Problems - Cosmology, TAE 2022

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Problem 1: Photon geodesics

The trajectory and energy of a photon propagating through the Universe will be affected by perturbations in the metric. For a perturbed FRW in conformal coordinates, and using the Newtonian gauge, i.e.

$$d\tau^2 = a^2(\eta)\,\left[(1+2\psi)d\eta^2 - (1-2\phi)|d\mathbf{x}|^2\right], \tag{1}$$

show that the redshift of a photon propagating from a source with velocity $\mathbf{v}_s$ to an observer with velocity $\mathbf{v}_0$ is given by

$$1+z = \frac{1}{a}\left[1 - \psi + \psi_0 + \hat{\mathbf{n}}\cdot(\mathbf{v}_s - \mathbf{v}_0) + \int_{\eta_0}^{\eta} d\eta'\,(\phi' + \psi')\right], \tag{2}$$

where $\hat{\mathbf{n}}$ is a unit vector along the line of sight, and the subscript $_0$ denotes quantities evaluated at the observer. What is the physical interpretation of the different terms entering this equation?

Show also that the photon follows a trajectory

$$\mathbf{x}(\eta) = \hat{\mathbf{n}}\int_{\eta}^{\eta_0} d\eta'\,(1+\phi+\psi) - \int_{\eta}^{\eta_0} d\eta'(\eta'-\eta)\nabla_{\perp}(\phi+\psi). \tag{3}$$

Hints:

- Start by writing down the geodesic equation for the photon 4-momentum $p^\mu = dx^\mu/d\lambda$:

  $$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\nu\sigma}p^\nu p^\sigma = 0, \tag{4}$$

  as well as the null condition

  $$p^\mu p^\nu g_{\mu\nu} = 0 \tag{5}$$

  to first order in the perturbations.

- Change variables to the directional vector $\hat{\mathbf{e}} \equiv \mathbf{p}/|\mathbf{p}|^2$, and the comoving energy $\epsilon \equiv a\,p_\mu u_c^\mu$, where $u_c^\mu = a^{-1}(1-\psi)\delta_0^\mu$ is the 4-velocity of an observer with constant comoving coordinates (why?). Integrate the resulting two equations to find $\epsilon(\eta)$ and $\mathbf{x}(\eta)$ along the photon trajectory.

- Noting that the frequency measured by an observer with 4-velocity u^μ is $h_P\nu = p^\mu u_\mu$, write the 4-velocity of source and observer as $u_q^\mu = a^{-1}(1-\psi, \mathbf{v})$ (why?), and use the equation for ϵ to obtain Eq. 2, where the redshift is defined as $1+z \equiv (u_s^\mu p_\mu)/(u_0^\mu p_\mu)$.

Problem 2: Perturbations during radiation domination

In the lectures we showed that the Newtonian potential ψ stays constant during matter domination. Let's now examine the solution during radiation domination. Solve Einstein's equations in the Newtonian gauge during radiation domination to show that the gravitational potential evolves as

$$\psi(k,\eta) \propto \frac{j_1(c_s k\eta)}{c_s k\eta}, \tag{6}$$

where $c_s = 1/\sqrt{3}$ is the sound speed for radiation, and $j_1(x)$ is the spherical Bessel function of the first kind:

$$j_1(x) = \frac{\sin x - x\cos x}{x^2}. \tag{7}$$

What is the behaviour on super-horizon and sub-horizon scales?

Hints:

- Recall Einstein's equations for a perfect fluid in the Newtonian gauge:

$$k^2\psi + 3\mathcal{H}(\psi' + \mathcal{H}\psi) = -4\pi G a^2\,\bar{\rho}\,\delta, \tag{8}$$

$$k^2(\psi' + \mathcal{H}\psi) = 4\pi G a^2\,(\bar{\rho} + \bar{p})\theta, \tag{9}$$

$$\psi'' + 3\mathcal{H}\psi' + (2\mathcal{H}' + \mathcal{H}^2)\psi = 4\pi G a^2\, c_s^2 \bar{\rho}\,\delta. \tag{10}$$

  Combine two of these to write an equation involving ψ alone.

- Use the fact that $a \propto t^{1/2} \propto \eta$ during radiation domination to simplify this equation.

- Solve the resulting equation by noting that the spherical Bessel functions satisfy the equation

$$x^2 j_n'' + 2x j_n' + (x^2 - n(n+1))j_n = 0. \tag{11}$$

Problem 3: Curvature perturbations

In the lectures we argued that ψ stays constant on super-horizon scales as long as the equation of state doesn't vary. Show that the following quantity (the "curvature perturbation") does stay constant even if the equation of state varies:

$$\mathcal{R} \equiv -\phi - \frac{\mathcal{H}(\psi' + \mathcal{H}\phi)}{4\pi G a^2(\bar{\rho} + \bar{p})}. \tag{12}$$

Hints:

- Use Einstein's equations (Eqs. 8-10) to write $\mathcal{R}$ in terms of ϕ and δ alone, on super-horizon scales ($k \ll \mathcal{H}$), as

$$\mathcal{R} \simeq -\phi - \frac{\bar{\rho}\delta}{3(\bar{\rho} + \bar{p})}. \tag{13}$$

- Remember that the continuity equation reads:

$$\delta' = -(1 + w)(\theta - 3\phi') - 3\mathcal{H}(c_s^2 - w)\delta. \tag{14}$$

  Show that the term involving θ can be dropped (why?), and use Eq. 13 to replace ψ with $\mathcal{R}$. This gives you an ODE for $\mathcal{R}$.

