Analogue Quantum Backreaction:

-0.1cm

1D Finite-size homogeneous Bose-Einstein Condensates

PhysRev. A 106, 053319 (2022)

Sang-Shin Baak, Caio Cesar Holanda Ribeiro†, Uwe R. Fischer

June 1, 2023

Department of Physics and Astronomy
Seoul National University
† University of Brasilia

Backreactions in Analogue Gravity

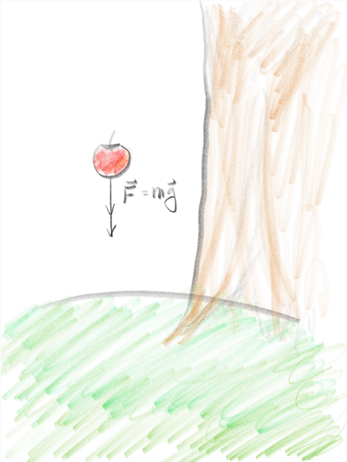
- S. Liberati, G. Tricella, and A. Trombettoni, Back-Reaction in Canonical Analogue Black Holes, Applied Sciences 10, 10.3390/app10248868 (2020).
- S. Patrick, H. Goodhew, C. Gooding, and S. Weinfurtner, Backreaction in an Analogue Black Hole Experiment, Phys. Rev. Lett. 126, 041105 (2021)
- R. Schützhold, M. Uhlmann, Y. Xu, and U. R. Fischer, Mean-field expansion in Bose-Einstein condensates with finite-range interactions, International Journal of Modern Physics B 20, 3555 (2006)
- R. Balbinot, S. Fagnocchi, and A. Fabbri, Quantum effects in acoustic black holes: The backreaction, PhysRev. D 71, 064019 (2005).

Outline

Introduction of Backreaction

Backreaction?

We are interested in **apple**!

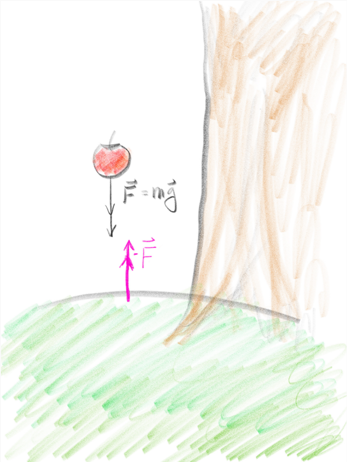


Backreaction?

The motion of the apple is determined by the **background** gravitational field (Earth). The earth may move in the solar system.

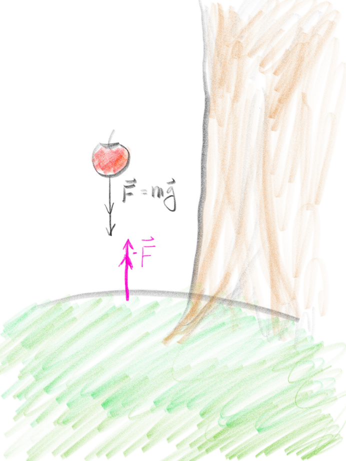
Backreaction?

In fact, the earth is attracted by the apple (Weakly).



Backreaction?

We are interested in **quantum field**!
The kinematics of the quantum field is determined by the **background** metric. The metric follows the classical equation of motion (e.g., Einstein equation or Euler equation).
In fact, there is a **backreaction**.



Backreaction

The Backreaction can be classified in three categories [a]

- Classical Backreaction: Coming from the averaging procedure.
- Semi-Classical Backreaction: Classical equations of motion with quantum source. Semi-classical Einstein equation

$$R_{ab} + \frac{1}{2} R g_{ab} = 8\pi G \langle T_{ab} \rangle$$

- Quantum Backreaction: Purely Quantum effect. Quantum equations of motion with Quantum source. Quantum gravity effect.

