Asymptotic quantum correlations of field modes in time-dependent backgrounds

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Motivation : Thermodynamic properties from quantum foundations

- Can thermodynamic properties emerge in isolated quantum systems via entanglement?

- Entanglement structure of quantum field $\Longleftrightarrow$ thermodynamic structure of horizons:

$$E = \frac{c_e}{a^2} E_{Komar}; \quad S = \frac{c_s}{\pi a^2} S_{BH}$$

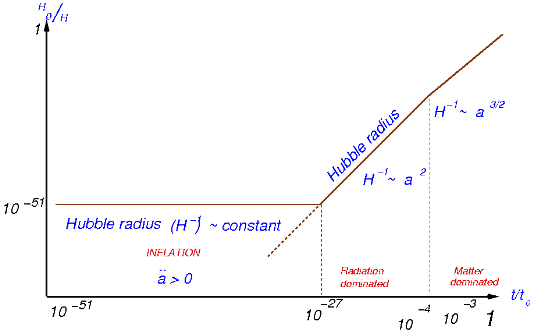
. [SMC & SS '20]

- The above relation preserves the Smarr Formula, which acts as an equation of state for BH thermodynamics:

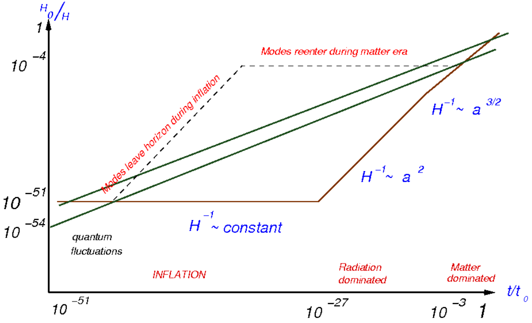
$$E = 2 T_H S \quad \Leftrightarrow \quad E_{Komar} = 2 T_H S_{BH} \Rightarrow M = 2 T_H S_{\text{I}}$$

Motivation : Thermodynamic properties from quantum foundations

- Dynamical treatment is required to model BH evaporation, early-Universe expansion, etc.
- Eg: Inflation: $a(t) \propto e^{Ht}$ (modes are exponentially stretched); Hubble radius $H^{-1} = (\dot{a}/a)^{-1}$.
- Can entanglement help us understand quantum-classical transition of fluctuations?



(a)



(b)

Model : Scalar field with time-dependent mass

- The Hamiltonian of a time-dependent massive scalar field (φ) in $(1+1)-$dimensions is:

$$\tilde{H} = \frac{1}{2}\int dx\,\left[\pi^2 + (\nabla\varphi)^2 + m_f^2(t)\,\varphi^2\right] \tag{1}$$

- Upon lattice-regularization using $\varphi = ja$ (UV cutoff) and $L = (N+1)a$ (IR cutoff):

$$H = \frac{1}{2}\sum_j \left[\pi_j^2 + \Lambda(t)\varphi_j^2 + (\varphi_j - \varphi_{j+1})^2\right] = \frac{1}{2}\left[\sum_{j=1}^{N}\pi_j^2 + \sum_{i,j=1}^{N} K_{ij}(t)\varphi_i\varphi_j\right] \quad ; \quad \Lambda(t) = a^2 m_f^2(t) \tag{2}$$

- Fluctuations propagating in an FLRW background reduce to the above form.

Model : Scalar field with time-dependent mass

- The normal modes of the system are as follows:

$$\omega^2_{k=1,..N}(t) = \begin{cases} \Lambda(t) + 4\sin^2 \frac{k\pi}{2(N+1)} & \text{Dirichlet } \varphi(0) = \varphi(L) = 0 \\ \Lambda(t) + 4\sin^2 \frac{(k-1)\pi}{2N} & \text{Neumann } \partial_x\varphi(0) = \partial_x\varphi(L) = 0 \end{cases} \tag{3}$$