- Use the continuity equation for the background ($\bar{\rho}' = -3\mathcal{H}(\bar{\rho} + \bar{p})$) to simplify this equation, and show that

$$\mathcal{R}' = -\frac{\mathcal{H}\bar{\rho}}{\bar{\rho} + \bar{p}} \left(\frac{\delta p}{\bar{\rho}} - \frac{\bar{p}'}{\bar{\rho}'}\delta \right). \tag{15}$$

  Can we say that the right hand side is zero, thus proving what we wanted?

Problem 4: From 3D to 2D.

Consider a quantity defined on the sphere $f(\hat{\mathbf{n}})$ in terms of a radial lightcone integral over a three-dimensional quantity $F(\mathbf{x},\eta)$:

$$f(\hat{\mathbf{n}}) = \int^{\eta_0} d\eta\, q_f(\chi) F(\chi\hat{\mathbf{n}},\eta), \tag{16}$$

where $q_f(\chi)$ is a radial kernel. Assume now that the three-dimensional quantity is linearly related to the primordial curvature perturbation $\mathcal{R}_{\mathbf{k}}$ through a transfer function $T_F(k,\eta)$:

$$F_{\mathbf{k}}(\eta) = T_F(k,\eta)\,\mathcal{R}_{\mathbf{k}}. \tag{17}$$

Show that the harmonic coefficients of f are then related to $\mathcal{R}_{\mathbf{k}}$ through:

$$f_{\ell m} = \int \frac{dk}{2\pi^2} k^2\, \Delta_\ell^f(k) i^\ell \int d\hat{\mathbf{n}}_{\mathbf{k}}\, Y^*_{\ell m}(\hat{\mathbf{n}}_{\mathbf{k}})\,\mathcal{R}_{\mathbf{k}}, \tag{18}$$

where $\hat{\mathbf{n}}_{\mathbf{k}} \equiv \mathbf{k}/k$, and

$$\Delta_\ell^f(k) = \int d\chi\, q_v(\chi)\, T_F(k,\eta)\, j_\ell(k\chi). \tag{19}$$

Then, show that the angular power spectrum of f is related to the primordial power spectrum $\Delta^2_{\mathcal{R}}(k)$ through

$$C_\ell^f = \frac{2}{\pi}\int \frac{dk}{k}\,|\Delta_\ell^f(k)|^2\,\Delta^2_{\mathcal{R}}(k). \tag{20}$$

Hints:

- Use the plane-wave expansion to relate the harmonic coefficients of f to the Fourier coefficients of F:

  $$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi\sum_{\ell m} i^\ell\, j_\ell(k\chi)\, Y_{\ell m}(\hat{\mathbf{n}})\, Y^*_{\ell m}(\hat{\mathbf{n}}_{\mathbf{k}}), \tag{21}$$

  where $j_\ell(x)$ is the spherical Bessel function of order ℓ.

- Remember the definition of the primordial power spectrum

  $$\langle \mathcal{R}^*_{\mathbf{k}}\mathcal{R}_{\mathbf{k}'}\rangle \equiv \frac{2\pi^2}{k^3}\Delta^2_{\mathcal{R}}(k)\,(2\pi)^3\delta^D(\mathbf{k}-\mathbf{k}'). \tag{22}$$

Problem 5: The growth factor

The evolution equation for the overdensity of non-relativistic, pressureless matter, neglecting all perturbations in any other species can be written, in the Newtonian limit, as:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_M(t)\,\delta = 0,$$

where $H = \dot{a}/a$ is the expansion rate and ρ_M is the non-relativistic matter density.

1. Change the time variable from t to a to change the form of this equation to:

$$\delta'' + \left(\frac{H'}{H} + \frac{3}{a}\right)\delta' - \frac{3}{2}\frac{H_0^2\Omega_M}{H^2\,a^5}\delta = 0, \tag{23}$$

   where $f' \equiv df/da$ and Ω_M and H_0 are the matter parameter and expansion rate today.

2. Show that, for a matter dominated universe, a possible solution to this equation is $\delta_1 = H$. For this you'll need to use the fact that in this case $H^2 \simeq H_0^2\Omega_M/a^3$. Is this a growing-mode solution?

3. Consider the more general case where $H^2 = H_0^2\left(\Omega_M/a^3 + \sum_i \Omega_i\, a^{-3(1+w_i)}\right)$. For what values of the equation of state parameter w_i is $\delta_1 = H$ still a solution? What components do these values correspond to?

4. For a second-order equation with two independent solutions δ_1 and δ_2 the Wronskian W is defined as $W \equiv \delta_1'\delta_2 - \delta_1\delta_2'$. Prove that for equation 23 the Wronskian satisfies

$$\frac{W'}{W} = -\left(\frac{H'}{H} + \frac{3}{a}\right). \tag{24}$$

   Integrate this equation to show that $W = C/(a^3\,H)$, where C is an integration constant.

5. Using the previous result and under the ansatz $\delta_2(a) = \delta_1(a)\,g(a)$ prove that the second solution is

$$\delta_2 = -C\,H(a)\int_0^a \frac{da'}{[a'\,H(a')]^3}. \tag{25}$$

6. We know that at early times ($a << 1$), during matter domination, $\delta = a$. Find the value of the integration constant C that gives this normalization.