Free fermions from graphs

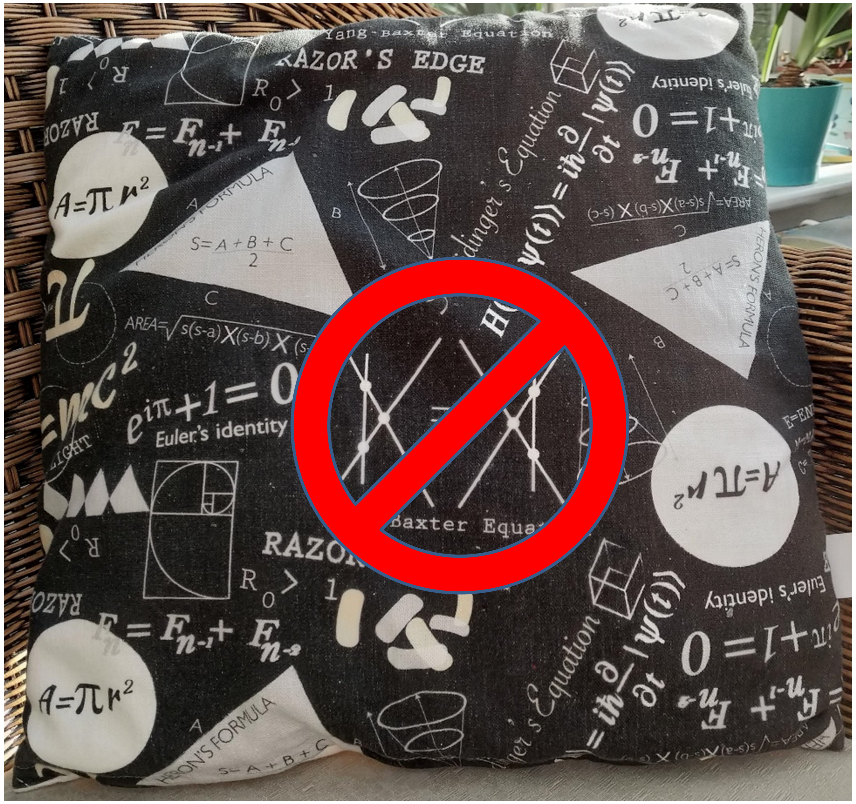
Paul Fendley

All Souls College, University of Oxford

(Quantum) many-body physics

- Always looking for many-body systems where we can do exact computations
- $\text{exact} \neq \text{rigorous}$
- Integrable systems make possible exact (usually non-rigorous) computations
- They have an extensive number of conversation laws
- There are intermediate (e.g. supersymmetry) cases not addressed in complexity theory. Not integrable, but not generic.
- Today I'll discuss systems where constraints are stronger than integrability, giving results both exact and rigorous

I’ll describe how to construct free-fermion raising and lowering operators in some special interacting models using elementary algebra.



Outline

1. What is a free fermion?
2. Solving the Ising chain using free fermions
3. Algebras and graphs for fermion bilinears
4. Free fermions in disguise
5. Claw-free graphs
6. A bit of physics

