|^2 = 2\int_0^t \|B^* z\|_U^2 \mathrm{d}\tau \geqslant 0,$$

if take $u = -B^* z$, i.e. colocated feedback, s.t.
$\Longrightarrow$ energy non-increasing !

(1)+ colocated feedback:

$$\begin{cases} \dot{z}(t) = \mathcal{A}z = (A - BB^*)z(t), \\ z(0) = z_0. \end{cases} \tag{2}$$

- Types of stability:

1. Exponential stability: $\exists M, \alpha > 0$, s.t.

$$\|z(t)\| \leqslant M\mathrm{e}^{-\alpha t}\|z_0\| \quad \forall\, z_0 \in X;$$

2. Weak stability: $\lim_{t\to\infty}\langle z(t), x\rangle = 0, \quad \forall x,\, z_0 \in X;$
3. Strong stability: $\lim_{t\to\infty} z(t) = 0;$
4. "Polynomial" stability: $\exists\ f(t)$ with $\lim_{t\to\infty} f(t) = 0$, s.t.

$$\|z(t)\| \leqslant f(t)\|z_0\|_{\mathcal{D}(A)} \quad z_0 \in \mathcal{D}(A).$$

The undamped system:

$$\begin{cases} \dot{w}(t) = Aw(t), \\ w(0) = z_0. \end{cases} \tag{3}$$

Proposition 1 (Ammari & Tucsnak '01) If there exists $T > 0$, such that $\forall\ z_0 \in \mathcal{D}(\mathcal{A})$,

$$\int_0^T \|B^* w\|_U^2 \mathrm{d}\tau \geqslant C\|z_0\|_W^2 \quad \text{with} \quad \mathcal{D}(\mathcal{A}) \subset X \subset W,$$

then for every $z_0 \in \mathcal{D}(\mathcal{A})$,

$$\|z(t)\|^2 \leqslant t^{-\frac{\theta}{1-\theta}} \|z_0\|_{\mathcal{D}(\mathcal{A})}^2, \quad \text{where} \quad [\mathcal{D}(\mathcal{A}), W]_\theta = X.$$

Proposition 2 (Borichev & Tomilov '10) $\mathcal{A}$ generates a bounded C_0-semigroup $\mathbb{T}$ on X and $\mathrm{i}\mathbb{R} \subset \rho(\mathcal{A})$, then

$$\|(\mathrm{i}\omega - \mathcal{A})^{-1}\| = O(|\omega|^s) \Longleftrightarrow \|\mathbb{T}_t z_0\| = O(t^{-1/s}) \quad s > 0.$$

[1] Chill, Paunonen, Seifert, Stahn, Tomilov, Analysis & PDE (accepted), 2019.
[2] Su, Control of water waves and floating body system, PhD thesis, *Université de Bordeaux*, Dec. 2021.

Statement of the main result:

Let the eigenpair of A be $(\mathrm{i}\mu_k, \phi_k)_{k\in J}$. We assume that for every $k, l \in J$ and $k \neq l$, there exist α, $\beta > 0$, s.t.

$[H_1]$ $\mu_k \neq \mu_l$ $(\forall k \neq l)$, $\quad \mu_k = k^\alpha + O(k^q)$ $(q < \alpha)$ as $k \to \infty$;

$[H_2]$ $\|B^*\phi_k\|_U \geqslant \frac{1}{k^\beta}$;

Theorem 1 (Su & Tucsnak 2019)

Assume that (A, B) satisfy $[H_1]$ and $[H_2]$ with

1. $0 < \alpha < 1$ and $\beta \geqslant 0$, then the solution of the damped system (2) satisfy

$$\|z(t)\|^2 \leqslant t^{-\frac{\alpha}{\beta-\alpha+1}} \|z_0\|^2_{\mathcal{D}(A)}, \qquad \forall\, z_0 \in \mathcal{D}(A);$$

2. $\alpha \geqslant 1$ and $\beta > 0$, then we have

$$\|z(t)\|^2 \leqslant t^{-\frac{\alpha}{\beta}} \|z_0\|^2_{\mathcal{D}(A)}, \qquad \forall\, z_0 \in \mathcal{D}(A).$$

Remark: $\alpha \geqslant 1$ and $\beta = 0 \Longrightarrow$ exponential stability !