- For each normal mode oscillator $\{y_j\}$, we obtain a form-invariant Gaussian state:

$$\Psi_{\mathrm{GS}}(\{y_j\}, t) = \prod_j \left(\frac{\omega_j(0)}{\pi b_j^2(t)} \right)^{1/4} \exp\left\{ - \left(\frac{\omega_j(0)}{b_j^2(t)} - i\frac{\dot{b}_j(t)}{b_j(t)} \right) \frac{y_j^2}{2} - \frac{i}{2}\omega_j(0) \int b_j^{-2}(t) dt \right\}, \tag{4}$$

- The wave-function can also be written as:

$$\Psi_{\mathrm{GS}}(\{y_j\}, t) = \prod_j \psi^{(j)}(y_j, t) \neq \prod_j \phi^{(j)}(x_j, t) \Rightarrow \text{entangled in physical coordinates} \tag{5}$$

- The scaling parameters b_j for each mode are solutions of the Ermakov-Pinney equation:

$$\ddot{b}_j(t) + \omega_j^2(t) b_j(t) = \frac{\omega_j^2(0)}{b_j^3(t)} \quad ; \quad j = +, - \tag{6}$$

- Three distinct stability regimes:

$$b_j(t) \propto \begin{cases} \text{oscillatory} & \omega_j^2(t) > 0 \text{ (stable)} \\ \omega_j(0)t & \omega_j^2(t) = 0 \text{ (metastable/zero-mode)} \\ \exp\{|\nu_j|t\} & \omega_j^2(t) < 0 \text{ (unstable/inverted mode)} \quad ; \quad \nu_j = i\omega_j \end{cases} \tag{7}$$

- During inflation, modes that remain within the Hubble radius are stable/oscillatory and those that cross the radius get squeezed/inverted.

Quantum correlation measures

- **Entanglement entropy** : Measures how "entangled" subsystems A and B are:

$$S = -Tr\,\rho_{red} \ln \rho_{red} \quad ; \quad \rho_{red} = \mathrm{Tr}_A \left|\Psi\right\rangle\left\langle\Psi\right| \; or \; \mathrm{Tr}_B \left|\Psi\right\rangle\left\langle\Psi\right|. \tag{8}$$

- **Loschmidt echo** : Deviation of the state from slightly different $H(t)$ evolutions:

$$\mathscr{M}(t) = \left|\langle\Psi_0|e^{i\int H' dt} e^{-i\int H dt}|\Psi_0\rangle\right| = |\langle\Psi_0|\Psi_2\rangle| \tag{9}$$

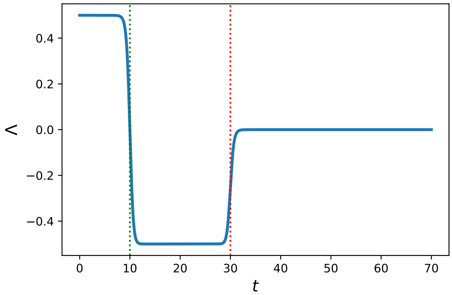
- Instabilities are quantified in terms of quantum Lyapunov exponents:

$$\lambda_L^{(k)} = v_k \tag{10}$$

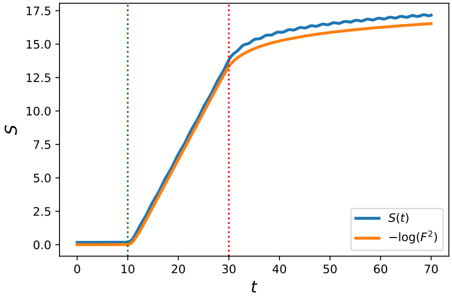
- **General behavior of EE** for sub-system size n:

$$S_n \sim \underbrace{\left(\sum_{j=1}^{2n} \lambda_L^{(j)}\right) t}_{\text{inverted mode}} + \underbrace{\log(t)}_{\text{zero-mode}} + \underbrace{S_0(t)}_{\text{stable mode}} \tag{11}$$

Entanglement Dynamics [Numerical]