1. What is a free fermion?

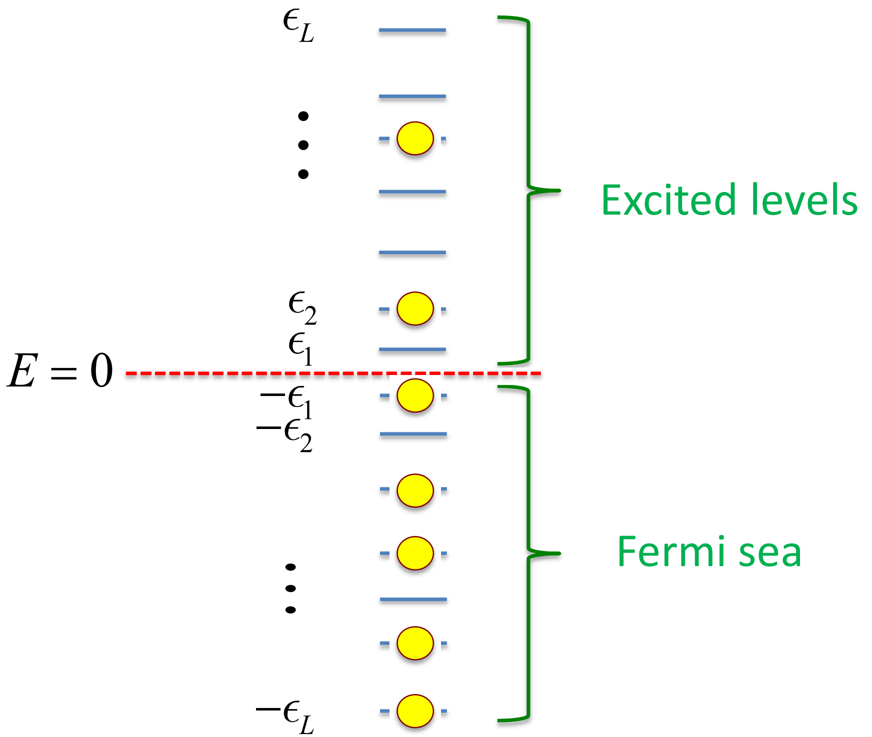
Forget statistics, forget operators, forget fields…

The basic property of free fermions is that their (energy) spectrum is of the form

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \ \ldots \ \pm \epsilon_L$$

The choices of a given $\pm$ are independent, and do not affect the values of ϵ_l .

Levels are either filled or empty.



The usual story

Automatically find such a spectrum when Hamiltonian is a bilinear in fermions.
On the lattice:

$$H = \sum_{a,b} \mathcal{H}_{ab}\psi_a\psi_b$$

where $\mathcal{H}_{ab}$ is an antisymmetric matrix, and the (Majorana) fermions obey the (Clifford) algebra

$$\{\psi_a,\, \psi_b\} = 2\delta_{ab}$$

Examples of non-obvious free-fermionic systems:

1d quantum transverse-field/2d classical Ising

Kauffman, Onsager; its fermionic version now known as the ``Kitaev chain''

1d quantum XY

Jordan-Wigner; Lieb-Schultz-Mattis

2d Kitaev honeycomb model

In field theory: sine-Gordon at special point

Coleman; Luther-Emery

2. Canonical example: the Ising/Kitaev chain

transverse field favouring disorder

interaction favouring order

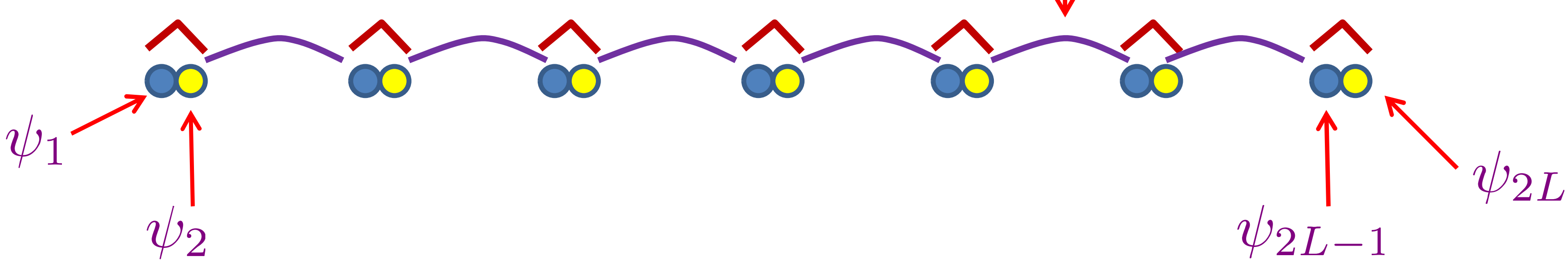
$$H = -h \sum_{j=1}^{L} X_j - J \sum_{j=1}^{L-1} Z_j Z_{j+1}$$

The X_j, Z_j are Pauli matrices acting on site j of the L-site chain.

