Free-fermion entanglement

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Workshop Entangle This V

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Entanglement

General Situation

- System in quantum state $|\Phi>$
- Divide into a subsystem A and a remainder B
- Entanglement information in reduced density matrix for subsystem

$$\rho = \mathrm{tr}_{\,\mathrm{B}}\,(|\Phi><\Phi|) = \frac{1}{Z}\exp(-\mathcal{H}), \quad \mathcal{H}: \text{Entanglement Hamiltonian}$$

Question: What are the special features of free-fermion systems ?

This talk: lattice systems, one dimension

- Nature of the problem
- Two examples
- Form of $\mathcal{H}$
- Relation to signal theory
- Commuting operators

Free fermions 1

General

- Eigenstates of Hamiltonian are Slater determinants
- Then higher correlation functions factorize
- This also must hold in subsystem A using ρ
- Therefore $\mathcal{H}$ must be quadratic in fermions.

  For N sites i, j in A

$$\mathcal{H} = \sum_{i,j=1}^{N} H_{i,j} c_i^\dagger c_j$$

- $H_{i,j}$ determined by the correlation matrix $C_{i,j} = \langle c_i^\dagger c_j \rangle$ in A

$$C = 1/(e^H + 1) \quad \leftrightarrow \quad H = \ln\left[\,(1 - C)/C\,\right]$$

  With the eigenfunctions $\phi_k(i)$ and eigenvalues ζ_k of C

$$H_{i,j} = \sum_k \phi_k(i)\,\varepsilon_k\,\phi_k(j)\,, \qquad \varepsilon_k = \ln\frac{1-\zeta_k}{\zeta_k}$$

Result: $(N \times N)$ matrix problem with C as input

Free fermions 2

Diagonal form of $\mathcal{H}$

$$\mathcal{H} = \sum_{k=1}^{N} \varepsilon_k c_k^\dagger c_k$$

Entanglement entropy S

$$S = -\mathrm{tr}(\rho \ln \rho) = -\sum_n w_n \ln w_n$$

Inserting $\mathcal{H}$ gives

$$S = \sum_k \ln\left(1 + e^{-\varepsilon_k}\right) + \sum_k \frac{\varepsilon_k}{e^{\varepsilon_k} + 1}$$

- Fully determined by single-particle spectrum
- Small ε_k most important
- In continuum approximation $\sum_k \to \int d\varepsilon\, n(\varepsilon)$

Summary: Simple general scheme, for any Slater determinant, for thermal states, and for systems with pair terms.

Simple hopping chain 1

Hamiltonian

$$\hat{\mathcal{H}} = -\frac{1}{2}\sum_n t\,(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) + \sum_n d\, c_n^\dagger c_n$$

Consider ring with M sites ($M \to \infty$). Diagonalize $\hat{\mathcal{H}}$ with

$$c_n = \frac{1}{\sqrt{M}}\sum_q e^{iqn} c_q$$

Result

$$\hat{\mathcal{H}} = \sum_q \omega_q c_q^\dagger c_q\,, \qquad \omega_q = -t\cos q + d$$

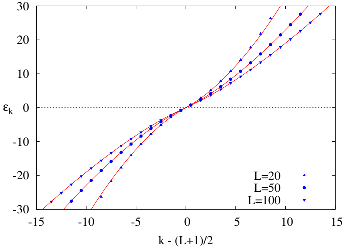
Choose state: Fermi sea, momenta between $\pm q_F$ occupied
Choose subsystem: Segment, L consecutive sites i, j

Correlation matrix

$$C_{i,j} = \langle c_i^\dagger c_j\rangle = \int_{-\pi}^{\pi}\frac{dq}{2\pi}\, e^{-iq(i-j)}\langle c_q^\dagger c_q\rangle = \frac{\sin q_F(i-j)}{\pi(i-j)}$$

Simple hopping chain 2

Eigenvalues ε_k for half filling



Discussion

- Slope near zero decreases slowly with L
- Entanglement entropy must increase
- Result for large L (numerical and analytical)

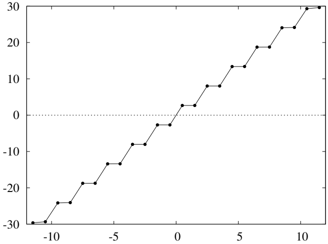
$$S = \frac{1}{3} \ln L + 0.726$$

- General behaviour in critical chains, see conformal result

Dimerized hopping chain

Alternating hopping $t(1 \pm \delta)$, half filling, segment $L = 100$

Eigenvalues ε_k for $\delta = 0.1$



Features

- Linear dispersion, equidistant levels
- Degeneracy reflects the two ends
- Slope depends on δ, but *not* on L
- Entanglement entropy a finite constant $S = S(\delta)$
- Typical situation in non-critical chains

Studies

Systems

- Non-critical/critical chains
- Chains with defects/inhomogeneous chains
- Higher dimensions

