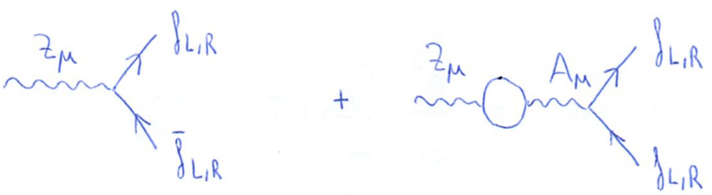
We define $s_{eff}^{2\,(th)}$ via the expression $A_{LR}^{\ell} = \dfrac{\left(\frac{1}{2} - s_{eff}^{2\,(th)}\right)^2 - \left(s_{eff}^{2\,(th)}\right)^2}{\left(\frac{1}{2} - s_{eff}^{2\,(th)}\right)^2 + \left(s_{eff}^{2\,(th)}\right)^2}$

Then, to obtain $s_{eff}^{2\,(th)}$ we need to compute the 1-loop corrections to C_L and C_R:

$\Rightarrow$ Π_{ZZ} contributions can be neglected (at linear order, we will see below), and we have



$$\Rightarrow \; C_L = \frac{e}{sc}\left(T_\ell^3 - Q_\ell s^2\right) + i\,\Pi_{\gamma Z}(q^2)\,\frac{-i}{q^2}\Bigg|_{q^2 = m_Z^2} (e Q_\ell)$$

$$= \frac{e}{sc}\left[T_\ell^3 - Q_\ell\left(s^2 - sc\,\frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2}\right)\right]$$

$$C_R = -\frac{e}{sc}\,Q_\ell s^2 + i\,\Pi_{\gamma Z}(q^2)\,\frac{-i}{q^2}\Bigg|_{q^2 = m_Z^2} (e Q_\ell)$$

$$= -\frac{e Q_\ell}{sc}\left(s^2 - sc\,\frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2}\right)$$

We can then quickly verify that $s^{2\,(th)}_{eff} = s^2 - sc\,\dfrac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2}$

Let me pause at this point for a moment to view (some of) these results from a different (in appearence) perspective:

Consider a (4-fermion) neutral current interaction, with matrix element / amplitude given by:

$$\mathcal{M}_{NC} = e^2 Q_e Q_{e'}\, G_{\gamma\gamma} + \frac{e^2}{sc}\left[Q_e (T^3_{e'} - s^2 Q_{e'}) + (T^3_e - s^2 Q_e) Q_{e'}\right] G_{\gamma Z}$$

$$+ \frac{e^2}{s^2c^2}(T^3_e - s^2 Q_e)(T^3_{e'} - s^2 Q_{e'})\, G_{ZZ}$$

with G_{ab} the corresponding propagators (resummed)

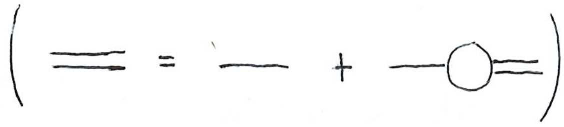
- To leading order, $G_{ab} = D_{ab}$, with

$$D_{\gamma Z} = 0 \;; \quad D_{\gamma\gamma} = \frac{1}{q^2} \;; \quad D_{ZZ} = \frac{1}{q^2 - m_{Z0}^2}$$

$$m_{Z0}^2 = \frac{e^2}{s^2c^2}\frac{v^2}{4}$$

- We can resum in G_{ab} the effects of the self-energies $\Pi_{ab}(q^2)$ by solving the (Dyson) equns:

$$G_{\gamma\gamma} = D_{\gamma\gamma} + D_{\gamma\gamma}\Pi_{\gamma\gamma}G_{\gamma\gamma}$$