[a] S.Schander, T.Thiemann, Front. Astron. Space Sci 8:692198

Effective Action Approach

Background choice

- Using (density,phase) variable (ρ, θ)[a],

$$\hat{\theta} = \langle \hat{\theta} \rangle + \delta\hat{\theta} = \theta_b + \delta\hat{\theta},$$
$$\hat{\rho} = \langle \hat{\rho} \rangle + \delta\rho = \rho_b + \delta\rho.$$

where $\hat{\Psi} = e^{i\hat{\theta}}\hat{\rho}$.

- Action (neglecting quantum pressure)

$$\mathcal{L}^{(\rho,\theta)} = -\rho\left(\partial_t\theta + \frac{1}{2}(\partial_x\theta)^2\right) - \epsilon[\rho] - U\rho.$$

- Euler equation (with quantum backreaction)

$$\partial_t\theta_b + \frac{1}{2}(\nabla\theta_b)^2 + \frac{\mathrm{d}\epsilon}{\mathrm{d}\rho} + U - \frac{\delta A_{\mathrm{eff}}}{\delta\rho} = 0$$

[a] R.Schützhold, PoSQG-Ph. 43 036 (2007)

Effective action

- The effective action is[a]

$$A_{\text{eff}} = \frac{1}{2}\int \mathrm{d}x\sqrt{-g}g^{ab}\partial_a\theta\partial_b\theta.$$

- Backreaction

$$\frac{\delta A_{\text{eff}}}{\delta\rho} = \frac{\delta A_{\text{eff}}}{\delta g^{ab}}\frac{\delta g^{ab}}{\delta\rho} = \frac{\sqrt{-g}}{2}\langle\hat{T}_{ab}\rangle\frac{\delta g^{ab}}{\delta\rho} = -\frac{1}{2}\langle(\nabla\hat{\theta})^2\rangle.$$

[a] U.R.Fischer, in *Quantum Analogues from phase transitions to black holes and cosmology*, edited by W.G.U.ruh, R.Schutzhold (Springer, Berlin 2007)

Number Conserving Bogoliubov Expansion

$U(1)$-SYMMETRY BREAKING APPROACH

- Lagrangian

$$\mathcal{L}(\partial_\mu \Psi, \Psi) = \frac{i}{2}(\Psi^* \partial_t \Psi - \partial_t \Psi^* \Psi) - \frac{1}{2m}|\partial_x \Psi|^2 - U|\Psi|^2 - \frac{g}{2}|\Psi|^4.$$

- Field expansion with total particle number $N \gg 1$

$$\Psi = \phi_0 + \chi + \mathcal{O}(N^{-1/2}),$$

where $\phi_0 = \mathcal{O}(N^{1/2})$, $\chi = \mathcal{O}(N^0)$.

- Action

$$S = \int \mathrm{d}^2 x \mathcal{L} = S_0 + S_1 + S_2 + \mathcal{O}(N^{-1/2}).$$

Quantization

◇ Canonical quantization

$$[\chi(t,x),\chi^\dagger(t,x')] = \delta(x-x'), \qquad [\chi(t,x),\chi(t,x')] = 0.$$

◇ Vacuum choice

$$\langle\chi\rangle = 0.$$

◇ Density and Current

$$\rho := \langle\Psi^\dagger\Psi\rangle = \rho_0 + \rho_\chi + \mathcal{O}(N^{-1/2}),$$
$$J := \frac{1}{m}\Im[\langle\Psi^\dagger\partial_x\Psi\rangle] = J_0 + J_\chi + \mathcal{O}(N^{-1/2}).$$

Particle Number Conservation Fails

- Continuity equation (leading order)

$$\partial_t \rho_0 + \partial_x J_0 = 0.$$

- Continuity equation (subleading order)

$$\begin{aligned}\partial_t \rho_\chi + \partial_x J_\chi &= -ig(\phi_0^2 \langle \hat{\chi}^{\dagger 2} \rangle - \phi_0^{*2} \langle \hat{\chi}^2 \rangle) \\ &\neq 0.\end{aligned} \tag{1}$$

Number Conserving Bogoliubov Expansion (Castin)

- One can expand [a]

$$\hat{\Psi} = (\phi_c + \hat{\chi} + \hat{\zeta}) \frac{\hat{A}}{\sqrt{\hat{N}}}$$