Main idea of the proof:

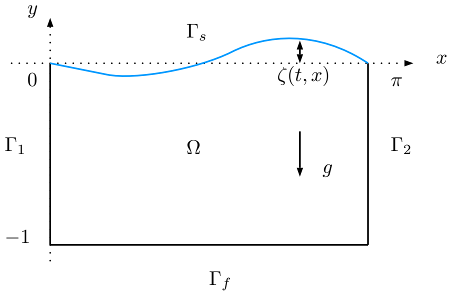
1. $0 < \alpha < 1$, $\beta \geqslant 0$: (Resolvent estimate)
   - $[H_1] + [H_2] \Longrightarrow \mathrm{i}\mathbb{R} \subset \rho(\mathcal{A})$
   - The structure of the eigenvalues: There exist M, $\gamma > 0$, s.t. for every $\omega \in \mathbb{R}$ and $|\omega| \geqslant M$, the interval
   $$\left[\omega - \gamma\omega^{\frac{\alpha-1}{\alpha}}, \omega + \gamma\omega^{\frac{\alpha-1}{\alpha}}\right]$$
   contains at most one element of $(\mu_k)_{k\in J}$.
   - Resolvent estimate:
   $$\|(\mathrm{i}\omega - \mathcal{A})^{-1}\| = O(|\omega|^{\frac{2(\beta-\alpha+1)}{\alpha}}).$$
2. $\alpha \geqslant 1$, $\beta > 0$: (Observability inequality)
$$\int_0^t \|B^* w\|_U^2 \mathrm{d}\tau \geqslant C\|z_0\|_{X_{-\beta/\alpha}}^2;$$

$Y = \mathcal{D}(A) = X_1$, $W = X_{-\beta/\alpha} \Longrightarrow \theta = \frac{\alpha}{\alpha+\beta}$.

Remark: Resolvent estimate also works in case 2!!

Problem setting:

- A rectangular domain Ω filled with water:



- Question: How to describe the change of the free surface when imposing a wave maker at the lateral boundary?
- Assumptions:
  Small-amplitude gravity water waves;
  Rigid wave maker;
  Incompressible, irrotational, inviscid...

2.1 Governing equations

- Notation: $\zeta(t,x)$: The elevation of the free surface; $\phi(t,x,y)$: The velocity potential of the fluid; $v(t)$: The velocity produced by the wave maker; $h(y)$: Shape function (the profile of the velocity field imposed by the wave maker).
- The governing equations of the water waves system, for every $t \geqslant 0$, $x \in [0,\pi]$ and $y \in [-1,\zeta(t,x)]$, are

$$\begin{cases} \Delta\phi(t,x,y) = 0, \\ (\partial_t\phi)\big|_{y=\zeta(t,x)} + \frac{1}{2}\left|\partial_x\phi\right|^2\big|_{y=\zeta(t,x)} + \zeta(t,x) = 0, \\ (\partial_{\vec{n}}\phi)\big|_{y=\zeta(t,x)} = \partial_t\zeta(t,x), \\ (\partial_{\vec{n}}\phi)\big|_{x=0} = -h(y)v(t), \\ (\partial_{\vec{n}}\phi)\big|_{y=-1} = 0 = (\partial_{\vec{n}}\phi)\big|_{x=\pi}. \end{cases} \tag{4}$$

2.2 Reformulation of the equations

- Dirichlet-Neumann operator $A_0[\zeta]$:

$$\begin{cases} \Delta\Psi(x,y) = 0, \\ \Psi\Big|_{y=\zeta(t,x)} = \psi(x), \\ (\partial_{\vec{n}}\Psi)_{x=0} = 0 = (\partial_{\vec{n}}\Psi)\Big|_{y=-1} = (\partial_{\vec{n}}\Psi)\Big|_{x=\pi}. \end{cases} \tag{5}$$

$$A_0[\zeta] = \gamma_1 D : \psi \longmapsto (\partial_{\vec{n}}\Psi)\Big|_{y=\zeta(t,x)}.$$