(resummations)

$$G_{Z\gamma} = D_{ZZ}\Pi_{\gamma Z}G_{\gamma\gamma}$$

$$G_{ZZ} = D_{ZZ} + D_{ZZ}\Pi_{ZZ}G_{ZZ}$$

$$\Downarrow$$

$$G_{\gamma\gamma} = \frac{1}{q^2 - \Pi_{\gamma\gamma}(q^2)}$$

$$G_{\gamma Z} = \frac{1}{q^2 - m_{Z_0}^2}\,\frac{\Pi_{\gamma Z}(q^2)}{q^2 - \Pi_{\gamma\gamma}(q^2)}$$

$$G_{ZZ} = \frac{1}{q^2 - m_{Z_0}^2 - \Pi_{ZZ}(q^2)}$$

Substituting these on the M_{NC} amplitude one gets:

$$M_{NC} = \frac{e^2 Q_\ell Q_{\ell'}}{q^2 - \Pi_{\gamma\gamma}} + \frac{e^2}{s^2c^2}\left\{\left[Q_\ell(T^3_{\ell'} - s^2Q_{\ell'}) + (T^3_\ell - s^2Q_\ell)Q_{\ell'}\right]\frac{1}{q^2 - m_{Z_0}^2}\,\frac{sc\,\Pi_{\gamma Z}}{q^2 - \Pi_{\gamma\gamma}} + (T^3_\ell - s^2Q_\ell)(T^3_{\ell'} - s^2Q_{\ell'})\,\frac{1}{q^2 - m_{Z_0}^2 - \Pi_{ZZ}}\right\}$$

The first term, $\dfrac{e^2}{q^2 - \Pi_{\gamma\gamma}(q^2)} = \dfrac{e^2}{q^2(1-\Pi'_{\gamma\gamma}(q^2))} \simeq \dfrac{e^2}{q^2}\left(1+\Pi'_{\gamma\gamma}(q^2)\right)$

(recall the definition of α^{th} !)

For the second term:

$$\left[Q_e(T^3_{e'} - s^2 Q_{e'}) + (T^3_e - s^2 Q_e) Q_{e'}\right] \cdot \frac{sc\,\Pi_{\gamma Z}(q^2)}{q^2 - \Pi_{\gamma\gamma}} \cdot \frac{1}{q^2 - m^2_{Z_0} - \Pi_{ZZ}} \cdot \frac{q^2 - m^2_{Z_0} - \Pi_{ZZ}}{q^2 - m^2_{Z_0}}$$

$$+ (T^3_e - s^2 Q_e)(T^3_{e'} - s^2 Q_{e'}) \frac{1}{q^2 - m^2_{Z_0} - \Pi_{ZZ}} =$$

$$= \frac{1}{q^2 - m^2_{Z_0} - \Pi_{ZZ}} \left\{ \left[T^3_e - \underbrace{\left(s^2 - sc\frac{\Pi_{\gamma Z}}{q^2 - \Pi_{\gamma\gamma}}\right)}_{\Downarrow} Q_e\right]\left[T^3_{e'} - \left(s^2 - sc\frac{\Pi_{\gamma Z}}{q^2 - \Pi_{\gamma\gamma}}\right) Q_{e'}\right]\right\}$$

$$+ O(\Pi^2)$$

(recall the definition of $s^{2\,(th)}_{eff}$ @ m_Z (Z-pole))

At this point, we could in principle compute the various self-energies and do a χ^2-fit like for the tree-level case. Instead, we are going to proceed as in the tree-level case by expressing some observables in terms of other observables (this cannot be done in the full 1-loop SM renormalization program, but in this simple case we are looking at, we can [we follow J. Wells review])

Before we proceed, one remark (that we will need)

We have $\alpha^{th} = \dfrac{e^2}{4\pi}\left(1 + \Pi'_{\gamma\gamma}(q^2=0)\right)$

But $\Pi'_{\gamma\gamma}(q^2=0)$ is not calculable!