- Amended Gross-Pitaevskii equation

$$i\partial_t \phi_{\mathrm{c}} = \left(-\frac{\partial_x^2}{2m} + U + g\rho_{\mathrm{c}} + 2g\langle \hat{\chi}^\dagger \hat{\chi} \rangle \right) \phi_{\mathrm{c}} + g\langle \hat{\chi}^2 \rangle \phi_{\mathrm{c}}^*.$$

- Bogoliubov de Gennes equation

$$i\partial_t \hat{\chi} = \left(-\frac{\partial_x^2}{2m} + U + 2g\rho_c \right) \hat{\chi} + g\phi_c^2 \hat{\chi}^\dagger,$$

[a] Y.Castin and R. Dum, Phys. Rev. A 57, 3008 (1998)

Number Conserving Bogoliubov Expansion

- Lagrangian

$$\mathcal{L}(\partial_\mu\Psi,\Psi) = \frac{i}{2}(\Psi^*\partial_t\Psi - \partial_t\Psi^*\Psi) - \frac{1}{2m}|\partial_x\Psi|^2 - U|\Psi|^2 - \frac{g}{2}|\Psi|^4.$$

- Field expansion with total particle number $N \gg 1$

$$\Psi = \phi_0 + \chi + \zeta + \mathcal{O}(N^{-1}),$$

where $\phi_0 = \mathcal{O}(N^{1/2})$, $\chi = \mathcal{O}(N^0)$, $\zeta = \mathcal{O}(N^{-1/2})$, and $g = \mathcal{O}(N^{-1})$.

- Action

$$S = \int \mathrm{d}^2x\,\mathcal{L} = S_0 + S_1 + S_2 + S_3 + \mathcal{O}(1/N).$$

Number Conserving Bogoliubov Expansion

- Gross-Pitaevskii equation (assume)

$$i\partial_t\phi_0 = \left(-\frac{\partial_x^2}{2m} + U + g\rho_0\right)\phi_0.$$

- Bogoliubov de Gennes (BdG) equation

$$i\partial_t\chi = \left(-\frac{\partial_x^2}{2m} + U + 2g\rho_0\right)\chi + g\phi_0^2\chi^*,$$

- correction on GP equation

$$i\partial_t\zeta = \left(-\frac{\partial_x^2}{2m} + U + 2g\rho_0\right)\zeta + g\phi_0^2\zeta^* + 2g|\chi|^2\phi_0 + g\chi^2\phi_0^*.$$

QUANTIZATION

◇ Canonical quantization

$$[\chi(t,x),\chi^\dagger(t,x')] = \delta(x-x'), \qquad [\chi(t,x),\chi(t,x')] = 0.$$

◇ Vacuum choice

$$\langle\chi\rangle = 0.$$

◇ Density and Current

$$\rho := \langle\Psi^\dagger\Psi\rangle = \rho_0 + (\rho_\chi + \rho_\zeta) + \mathcal{O}(N^{-1/2}),$$
$$J := \frac{1}{m}\Im[\langle\Psi^\dagger\partial_x\Psi\rangle] = J_0 + (J_\chi + J_\zeta) + \mathcal{O}(N^{-1/2}).$$

Density and Current

- Density

$$\rho_0 = |\phi_0|^2 = \mathcal{O}(N)$$
$$\rho_\chi = \langle \chi^\dagger \chi \rangle = \mathcal{O}(N^0)$$
$$\rho_\zeta = 2\Re[\phi_0^* \zeta] = \mathcal{O}(N^0).$$

- Current

$$J_0 = \frac{1}{m}\Im[\phi_0^* \partial_x \phi_0] = \mathcal{O}(N)$$
$$J_\chi = \frac{1}{m}\Im[\langle \chi^\dagger \partial_x \chi \rangle] = \mathcal{O}(N^0)$$
$$J_\zeta = \frac{1}{m}\Im[\phi_0^* \partial_x \zeta + \zeta^* \partial_x \phi_0] = \mathcal{O}(N^0),$$