Geometries

- Open/closed chains
- Singly/multiply connected subsystems

States

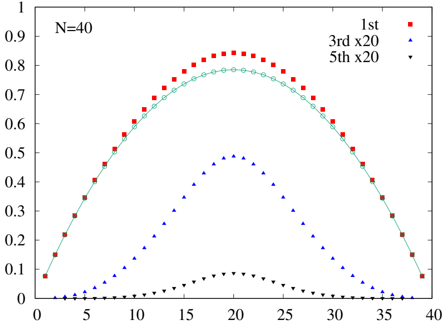
- Ground states
- Excited states

Time phenomena

- Time-evolution after a quench

Entanglement Hamiltonian

Simple hopping chain, matrix elements in $-H/N$ for half filling

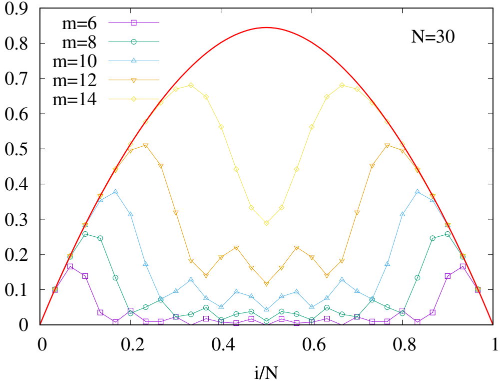


Features

- Hopping to NN in H *almost, but not exactly* parabolic
- Hopping to more *distant* neighbours exists, smaller, profiles different
- Continuum limit including *all* hoppings gives conformal result

Mechanism for hopping profile

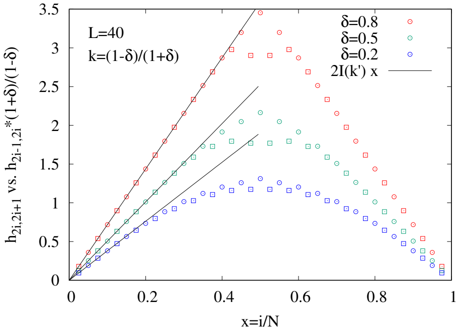
Nearest-neighbour hopping in $-H/N$ for half filling
Contribution of the lowest $2m$ states, positive and negative ε_k



Feature: Slopes given by smallest ε_k, maximum by largest one

Non-critical system

Chain with alternating hopping $t(1 \pm \delta)$. Matrix elements in $-H/N$

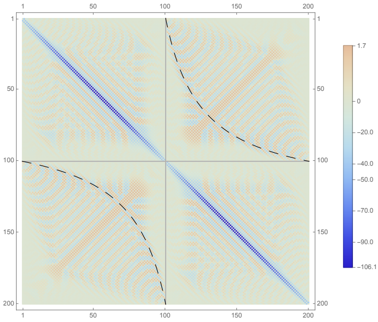


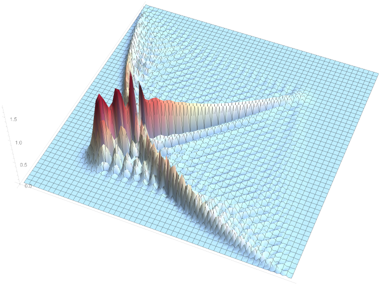
Features

- Hopping to NN increasingly *triangular* for larger δ
- Alternating structure as in physical Hamiltonian
- Slopes given by CTM result for half-infinite subsystem

Disjoint subsystem

Elements in H/N for two separated intervals (100/51/100 sites)





New feature

- Small hopping in H *between* intervals
- Enhanced for narrow region in other interval $\leftrightarrow$ ridge
- In continuum limit hopping to just *one* point $\leftrightarrow$ hyperbola

Relation to signal theory

Correspondence between free-fermion entanglement and time- and band-limited signals

Hopping model:

- Subsystem specified in real space
- Fermi sea specified in momentum space

Signals with limitations:

- Duration specified in time domain
- Bandwidth specified in frequency domain

Consequence

- Similar formulae
- Many early results in signal theory
- In particular at Bell Labs in sixties and seventies

Figure: David Slepian (1923-2007)

Signal formulae

Signal $f(t)$, total energy

$$E = \int_{-\infty}^{\infty} dt\; f(t)^2 = 1$$

Fourier representation

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\; f(\omega)\, \mathrm{e}^{-i\omega t}$$

Energy fraction in $(-T/2, T/2)$

$$\zeta = \int_{-T/2}^{T/2} dt\; f(t)^2$$

Limit to band $(-\Omega, \Omega)$

$$f(t) = \int_{-\Omega}^{\Omega} \frac{d\omega}{2\pi}\; f(\omega)\, \mathrm{e}^{-i\omega t}$$