- Neumann-Neumann operator $B_0[\zeta]$:

$$\begin{cases} \Delta\Phi(x,y) = 0, \\ \Phi\Big|_{y=\zeta(t,x)} = 0, \\ (\partial_{\vec{n}}\Phi)_{x=0} = h(y)v(t) = \mathcal{V}(t,y), \\ (\partial_{\vec{n}}\Phi)\Big|_{y=-1} = 0 = (\partial_{\vec{n}}\Phi)\Big|_{x=\pi}. \end{cases} \tag{6}$$

$$B_0[\zeta] = \gamma_1 N : \mathcal{V} \longmapsto (\partial_{\vec{n}}\Phi)\Big|_{y=\zeta(t,x)}.$$

2.2 Reformulation of the equations

Remark: Let $\psi(t,x) := \phi(t,x,\zeta(t,x))$, the water waves system (4) is decomposed into (5) and (6), by verifying

$$\phi(t,x,y) = \Psi(t,x,y) + \Phi(t,x,y).$$

The Zakharov/Craig-Sulem formulation (ZCS): (ζ,ψ)-formulation:

$$\begin{cases} \partial_t\zeta - A_0[\zeta]\psi = B_0[\zeta]\mathcal{V}, \\ \partial_t\psi + \zeta + \frac{1}{2}|\partial_x\psi|^2 - \frac{(A_0[\zeta]\psi + B_0[\zeta]\mathcal{V} + \partial_x\zeta\,\partial_x\psi)^2}{2(1+|\partial_x\zeta|^2)} = 0. \end{cases} \tag{7}$$

▷ Related results for ZCS:

1. In an infinite strip domain without control, i.e. $x \in \mathbb{R}$, $\mathcal{V} = 0$:

Local well-posedness: D. Lannes '05;

2. Control of ZCS from the free surface:

T. Alazard et al. '15 (2π-peridoc domain, pressure as control on $\omega \subset \mathbb{R}$.)

Boundary control system of ZCS (7) ? Completely open !!!

Previous work.

- Reid & Russell '85: Null controllability for infinite time;
- Reid '86: Null controllability in an irregular domain;
- Reid '95: Null controllability in finite time for gravity-capillary;
- Mottelet '00:

Flexible generator: Approximate controllability holds (finite);
Rigid generator: No approximate controllability in finite time;
strong stability in non-uniform way.

Zero mean spaces:

$$H = \left\{ f \in L^2[0,\pi] \middle| \int_0^\pi f(x)\,\mathrm{d}x = 0 \right\},$$

$$\varphi_k(x) = \sqrt{\frac{2}{\pi}}\cos(kx) \qquad k \in \mathbb{N},\ x \in [0,\pi],$$

$$H_\alpha = \{ f \in H | \sum_{k \geqslant 1} k^{2\alpha} |\langle f, \varphi_k \rangle|^2 < \infty \} \qquad \forall\, \alpha \in \mathbb{R},$$

$$\mathcal{H}^1_{\mathrm{top}}(\Omega) = \left\{ f \in \mathcal{H}^1(\Omega)\ \middle| f(x,0) = 0,\ x \in (0,\pi) \right\}.$$

2.3 Linearized water waves equations

- Linearized Dirichlet-Neumann operator $A_0 := A_0[0]$:

$$A_0 = \gamma_1 D : \psi \longmapsto \partial_y \Psi(x,0)$$

Proposition 3: Dirichlet operator $D \in \mathcal{L}(H_{\frac{1}{2}}, \mathcal{H}^1(\Omega))$ and $A_0 : H_1 \to H$ is strictly positive.

$$A_0 \varphi_k = \lambda_k \varphi_k \quad \lambda_k = k \tanh(k) \qquad \forall\, k \in \mathbb{N}$$

$$A_0 \psi = \sum_{k \in \mathbb{N}} \lambda_k \langle \psi, \varphi_k \rangle \varphi_k \qquad \forall\, \psi \in H_1.$$