Number Conserving Bogoliubov Expansion

- Continuity equation (leading order)

$$\partial_t \rho_0 + \partial_x J_0 = 0.$$

- Continuity equation (subleading order)

$$\partial_t \rho_\chi + \partial_x J_\chi = -ig(\phi_0^2 \langle \hat{\chi}^{\dagger 2} \rangle - \phi_0^{*2} \langle \hat{\chi}^2 \rangle),$$
$$\partial_t \rho_\zeta + \partial_x J_\zeta = ig(\phi_0^2 \langle \hat{\chi}^{\dagger 2} \rangle - \phi_0^{*2} \langle \hat{\chi}^2 \rangle).$$

- Continuity equation in working order.

$$\partial_t \rho + \partial_x J = \mathcal{O}(N^{-1/2}) \simeq 0.$$

Backreaction in BEC

Depletion

Depletion $\rho_\chi = \langle \chi^\dagger \chi \rangle$ is the number of noncondensed particles in a unit length. In our system, the only **quantized field** is the Bogoliubov field χ. Hence, it is related to the created particles. The smallness of the number of depleted particles compared to N is the **condition for the Bogoliubov approximation valid.**

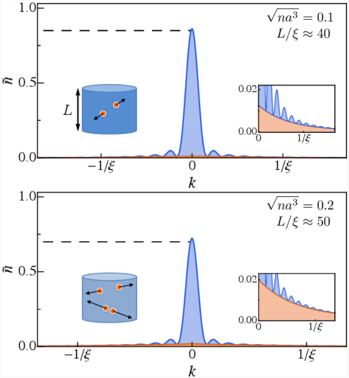


Figure 1: Depletion measurement[a]

[a] R.Lopes1, C.Eigen, N.Navon, D.Clément, R.P.Smith, Z.Hadzibabic, Phys. Rev. Lett. 119, 190404 (2017)

Quantum Backreaction

The **quantum backreaction** force f_q we define is a **deviation from the classical** Euler equation caused by the quantum fluctuation.

- ◇ Classical Euler equation

$$\partial_t J = f_\mathrm{cl} := -\partial_x(\rho v^2) - \frac{\rho}{m}\left(\frac{-\partial_x^2\sqrt{\rho}}{2m\sqrt{\rho}} + U + g\rho\right)$$

- ◇ Quantum Effect

$$\partial_t J = f_\mathrm{cl}(\rho, J) + f_\mathrm{q}.$$

Measurable quantities are not fundamental variables.[a]

[a] R.Schützhold, M.Uhlmann, Y.Xu, U.R.Fischer, Quantum backreaction in dilute Bose-Einstein condensates. Phys. Rev. D, 72, 105005 (2005)

Quantum Backreaction force

- Quantum Backreaction force

$$f_{\rm q} = \partial_t J_\chi - v_0 \partial_t \rho_\chi + \partial_x (J_\chi v_0 - \rho_\chi v_0^2) - J_\chi \partial_x v_0$$
$$- \frac{\rho_0}{2m} \partial_x \left(\frac{g G^{(2)}}{\rho_0} \right) + \frac{\rho_\chi}{m} \partial_x \left[- \frac{\partial_x^2 \sqrt{\rho_0}}{2m\sqrt{\rho_0}} + U + g\rho_0 \right]$$
$$- \frac{\rho_0}{4} \partial_x \left[\frac{1}{2} \frac{1}{\sqrt{\rho_0}} \partial_x^2 \left(\frac{\rho_\chi}{\sqrt{\rho_0}} \right) - \frac{\rho_\chi}{\rho_0^{3/2}} \partial_x^2 \sqrt{\rho_0} \right].$$

- For condensate at rest

$$f_{\rm cl} = -\partial_x \left[\left(1 - \frac{\partial_x^2}{4} \right) (\rho_\chi + \rho_\zeta) \right].$$