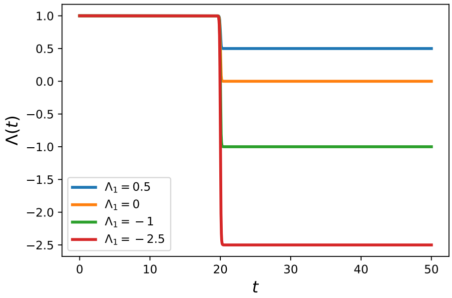
(a)



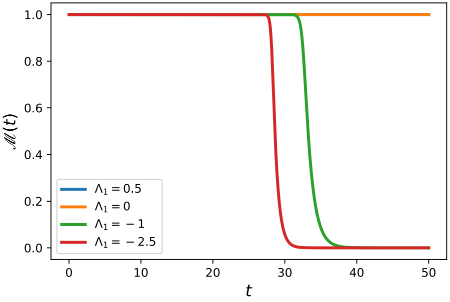
(b)

Figure: (a) Evolution of rescaled frequency $\Lambda(t)$, (b) Dynamics of entanglement entropy $S(t)$ and logarithmic fidelity due to the quench.

Loschmidt Echo [Numerical]



(a)



(b)

Figure: (a) Quench functions $\Lambda(t)$ and (b) Loschmidt Echo $\mathscr{M}(t)$ for the corresponding quenches. Here, $\delta\Lambda = 10^{-10}$.

Results : Classicality, Asymptotic convergence

- **Classicality of EE** : When subsystem size $n \geq \frac{m}{2}$ where $m = dim\{\lambda_L^{(j)}\}$, EE growth saturates: [*Hackl et. al. '18*]

$$S_n \sim h_{KS} t \quad ; \quad h_{KS} = \sum_{j=1}^{m} \lambda_L^{(j)} \tag{12}$$

where h_{KS} is the Kolmogorov-Sinai rate, associated with a classically chaotic system.

- Other measures : Ground state fidelity $\mathscr{F}_0$, Circuit complexity C_{CM}.

- **Asymptotic convergence** : Subsystem measure EE converges with full-system measures:

$$S^{\text{inv}}_{n \geq \frac{m}{2}} \sim -\log \mathscr{F}_0^2 \sim -\log \mathscr{M} \sim C_{\text{CM}} \sim h_{\text{KS}} t \tag{13}$$

$$S_n^{\text{zero}} \sim -\log \mathscr{F}_0^2 \sim -\log \mathscr{M}^2 \sim C_{CM} \sim \log t \tag{14}$$

Thermality of EE : Subsystem scaling & Particle creation

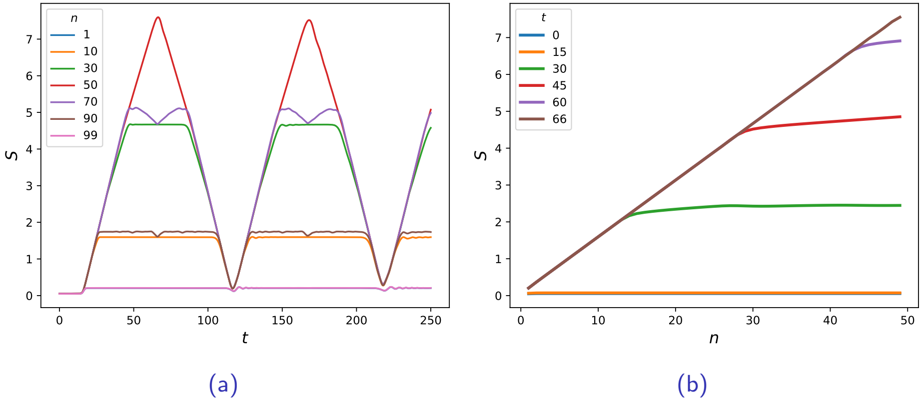
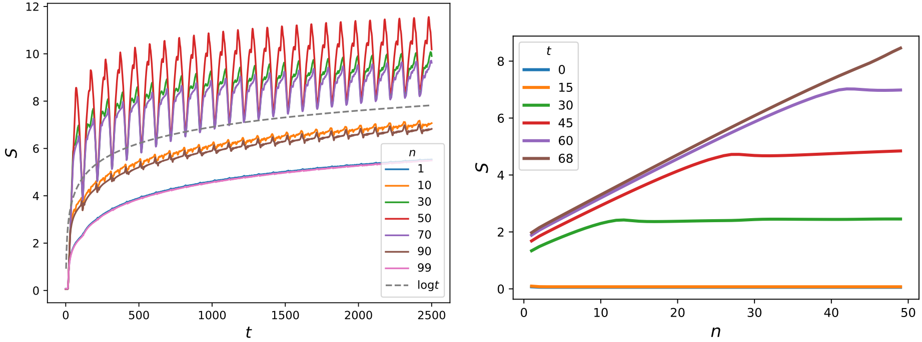