- Linearized Neumann to Neumann operator B_1:

$$B_1 = \gamma_1 N : \mathcal{V} \longmapsto \partial_y \Phi(x,0)$$

Proposition 4: Neumann operator $N \in \mathcal{L}(L^2[-1,0], \mathcal{H}^1_{\mathrm{top}}(\Omega))$, $B_1 \in \mathcal{L}(L^2[-1,0], L^2[0,\pi])$. Let $h \in L^2[-1,0]$ with $\int_{-1}^0 h(y)\,\mathrm{d}y = 0$, then, for $v \in \mathbb{C}$,

$$B_0 v = v B_1 h \Longrightarrow B_0 \in \mathcal{L}(\mathbb{C}, H).$$

Let
$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix},$$
the linearized water waves equations can be recast into
$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t) \\ z(0) = z_0 \end{cases} \tag{8}$$
$u(t) = \dot{v}(t)$, $z(t) = \begin{bmatrix} \zeta & \dot{\zeta} \end{bmatrix}^\intercal$, $X = H_{1/2} \times H$ and $B \in \mathcal{L}(\mathbb{C}, X)$, $X_1 = \mathcal{D}(A) = H_1 \times H_{1/2}$.

Definition 1: Solution of the linearized water waves (ζ, ϕ):

Given $u \in L^2_{\mathrm{loc}}[0,\infty)$ and $h \in L^2[-1,0]$, $\int_{-1}^0 h(y)\,\mathrm{d}y = 0$,
$$\phi \in \mathcal{H}^1_{\mathrm{loc}}([0,\infty); \mathcal{H}^1(\Omega)), \quad \zeta \in C([0,\infty); H_{\frac{1}{2}}) \cap C^1([0,\infty); H),$$
$$\partial_t \phi(t,\cdot,0) + \zeta(t,\cdot) = 0, \text{ in } L^2_{\mathrm{loc}}([0,\infty); L^2[0,\pi]),$$
$\forall\ \Psi \in \mathcal{H}^1(\Omega)$:
$$\int_0^\pi \dot{\zeta}(t,x)\overline{\Psi(x,0)}\mathrm{d}x - \int_0^\pi \dot{\zeta}(0,x)\overline{\Psi(x,0)}\mathrm{d}x =$$
$$\int_0^t \int_\Omega \nabla(\partial_t\phi)(\sigma,x,y) \cdot \overline{\nabla\Psi(x,y)}\mathrm{d}x\mathrm{d}y\mathrm{d}\sigma - \int_0^t u(\sigma) \int_{-1}^0 h(y)\overline{\Psi(0,y)}\mathrm{d}y\mathrm{d}\sigma.$$

Theorem 2 (Su, Tucsnak and Weiss 2020)

Let $h \in L^2[-1,0]$ be such that $\int_{-1}^{0} h(y)\,\mathrm{d}y = 0$. Then $\forall\ u \in L^2_{\mathrm{loc}}[0,\infty)$, $\zeta_0 \in H_{\frac{1}{2}}$ and $w_0 \in H$, the linearized equations admits a unique solution with $\zeta(0) = \zeta_0$ and $\dot{\zeta}(0) = w_0$. Moreover, there exists a well-posed linear control system $(\mathbb{T}, \Phi)$ with $X = H_{\frac{1}{2}} \times H$ and $U = \mathbb{C}$ s.t., setting $z_0 = \left[\begin{smallmatrix} \zeta_0 \\ w_0 \end{smallmatrix}\right]$, we have

$$z(\tau) = \mathbb{T}_\tau z_0 + \Phi_\tau u \qquad \forall\ \tau \geqslant 0.$$

Finally, the generator A of $\mathbb{T}$ is skew-adjoint, with domain $\mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}$, and there exists $B \in \mathcal{L}(\mathbb{C}, X)$ s.t. $\forall\ \tau \geqslant 0$,

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} B u(\sigma)\,\mathrm{d}\sigma \qquad \forall\ u \in L^2_{\mathrm{loc}}[0,\infty).$$

linear water waves $\Longleftrightarrow$ well-posed linear control system (A, B)

[1] Su, Tucsnak and Weiss, *Systems & Control Letters*, 2020

2.3.2 Stabilization

Theorem 3 (Stabilizability properties)