How to go around this? (See Wells review)

$\Rightarrow$ We rewrite $$\Pi'_{\gamma\gamma}(0) = \mathrm{Re}\left(\frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2}\right) - \left[\mathrm{Re}\left(\frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2}\right) - \Pi'_{\gamma\gamma}(0)\right]$$

(add and subtract self-energy $\Pi_{\gamma\gamma}$ at $q^2 = m_Z^2$)

$\Rightarrow$ $\Pi_{\gamma\gamma}(m_Z^2)$ is calculable ; $\mathrm{Re}\left[\dfrac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2}\right] - \Pi'_{\gamma\gamma}(0) \equiv \Delta\alpha(m_Z^2)$

is not calculable, but it is measurable experimentally:

$$\Delta\alpha(m_Z^2) = 0.0590 \pm 0.0010 \quad \text{(arxiv: 1908.00921)}$$

$$\Rightarrow \alpha(m_Z^2) = \frac{\alpha}{1 - \Delta\alpha(m_Z^2)} \simeq \alpha\left(1 + \Delta\alpha(m_Z^2)\right) \simeq \frac{e^2}{4\pi}\left(1 + \mathrm{Re}\left(\frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2}\right)\right)$$

$$\hat{\alpha}^{-1}(m_Z^2) = 128.948 \pm 0.013 \quad \text{(arxiv: 1908.00921)}$$

Then:

$$e^2 = \frac{4\pi\hat{\alpha}(m_Z^2)}{1+\frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2}} \quad ; \quad v^2 = \frac{1}{\sqrt{2}\,\hat{G}_F}\left[1-\frac{\Pi_{WW}(0)}{m_W^2}\right]$$

From the $(m_Z^{th})^2$ relation:

$$s^2c^2 = \frac{e^2v^2}{4}\left(\frac{1}{\hat{m}_Z^2-\Pi_{ZZ}(m_Z^2)}\right) =$$

$$= \frac{\pi\hat{\alpha}(m_Z^2)}{\sqrt{2}\,\hat{G}_F}\left(\frac{\left(1-\frac{\Pi_{WW}(0)}{m_W^2}\right)}{\left(\hat{m}_Z^2-\Pi_{ZZ}(m_Z^2)\right)}\cdot\frac{1}{1+\frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2}}\right) =$$

$$= \frac{\pi\hat{\alpha}(m_Z^2)}{\sqrt{2}\,\hat{G}_F\,\hat{m}_Z^2}\cdot\left(1+\frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2}-\frac{\Pi_{WW}(0)}{m_W^2}-\frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2}\right)$$

$$+\,O(\Pi^2)$$

$$\equiv \underbrace{\frac{\pi\hat{\alpha}(m_Z^2)}{\sqrt{2}\,\hat{G}_F\,\hat{m}_Z^2}}_{\hat{x}}(1+\delta_S)$$

$$s^2 = \tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-4\hat{x}(1+\delta_S)} = \overbrace{\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-4\hat{x}}}^{\hat{s}_0^2} + \delta_S\frac{\hat{x}}{\sqrt{1-4\hat{x}}}$$

$$+\,O(\delta_S^2)$$

$$\downarrow O(\Pi^2)$$

Note that this matches Wells' definitions $\hat{S}_0^2 \hat{C}_0^2 = \hat{X}$

and $S^2 = \hat{S}_0^2 + \dfrac{\hat{S}_0^2 \hat{C}_0^2}{1 - 2\hat{S}_0^2} \delta_S \quad (+ O(\delta_S^2) \ldots)$

Then :

$$m_W^{2\,th} = \frac{e^2 v^2}{4 S^2} + \Pi_{WW}(m_W^2) =$$

$$= \frac{\pi \hat{\alpha}(m_Z^2)}{\sqrt{2}\,\hat{G}_F\,\hat{S}_0^2} \; \frac{1}{\left(1 + \dfrac{\hat{C}_0^2}{\hat{C}_0^2 - \hat{S}_0^2}\delta_S\right)} \; \frac{\left(1 - \dfrac{\Pi_{WW}(0)}{m_W^2}\right)}{\left(1 + \dfrac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2}\right)} + \Pi_{WW}(m_W^2)$$

$$= \frac{\pi \hat{\alpha}(m_Z^2)}{\sqrt{2}\,\hat{G}_F\,\hat{S}_0^2} \left[1 + \frac{\Pi_{WW}(m_W^2)}{m_W^2} - \frac{\Pi_{WW}(0)}{m_W^2} - \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} - \frac{\hat{C}_0^2}{\hat{C}_0^2 - \hat{S}_0^2}\delta_S \right] + O(\Pi^2)$$