Then, with $\nu = \omega/\Omega$

$$\zeta = \int_{-1}^{1} \frac{d\nu d\nu'}{2\pi} \left[\frac{\sin c(\nu - \nu')}{\pi(\nu - \nu')} \right] f(\nu)\, f^*(\nu')\,, \quad c = \Omega T/2$$

- Sine kernel as for fermion correlation function
- Largest eigenvalue of kernel gives maximal fraction ζ

A particular property

The following matrix commutes with C and H for the infinite hopping chain

$$T = \begin{pmatrix} d_1 & t_1 & & & \\ t_1 & d_2 & t_2 & & \\ & t_2 & d_3 & t_3 & \\ & & \ddots & \ddots & \\ & & & t_{N-1} & d_N \end{pmatrix}$$

where

$$t_i = \frac{i}{N}\left(1 - \frac{i}{N}\right), \qquad d_i = -2\cos q_F \, \frac{i-1/2}{N} \left(1 - \frac{i-1/2}{N}\right)$$

Features of T

- Same eigenfunctions as C and H, but much simpler structure
- Elements t_i, d_i vary *exactly* parabolically
- Spectrum very similar to that of H
- Can be used to give an explicit formula for H

Formula for H

For half filling and large N,

$$-H/N = \sum_{m=0}^{\infty} \gamma_m T^{2m+1}$$

with

$$\gamma_m = \frac{2^{2m}\,\Gamma(m+1/2)\,\Gamma(2m+1/2)}{\Gamma(m+1)\,\Gamma(2m+2)}, \quad \gamma_0 = \pi$$

Features

- H expressed as a power series in T
- Higher powers of T give hopping over larger distances
- Matrix elements to leading order

$$(T^{2m+1})_{i,i+2p+1} = \binom{2m+1}{m-p} (t_{i+p})^{2m+1},$$

Number of paths from i to $i+(2p+1)$ in $(2m+1)$ steps times hopping amplitudes in the middle between the two points

Origin of T

Can one understand the existence of the quantity T ?

• Full correlation matrix : projector on Fermi sea F

$$\hat{C}_{m,n} = \sum_{q_k \in F} \hat{\phi}_k(m)\, \hat{\phi}_k(n) \quad \rightarrow \quad \hat{C} = P_F$$

• Restricted correlation matrix : project in addition on subsystem A

$$C = P_A \hat{C} P_A = P_A P_F P_A$$

Result: C is the product of projectors

Consequence

- A quantity commutes with C if it commutes with P_A and P_F
- Such a quantity exists if the eigenfunctions $\hat{\phi}_k$ of the chain have a certain property ("bispectrality", Crampé et al. 2019)
- The *explicit* form of C is not needed here

Bispectrality

- Consider an open hopping chain, M sites $n = 1, 2, \dots M$
- Eigenfunctions are: $\hat{\phi}_k(n) = C \sin(q_k n)$, $q_k = \pi k/(M+1)$
- These satisfy in real space

$$\hat{H}\,\hat{\phi}_k(n) = -\hat{\phi}_k(n-1) - \hat{\phi}_k(n+1) = \omega_k\,\hat{\phi}_k(n)$$

- But k and n enter symmetrically. Therefore, with $n \leftrightarrow k$,

$$\hat{K}\,\hat{\phi}_k(n) = -\hat{\phi}_{k-1}(n) - \hat{\phi}_{k+1}(n) = \omega_n\,\hat{\phi}_k(n)$$

- Now construct an operator from $\hat{H}$ and $\hat{K}$ (Heun operator)

$$\hat{T} = \hat{K}\hat{H} + \hat{H}\hat{K} + \mu\hat{K} + \nu\hat{H}$$

- $\hat{T}$ tridiagonal in real space and in momentum space
- $\hat{T}$ decomposes into preselected blocks in *both* spaces
  for proper choice of μ and $\nu \rightarrow$ subsystem A, Fermi sea F

Consequence: Commutation with P_A and P_F

Summary

General

- Free-fermion systems important since solvable in principle
- Entanglement set-up has particular features

Aspects

- Only single-particle correlations needed
- Entanglement Hamiltonian has free-fermion form
- Entanglement entropy from single-particle spectrum
- Set-up has a parallel in signal theory
- A commuting quantity exists for simple hopping models
- Explanation in terms of bispectrality

Supplement

Sources of figures

- p.6: J.Stat.Mech.(2013) P04028
- p.9: J.Phys.A: Math.Theor.50(2017) 284003
- p.11: J.Stat.Mech.(2020) 103102
- p.12: J.Stat.Mech.(2022) 083101
- p.7,10: V Eisler

Additional references

- p.13: Slepian D, Bell Syst.Tech.J. 57, 1371 (1978)
- p.14: Slepian D, SIAM Review 25, 379 (1983)
- p.15: see Ref. p.6
- p.16: see Ref. p.9
- p.17,18: Crampé N et al., J.Stat.Mech.(2019) 093101