For $\Sigma = (A, B)$ introduced in Theorem 2, we have

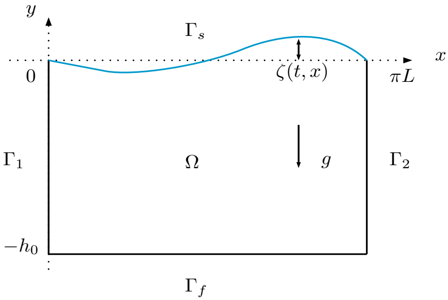
- Σ is not exponentially stabilizable;
- Σ is strongly stabilizable iff h is a *strategic profile*, i.e.
$$\int_{-1}^{0} h(y) \cosh\left[k(y+1)\right] \mathrm{d}y \neq 0 \qquad \forall\, k \in \mathbb{N};$$
- If
$$\inf_{k\in\mathbb{N}} \frac{k}{\cosh k} \left| \int_{-1}^{0} h(y) \cosh\left[k(y+1)\right] \mathrm{d}y \right| > 0,$$
then the system Σ is USSD. Precisely, $F = -B^*$ leads to the closed-loop semigroup $\mathbb{T}^{cl}$ satisfying
$$\|\mathbb{T}^{cl}_t z_0\|^2_X \leqslant C t^{-1/3} \|z_0\|^2_{\mathcal{D}(A)} \qquad \forall\, z_0 \in \mathcal{D}(A),\ t \geqslant 0.$$

Remark: gravity-capillary waves
$$\|\mathbb{T}^{cl}_t z_0\|^2_{H_{3/2}\times H} \leqslant C t^{-3/2} \|z_0\|^2_{H_3^b \times H_{3/2}}$$

L: The horizontal scale of the domain Ω;

h_0: The typical water depth; a: the order of the surface variation;

The shallowness parameter: $\mu = \frac{h_0^2}{L^2} \ll 1$ in shallow water regime;



Dimensionless quantities:

$$\tilde{x} = \frac{x}{L}, \quad \tilde{y} = \frac{y}{h_0}, \quad \tilde{t} = \frac{t}{L/\sqrt{gh_0}}, \quad \tilde{\zeta} = \frac{\zeta}{a}, \quad \tilde{\phi} = \frac{\phi}{aL\sqrt{g/h_0}},$$

"twisted" Laplace operator: $\Delta_\mu = \mu \frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial^2}{\partial \tilde{y}^2}$

Omitting the tildes, for all $x \in (0, \pi)$ and $y \in (-1, 0)$,

$$\text{dimensionless version}: \begin{cases} \Delta_\mu \phi_\mu(t,x,y) = 0, \\ \partial_t \zeta_\mu(t,x) - \dfrac{1}{\mu} \partial_y \phi_\mu(t,x,0) = 0, \\ \partial_t \phi_\mu(t,x,0) + \zeta_\mu(t,x) = 0, \\ \partial_x \phi_\mu(t,0,y) = -h(y)v(t), \\ \partial_y \phi_\mu(t,x,-1) = 0 = \partial_x \phi_\mu(t,\pi,y), \end{cases} \tag{9}$$

$$\iff \begin{cases} \partial_t^2 \zeta_\mu(t,x) + \dfrac{1}{\mu} A_\mu \zeta_\mu(t,x) = \dfrac{1}{\mu} B_\mu u(t), \\ \zeta_\mu(0,x) = \zeta_0(x), \quad \partial_t \zeta_\mu(0,x) = \zeta_1(x), \end{cases} \tag{10}$$

Remark: No zero-mean conditions now !

Dimensionless operators:

$$D - N : A_\mu; \quad N - N : B_\mu \in \mathcal{L}(\mathbb{C}, L^2[0,\pi])$$

Proposition 5: $A_\mu : \mathcal{H}^1[0,\pi] \to L^2[0,\pi]$ and $A_\mu \varphi_k = \lambda_{\mu,k}\varphi_k$

$$\lambda_{\mu,k} = \sqrt{\mu}k\tanh(\sqrt{\mu}k) \qquad \forall\, k \in \mathbb{N}\cup\{0\},$$

$$A_\mu \eta = \sum_{k\in\mathbb{N}} \lambda_{\mu,k}\langle \eta, \varphi_k\rangle \varphi_k \qquad \forall\, \eta \in \mathcal{H}^1[0,\pi].$$