The non-local Jordan-Wigner transformation defines $2L$ Majorana fermions

$$\psi_{2j-1} = Z_j \prod_{k=1}^{L-1} X_k \qquad \psi_{2j} = iX_j \psi_{2j-1} \qquad \{\psi_a, \psi_b\} = 2\delta_{ab}$$

$$X_j = i\psi_{2j-1}\psi_{2j} \qquad Z_j Z_{j+1} = i\psi_{2j}\psi_{2j+1}$$



Ising Hamiltonian is bilinear in fermions

$$H = -h\sum_{j=1}^{L} X_j - J\sum_{j=1}^{L-1} Z_j Z_{j+1} = ih\sum_{j=1}^{L}\psi_{2j-1}\psi_{2j} + iJ\sum_{j=1}^{L-1}\psi_{2j}\psi_{2j+1}$$

$$H = \sum_{a,b}\mathcal{H}_{ab}\psi_a\psi_b \qquad \mathcal{H} = i\begin{pmatrix} 0 & h & 0 & 0 & 0 & \cdots \\ -h & 0 & J & 0 & 0 & \cdots \\ 0 & -J & 0 & h & 0 & \cdots \\ 0 & 0 & -h & 0 & J & \cdots \\ & & & \vdots & & \end{pmatrix}$$

With little additional effort, allow random couplings, i.e.

$$H = i\sum_{j=1}^{L} t_{2j-1}\psi_{2j-1}\psi_{2j} + i\sum_{j=1}^{L-1} t_{2j}\psi_{2j}\psi_{2j+1} = i\sum_{m=1}^{2L-1} t_m\psi_m\psi_{m+1}$$

Just replace entries in matrix correspondingly.

Really easy to find spectra of such Hamiltonians

Because $\{\psi_a,\,\psi_b\} = 2\delta_{ab}$, commuting a linear in fermions with a bilinear yields a linear:

$$\left[H,\,\sum_{a=1}^{2L} r_a\psi_a\right] = \sum_{b=1}^{2L} s_b\psi_b$$ where $$\mathcal{H}_{ba}r_a = s_b$$

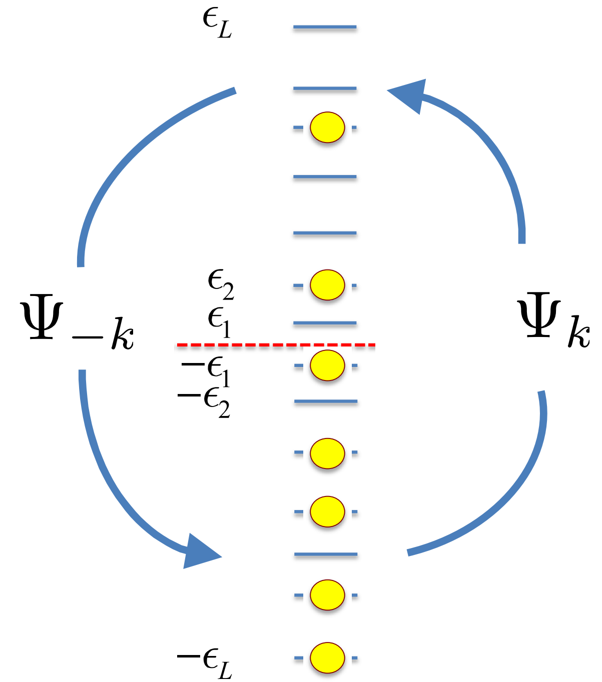
Define raising and lowering operators $$\Psi_{\pm k} = \sum_a v_a^{(\pm k)}\psi_a \qquad k = 1 \ldots L$$

using the eigenvectors of $\mathcal{H}$: $$\mathcal{H}v^{(\pm k)} = \pm 2\epsilon_k\, v^{(\pm k)}$$