Systems & Results

System : Homogeneous finite 1D gas

- Idealized potential

$$U = \mu - g\rho_0 + \frac{1}{2m}\partial_x[\delta(x-\ell/2)+\delta(x+\ell/2)],$$

allows the stationary (leading order) condensate field

$$\phi_0 = e^{i\mu t}\sqrt{\rho_0}.$$

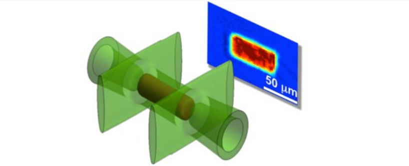


Figure 2: Experimental realization of Homogeneous gas[a]

[a] T.F.Schmidutz et al., PhysRevLett.112.040403 (2014)

Mode solution

- Let $\chi = e^{i\mu t}\psi$, then BdGE becomes

$$i\partial_t\psi = -\frac{1}{2}\partial_x^2\psi + \frac{g}{g_0}(\psi + \psi^*).$$

- Let $\Phi = (\psi, \psi^*)^{\mathrm{T}}$ and quantize the field.

$$\Phi(t,x) = \sum_{n=0}^{\infty}[a_n\Phi_x(t,x) + a_n^\dagger\sigma_1\Phi_n^*(t,x)].$$

where $\{\Phi_n\}_{n=0}^{\infty}$ is the set of positive field modes.

Mode Solution

To find a well-defined initial condition, we will start with the noninteracting particles and suddenly turn on the coupling $g = g_0\Theta(t)$. Using Neumann boundary condition at $x = \pm\ell/2$, one gets

- Noninteracting regime ($g = 0$),

$$\Phi_n(t,x) = \frac{e^{-i\Omega_n t}[e^{ik_n x} + (-1)^n e^{-ik_n x}]}{\sqrt{2\ell(1+\delta_{0,n})}}(1,0)^{\mathrm{T}}$$

where $\omega = \pm k^2/2$ with $\omega := \Omega_n = n^2\pi^2/2\ell^2$.

- Interacting regime ($g = g_0$)

$$\Pi_n = \frac{e^{i\omega_n t}[e^{ik_n x} + (-1)^n e^{-ik_n x}]}{\sqrt{2\ell[1-(\omega_n - k_n^2/2-1)^2]}}(1, \omega_n - k)_n^2 - 1)^{\mathrm{T}}.$$

where $\omega_n = \sqrt{k_n^2(k_n^2/4+1)}$.

Mode Solution

But, in the interacting regime, we have zero-norm mode.

$$\Pi_0 := (1, -1)^{\mathrm{T}}$$
$$\tilde{\Pi}_0 = \frac{1}{2}(1, 1)^{\mathrm{T}} - it\Pi_0.$$

Thus,

$$\Phi_n = \alpha_{n,0}\Pi_0 + \beta_{n,0}\tilde{\Pi}_0 + \sum_{j=1}^{\infty}[\alpha_{n,j}\Pi_j - \beta_{n,j}\sigma_1\Pi_j^*].$$

Using this, one can calculate the $\langle\rho\rangle$ and $\langle j\rangle$.

Condensate Correction

Set the ansatz $\zeta(t,x) = \exp(-i\mu t)f(t,x)/\sqrt{\rho_0}$, then the condensate correction equation is

$$i\partial_t f = -\frac{1}{2}\partial_x^2 f + (f + f^*) + 2\rho_\chi + \langle\hat{\psi}^2\rangle,$$

with $f = 0$ at $t = 0$, and $\partial_x f = 0$ at $x = \pm\ell/2$ for all times. Using spinor expression $F = (f, f^*)^{\mathrm{T}}$, the equation we need to solve is

$$\left(i\partial_t\sigma_3 + \frac{1}{2}\partial_x^2 F - \sigma_4\right)F = \begin{pmatrix} 2\rho_\chi + \langle\hat{\psi}^2\rangle \\ 2\rho_\chi + \langle\hat{\psi}^{\dagger 2}\rangle \end{pmatrix}.$$

Depletion

◇ Depletion

$$\rho_\chi = \frac{t^2}{\ell} + \frac{1}{2\ell}\sum_{n=1}^{\infty}\frac{(-1)^n}{\omega_n^2}\left[(-1)^n + \cos(2k_n x)\right]\left[1-\cos(2\omega_n t)\right]$$