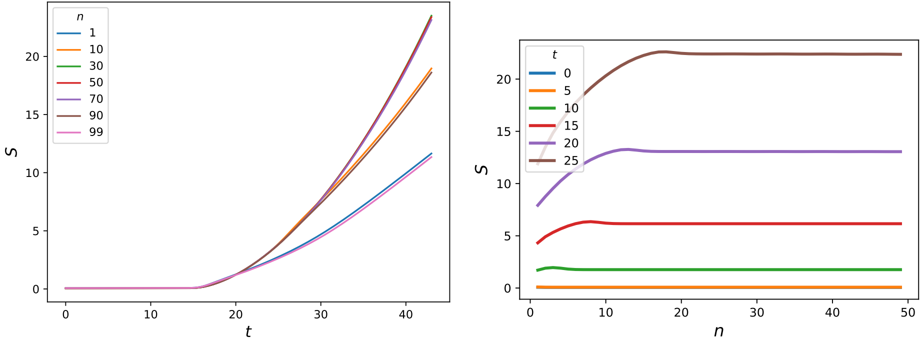
Figure: **Thermal signatures of EE** for stable modes — (a) Dynamics for various subsystem sizes, and (b) Area-law to volume-law transition.

Subsystem Scaling : Zero modes



(a) (b)

Figure: Entanglement dynamics after a mass quench ($\Lambda(t)$) resulting in late-time zero mode— (a) Time-evolution for various subsystem sizes, and (b) Area-law to volume-law transition of entanglement entropy. Here, $N = 100$.



(a) (b)

Figure: **Thermal signatures of EE** for inverted modes — (a) Dynamics for various subsystem sizes, and (b) Area-law to volume-law transition.

Conclusions

- EE (subsystem measure) and full-system correlation measures asymptotically converge in the presence of quantum instabilities. They serve as diagnostic tools for instability/chaos.

- Inverted modes cause EE to classicalize for sufficiently large subsystem size $n \geq m/2$.

- For stable/zero modes, EE-scaling oscillates between area-law and volume-law. For inverted modes, there is a progressive deviation from area-law, asymptotically approaching extensive behaviour.

Ongoing & Future Work

- Extension to scalar fluctuations in the **early-Universe expansion**.
- Quantum to classical transition measured in terms of relative width of Wigner function about classical trajectories. Work in progress.
- Generalization to complex fields and higher-spins.
- Black hole evaporation : Hawking radiation, Page curve.

Thank You!

Classical Chaos : Kolmogorov-Sinai Entropy Rate

- Starting from some initial ensemble of configurations in phase space, chaotic evolution leads to a fractal shaped deformation of the phase space volume of this ensemble, requiring more and more phase space cells to cover its shape, while Liouville's theorem ensures that its volume stays unchanged.
- Those directions in phase space for which the Lyapunov exponents are positive, will contribute to the growth of the number of needed cells and thus determine entropy growth (Pesin's theorem):

$$h_{\mathrm{KS}} = \sum_{i=1}^{N_l} \lambda_i = \frac{dS}{dt} \tag{15}$$

- Dynamics completely characterized by a single number h_{KS}.
- For systems that thermalize, entropy growth occurs only until system attains thermal equilibrium.