Remark: as $\mu \to 0$, $\frac{1}{\mu}A_\mu : \frac{1}{\mu}\lambda_{\mu,k} \to k^2 \dashrightarrow -\partial_x^2\,?$

- Wave equation with Neumann boundary control:

$$\begin{cases} \partial_t^2\zeta(t,x) - \partial_x^2\zeta(t,x) = 0, \\ \partial_x\zeta(t,0) = u(t), \quad \partial_x\zeta(t,\pi) = 0, \\ \zeta(0,x) = \zeta_0(x), \quad \partial_t\zeta(0,x) = \zeta_1(x). \end{cases} \tag{11}$$

$$(11) \iff \begin{cases} \partial_t^2\zeta(t,x) + A_w\zeta(t,x) = B_w u(t), \\ \zeta(0,x) = \zeta_0(x), \quad \partial_t\zeta(0,x) = \zeta_1(x). \end{cases} \tag{12}$$

with $B_w = -\delta_0$ and

$$A_w = -\partial_x^2, \qquad \mathcal{D}(A_w) = \left\{ f \in \mathcal{H}^2[0,\pi] \,\middle|\, f'(0) = f'(\pi) = 0 \right\},$$

2.4 Asymptotic analysis

Proposition 6: $\lim_{\mu\to 0} \frac{1}{\mu} B_\mu u = B_w u$ in $(\mathcal{H}^1[0,\pi])'$.

Theorem 4 (Su 2020)

$u \in L^2_{\text{loc}}[0,\infty)$ and $\zeta_0 \in \mathcal{H}^1[0,\pi]$, $\zeta_1 \in L^2[0,\pi]$, let ζ_μ be the solution of (10), satisfying

$$\zeta_\mu \in C([0,\infty);\mathcal{H}^{1/2}[0,\pi]) \cap C^1([0,\infty);L^2[0,\pi]).$$

Let ζ be the weak solution (12) satisfying

$$\zeta \in C([0,\infty);\mathcal{H}^1[0,\pi]) \cap C^1([0,\infty);L^2[0,\pi]).$$

$$\Longrightarrow \begin{array}{l} \lim\limits_{\mu\to 0} \sup\limits_{t\in[0,\tau]} \|\zeta_\mu - \zeta\|_{\mathcal{H}^{1/2}[0,\pi]} = 0, \\ \lim\limits_{\mu\to 0} \sup\limits_{t\in[0,\tau]} \|\partial_t\zeta_\mu - \partial_t\zeta\|_{L^2[0,\pi]} = 0. \end{array}$$

$$H_\alpha = \{f \in L^2[0,\pi] | \sum_{k\geqslant 0}(1+|\lambda_k|^{2\alpha})\,|\langle z,\varphi_k\rangle|^2 < \infty\} \Longrightarrow \begin{cases} H_{\mu,\alpha} \;\; \lambda_{\mu,k} = \dfrac{k\tanh(\sqrt{\mu}k)}{\sqrt{\mu}} \\ \mathbb{H}_\alpha \;\; \lambda_k = k^2 \end{cases}$$

Sketch of the proof:

Step 1. System transformation.

For $\zeta_0 \in \mathbb{H}_{\frac{1}{2}}$ and $\zeta_1 \in H$, (9) and (11) admit ζ_μ and ζ satisfying

$$\zeta_\mu \in C([0,\infty); H_{\mu,\frac{1}{2}}) \cap C^1([0,\infty); H),$$

$$\zeta \in C([0,\infty); \mathbb{H}_{\frac{1}{2}}) \cap C^1([0,\infty); \mathbb{H}).$$

Define

$$\alpha_\mu := \partial_t \zeta_\mu, \qquad \beta_\mu := \left(\frac{1}{\mu} A_\mu\right)^{1/2} \zeta_\mu,$$

$$\alpha := \partial_t \zeta, \qquad \beta := A_w^{1/2} \zeta,$$

$$\Longrightarrow \alpha_\mu, \beta_\mu, \alpha, \beta \in C([0,\infty); H).$$

Let

$$w_\mu(t) = \begin{bmatrix} \alpha_\mu(t,\cdot) \\ \beta_\mu(t,\cdot) \end{bmatrix} \quad \text{and} \quad w(t) = \begin{bmatrix} \alpha(t,\cdot) \\ \beta(t,\cdot) \end{bmatrix},$$