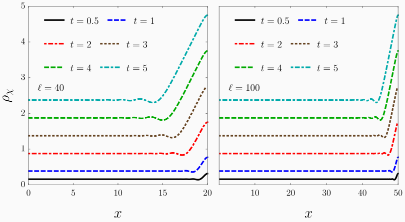


Figure 3: Depletion at the right half

Number of Depleted particles

◇ The number of total depleted particles

$$\delta N = \int_{-\ell/2}^{\ell/2} \mathrm{d}x \rho_\chi = t^2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\omega_n^2}[1-\cos(2\omega_n t)]$$

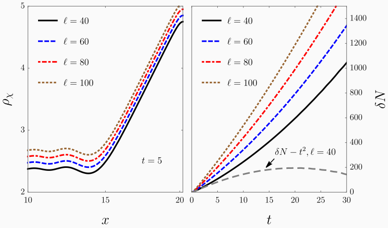


Figure 4: Depleted particles

Condensate Correction

◇ Density Correction

$$\rho_\zeta = -\frac{t^2}{\ell} - \frac{1}{4\ell}\sum_{n=1}^{\infty}\frac{(-1)^n}{\omega_n^2}\Bigg\{2(-1)^n[1-\cos(2\omega_n t)] + \cos(2k_n x)\left[\frac{2-k_n^2}{k_n^2+1} + 2\cos(2\omega_n t) - \frac{k_n^2+4}{k_n^2+1}\cos(\omega_{2n} t)\right]\Bigg\}.$$

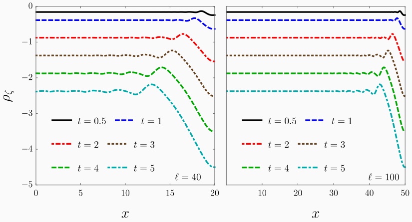
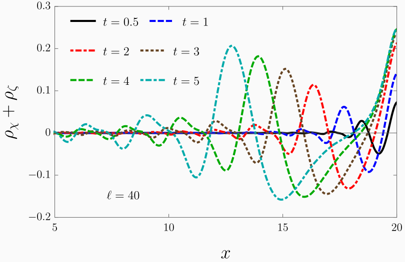


Figure 5: Condensate density Correction

Bragg Remainder

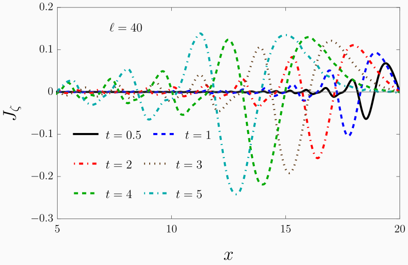
$$\rho_\chi + \rho_\zeta = -\frac{1}{4\ell}\sum_{n=1}^{\infty}\frac{(-1)^n}{\omega_n^2}\Bigg\{\cos(2k_n x) \times \left[\frac{2-k_n^2}{k_n^2+1} + 4\cos(2\omega_n t) - 2 - \frac{k_n^2+4}{k_n^2+1}\cos(\omega_{2n} t)\right]\Bigg\}.$$



Current

For our model, in which $J_0 = J_\chi = 0$,

$$J = J_\zeta = \frac{-2}{\ell} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(2k_n x)}{k_n} \left[\frac{\sin(2\omega_n t)}{2\omega_n} - \frac{\sin(\omega_{2n} t)}{\omega_{2n}} \right].$$