Then $$\left[H,\,\Psi_{\pm k}\right] = \pm 2\epsilon_k\Psi_{\pm k}$$

Acting with $\Psi_{\pm k}$ either annihilates a state or changes the energy by $\pm 2\epsilon_k$

$$\left[H,\ \Psi_{\pm k}\right] = \pm 2\epsilon_k \Psi_{\pm k}$$



Using the algebra only, easy to show that

$$\left\{\Psi_k,\ \Psi_{k'}\right\} = 2\delta_{k,-k'}$$

$$H = \sum_{k=0}^{L} \epsilon_k \Psi_k \Psi_{-k}$$

so that every level is filled or empty:

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \ \ldots \ \pm \epsilon_L$$

We thus have reduced the computation of the eigenvalues of a $2^L \times 2^L$ matrix to those of a $2L$ x $2L$ one!

$$\mathcal{H} = i \begin{pmatrix} 0 & h & 0 & 0 & 0 & \cdots \\ -h & 0 & J & 0 & 0 & \cdots \\ 0 & -J & 0 & h & 0 & \cdots \\ 0 & 0 & -h & 0 & J & \cdots \\ & & & \vdots & & \end{pmatrix}$$

By now many many models have been solved by J-W transformations. Do one on your fave chain, and if the Hamiltonian is quadratic in fermions, you win!

Using graph theory, Chapman and Flammia showed (rigorously) when a J-W transformation to a fermion-bilinear is possible

2003.05465

Is at all there is?

Since we did everything with the fermions algebraically, suggests that we don't even really need the fermions!

3. Algebras and graphs for free fermions

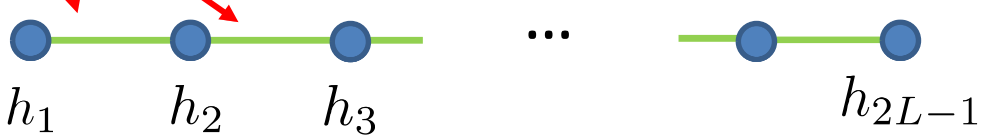
constants

Write $H = \sum_{m=1}^{2L-1} h_m$ $\qquad h_{2j-1} = t_{2j-1} X_j \qquad h_{2j} = t_{2j} Z_j Z_{j+1}$

These operators obey a very simple algebra

$$h_m^2 = t_m^2 \,, \qquad h_m h_{m+1} = -h_{m+1} h_m \,, \qquad h_m h_{m'} = h_{m'} h_m \quad \text{for } |m - m'| > 1$$

``frustration'' graph:



Can forget presentation as long as generators obey same algebra, e.g. instead take

$$h_{2j-1} = t_{2j-1} Z_{j-1} X_j Z_{j+1}$$

Although this approach looks unusual, it in essence is Onsager's original approach!

Conserved charges

adjacent h_m anticommute, others commute.

Find non-local conserved charges $Q^{(r)}$ involving h_m at least 2 sites apart:

$$Q^{(1)} = H = \sum_m h_m$$

$$\left[H,\, Q^{(r)}\right] = 0 \qquad Q^{(2)} = \sum_{m_1+1<m_2} h_{m_1} h_{m_2}$$

$$Q^{(3)} = \sum_{m_1+1<m_2<m_3-1} h_{m_1} h_{m_2} h_{m_3}$$

etc

The $Q^{(r)}$ commute among themselves as well.

``transfer matrix'' $T(u) = \sum_{r=0}^{L} u^r Q^{(r)}$

Note finite sum and $[T(u),\, T(u')]=0$

Local conserved charges follow from $\frac{d}{du}\ln(T(u)) = H + uH^{(2)} + u^2 H^{(3)} + \dots$

Raising and lowering operators

$T(u) = \sum_{r=0}^{L} u^r Q^{(r)}$ is an operator. However, a little algebra gives

$$T(u)T(-u) = P(-u^2)$$

where $P(-u^2) = \sum_{r=0}^{L} (-u^2)^r P^{(r)}$

is a polynomial constructed as with $T(u)$:

$$P^{(1)} = \sum_m t_m^2$$

$$P^{(2)} = \sum_{m_1+1<m_2} t_{m_1}^2 \, t_{m_2}^2$$

$$P^{(3)} = \sum_{m_1+1<m_2<m_3-1} t_{m_1}^2 \, t_{m_2}^2 \, t_{m_3}^2$$

¡

Less obviously, the roots of $P(-u^2)$ are $u_k = \pm \frac{1}{\epsilon_k}$

and the raising/lowering operators are $\Psi_{\pm k} = T(u_k) Z_1 T(-u_k)$

!