$$\begin{cases} \dot{w}_\mu(t) = \mathcal{A}_\mu w_\mu(t) + \mathcal{B}_\mu u(t), \\ w_\mu(0) = w_{\mu,0}, \end{cases} \qquad \begin{cases} \dot{w}(t) = \mathcal{A}_0 w(t) + \mathcal{B}_0 u(t), \\ w(0) = w_0, \end{cases}$$

with $X_\mu = X = H \times H$, $\mathcal{A}_\mu$ and $\mathcal{A}_0$ are skew-adjoint. Using Fourier series we have

$$\lim_{\mu \to 0} (I - \mathcal{A}_\mu)^{-1} \begin{bmatrix} f \\ g \end{bmatrix} = (I - \mathcal{A}_0)^{-1} \begin{bmatrix} f \\ g \end{bmatrix} \qquad \text{in} \quad X.$$

$$\lim_{\mu \to 0} (I - \mathcal{A}_\mu)^{-1} \mathcal{B}_\mu u = (I - \mathcal{A}_0)^{-1} \mathcal{B}_0 u \qquad \text{in} \quad X.$$

Step 2. Convergence of a scattering semigroup.

$$w_\mu = \mathbb{T}_{\mu,t} w_{\mu,0} + \Phi_{\mu,t} u \quad \text{and} \quad w = \mathbb{T}_t w_0 + \Phi_t u,$$

$$\mathscr{A} = \begin{bmatrix} \mathcal{A} & \mathcal{B}\delta_0 \\ 0 & \frac{d}{d\xi} \end{bmatrix} \longrightarrow \mathfrak{T}_t = \begin{bmatrix} \mathbb{T}_t & \Phi_t \\ 0 & S_t \end{bmatrix} \quad \text{on} \quad X \times L^2_\omega[0, \infty).$$

$$\mathcal{D}(\mathscr{A}) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in X \times \mathcal{H}^1_\omega[0, \infty) \,|\, \mathcal{A}x + \mathcal{B}u(0) \in X \right\},$$

By Trotter-Kato theorem,

$$\lim_{\mu\to 0}(\mathbb{T}_{\mu,t}\,x+\Phi_{\mu,t}\,u)=\mathbb{T}_t\,x+\Phi_t\,u \qquad \text{in} \quad X\times\mathcal{U}.$$

Step 3. $\lim_{\mu\to 0}\zeta_\mu=\zeta$ in $C^1([0,\tau];H)\cap C([0,\tau];H_{\frac{1}{2}})$.

$$\|\mathbb{T}_{\mu,t}\,w_{\mu,0}-\mathbb{T}_t\,w_0\|_X\leqslant\|\mathbb{T}_{\mu,t}\,w_{\mu,0}-\mathbb{T}_{\mu,t}\,w_0\|_X+\|\mathbb{T}_{\mu,t}\,w_0-\mathbb{T}_t\,w_0\|_X\,.$$

$$\Longrightarrow\lim_{\mu\to 0}w_\mu=w \quad \text{in} \quad X,$$

$$i.e. \quad \lim_{\mu\to 0}\alpha_\mu=\alpha \qquad \lim_{\mu\to 0}\beta_\mu=\beta \quad \text{in} \quad H.$$

$$\Longrightarrow\begin{cases}\lim_{\mu\to 0}\partial_t\zeta_\mu=\partial_t\zeta & \text{in} \quad C([0,\tau];H);\\ \lim_{\mu\to 0}\zeta_\mu=\zeta & \text{in} \quad C([0,\tau);H_{\frac{1}{2}}).\end{cases}$$

[1] Staffans & Weiss, Lax-Phillips semigroup *Trans.Amer.Math.Society*, 2002.

[2] Su, Asymptotic behaviour of a linearized water waves system in a rectangle, *Asymptotic Analysis*, in press, 2022.

3. Perspectives

★ What is the next?

▷ The system in a bounded domain

- Control of nonlinear system;
- Water waves system in a general convex domain;

▷ Shallow water convergence

- The controllability & the dispersive effect;
- Higher regularity and regular convergence;

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Thanks for your attention !

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