Potential in 1D finite homogeneous gas

- For uniform condensate at rest,

$$J = J_\zeta .$$

- The time evolution of J

$$\partial_t J_\zeta = f_{\mathrm{cl}} + f_{\mathrm{q}} = -\partial_x V$$

- The classical force

$$f_{\mathrm{cl}} = -\partial_x V_{\mathrm{cl}}(\rho)$$

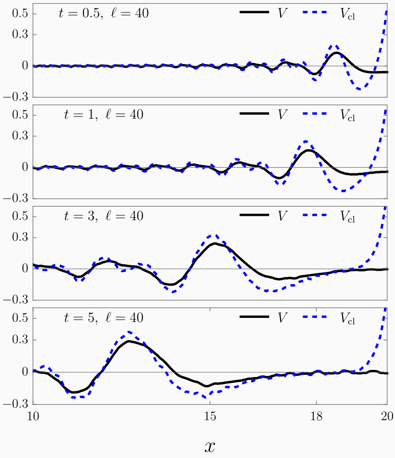


Figure 6: Potential, Quantum Potential

Summary

- We can find backreaction directly from the quantum dynamics.
- In analogue backreaction, one must carefully determine the background.
- One needs a clear initial vacuum choice for unstable theory.
- In general, condensate correction must be considered when one tries to measure the depletion.
- The quantum backreaction can be measured in the BEC at rest.
- The depletion can be used for testing the validity regime of Bogoliubov approximation holds, and it can especially be used for the stationarity of the system.

Thank you.

Group Homepage

https://physics.snu.ac.kr/fischer/

Analogue Gravity

- First order Lagrangian $L(\phi, \partial_\mu \phi)$.
- Field expansion around classical field ϕ_0. $(0 < \epsilon \ll 1)$

$$\phi = \phi_0 + \epsilon\phi_1 + \frac{\epsilon^2}{2}\phi_2 + \cdots$$

- Action is of the form

$$\begin{aligned} S = & \, S[\phi_0] \\ & + \frac{\epsilon^2}{2}\int \mathrm{d}^n x \left[\left\{ \frac{\partial^2 L}{\partial(\partial_\mu \phi)\partial(\partial_\nu \phi)} \right\} \partial_\mu \phi_1 \partial_\nu \phi_1 \right. \\ & \qquad \left. + \left(\frac{\partial^2 L}{\partial\phi\partial\phi} - \partial_\mu \left\{ \frac{\partial^2 L}{\partial(\partial_\mu \phi)\partial\phi} \right\} \right) \phi_1 \phi_1 \right] \\ & + \cdots . \end{aligned}$$

Analogue Gravity

- Euler-Lagrange equation

$$\partial_\mu \left(\frac{\partial^2 L}{\partial(\partial_\mu \phi)\partial(\partial_\nu \phi)} \partial_\nu \phi_1 \right) - \left(\frac{\partial^2 L}{\partial \phi \partial \phi} - \partial_\mu \left\{ \frac{\partial^2 L}{\partial(\partial_\mu \phi)\partial \phi} \right\} \right) \phi_1 = 0$$

- In the form

$$\partial_\mu(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi_1) - V(\phi_1).$$

where $g = \det g_{\mu\nu}$, and $g_{\mu\nu}$ is the analogue metric.

Action in order

$$S_0 = \int \mathrm{d}^2 x \frac{i}{2}(\phi_0^* \partial_t \phi_0 - \partial_t \phi_0^* \phi_0) - \frac{1}{2m}|\partial_x \phi_0|^2 - U|\phi_0|^2 - g|\phi_0|^4.$$

$$S_2 = \int \mathrm{d}^2 x \frac{i}{2}(\chi^* \partial_t \chi - \partial_t \chi^* \chi) - \frac{1}{2m}|\partial_x \chi|^2 - U|\chi|^2$$
$$-2g\rho_0|\chi|^2 - \frac{g}{2}(\phi_0^{*2}\chi^2 + \phi_0^2 \chi^{*2}),$$

$$S_3 = \int \mathrm{d}^2 x \frac{i}{2}(\zeta^* \partial_t \chi + \chi^* \partial_t \zeta - \partial_t \zeta^* \chi - \partial_t \chi^* \zeta)$$
$$-\frac{1}{2m}(\partial_x \chi^* \partial_x \zeta + \partial_x \zeta^* \partial_x \chi) - U(\chi^* \zeta + \zeta^* \chi) - 2g\rho_0(\chi^* \zeta + \zeta^* \chi)$$
$$-g(\phi_0^{*2}\chi\zeta + \phi_0^2 \chi^* \zeta^*) - g|\chi|^2(\chi^* \phi_0 + \phi_0^* \chi),$$