The S, T, U parameters

Let me start the introduction of S, T, U parameters by saying that, besides checking the self-consistency of the SM when confronted with precision experimental measurements of EW processes, this can also be used to probe / constrain the possible BSM effects on such observables via new contributions to the 1-loop self-energies.

One usually encounters $\Pi_{ab}(q^2) = \Pi_{ab}^{SM}(q^2) + \underbrace{\delta\Pi_{ab}(q^2)}_{\text{BSM contributions}}$

If the BSM physics is of very high-scale $M \gg v$, we can assume $q^2 \ll M^2$ (measurements are performed at q^2 much lower than M^2), and we can assume $\delta\Pi_{ab}(q^2) = A_{ab} + q^2 B_{ab} + \ldots$ (truncate at linear order)

- Then, there are 8 quantities describing the BSM physics:

$$A_{ab} = \delta\Pi_{ab}(0)$$

$$ab = \gamma\gamma, \gamma Z, ZZ, WW$$

$$B_{ab} = \delta\Pi'_{ab}(0)$$

- Two of those are identically 0 : $A_{\gamma\gamma} = 0$; $A_{\gamma Z} = 0$

(by gauge invariance)

- Three other linear combinations are absorbed in matching α, G_F, m_Z to their experimental values (renormalization of our 3 input parameters)

Then, there are 3 remaining (combinations of) parameters which completely describe the effect of new physics at a high scale on the EW precision observables / measurements [These are S, T, U] (for the $\delta\Pi$'s)

Going back to the SM, the same logic applies to $\Pi_{ab}^{SM}(q^2)$ (we can truncate @ first-order in q^2-expansion around $q^2=0$) for low-energy ($q^2 \ll m_W^2$) "EW precision measurements"

[e.g. μ-decay, low-energy e^+e^- scattering...]

and then the shift at 1-loop in any such observable will be entirely described by these S, T, U (for the Π^{SM}'s)

(we are also considering observables at $q^2 = m_W^2$ and $q^2 = m_Z^2$, but we will see that e.g. m_W^2 still only sees S, T, U)

- S, T, U (they must be combinations of self-energy differences to cancel-out the divergent parts of the self-energies; S, T, U are UV-finite!)

$$\frac{\alpha S}{4 s^2 c^2} = \left(\frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2} - \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} \right)$$

$$- \frac{(c^2 - s^2)}{sc} \left(\frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2} - \frac{\Pi_{\gamma Z}(0)}{m_Z^2} \right)$$

$$\alpha T = \frac{\Pi_{WW}(0)}{m_W^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2}$$

$$\frac{\alpha U}{4 s^2} = \frac{\Pi_{WW}(m_W^2)}{m_W^2} - \frac{\Pi_{WW}(0)}{m_W^2} - c^2 \left(\frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2} \right)$$

$$- s^2 \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} - 2sc \left(\frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2} - \frac{\Pi_{\gamma Z}(0)}{m_Z^2} \right)$$

We then have :

$$m_W^{2\,th} = \underbrace{\frac{\pi\, \hat{\alpha}(m_Z^2)}{\sqrt{2}\, \hat{G}_F\, \hat{S}_0^2}}_{m_W^{2\,th(0)}} \left[1 + \frac{\alpha}{4 S_0^2} U + \frac{C_0^2 \alpha}{C_0^2 - S_0^2} T - \frac{\alpha}{2(C_0^2 - S_0^2)} S \right]$$