Other models with this property?

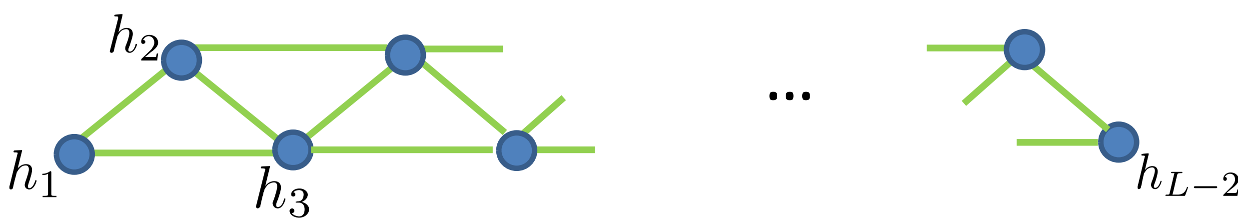
4. Free fermions in disguise

Fendley 2019

$$H = \sum_{m=1}^{L-2} h_m \,, \qquad h_m = t_m X_m X_{m+1} X_{m+2}$$

The generators anticommute two sites apart:

$$h_m^2 = t_m^2 \,, \quad h_m h_{m+1} = -h_{m+1} h_m \,, \quad h_m h_{m+2} = -h_{m+2} h_m \,,$$
$$h_m h_{m'} = h_{m'} h_m \quad \text{for } |m - m'| > 2$$



- Not solvable by Jordan-Wigner: $H = \sum_{m=1}^{L-2} t_m \, \psi_{2m-1}\psi_{2m}\psi_{2m+2}\psi_{2m+3}$
- Commutes with $\widetilde{H} = \sum_m \widetilde{h}_m \,, \quad \widetilde{h}_m = \widetilde{t}_m Z_j X_{j+1} X_{j+2}$
- Model has an $N{=}2$ supersymmetry, with generators made of fermion trilinears.

The same algebraic procedure works here

Non-local conserved charges $Q^{(r)}$ now involve h_m at least 3 sites apart:

$$[Q^{(r)}, Q^{(s)}] = 0$$

$$Q^{(1)} = H = \sum_m h_m \qquad Q^{(2)} = \sum_{m_1+2<m_2} h_{m_1} h_{m_2}$$

$$Q^{(3)} = \sum_{m_1+2<m_2<m_3-2} h_{m_1} h_{m_2} h_{m_3}$$

etc

Proof involves only the algebra. ¡Model is integrable even with random couplings!

$$T(u) = \sum_{r=0}^{L} u^r Q^{(r)} \qquad P(-u^2) = \sum_{r=0}^{L} (-u^2)^r P^{(r)}$$

$$T(u)T(-u) = P(-u^2)$$

$$P^{(1)} = \sum_m t_m^2 \qquad P^{(2)} = \sum_{m_1+2<m_2} t_{m_1}^2\, t_{m_2}^2 \qquad P^{(3)} = \sum_{m_1+2<m_2<m_3-2} t_{m_1}^2\, t_{m_2}^2\, t_{m_3}^2$$

The raising/lowering operators require another miracle

The construction here is not simple like for the J-W fermions. It requires one non-obvious identity, and only works for open chain.

Include an edge mode χ obeying

$$h_1\chi = -\chi h_1$$

$$h_m\chi = \chi h_m \qquad m > 1$$

Here $\chi = Z_1$ works.

From only the algebra follows $u\Big\{[H,\chi]\,,\,T(u)\Big\} = 2\big[\chi,\,T(u)\big]$

Then $\Psi_{\pm k} = T(u_k)Z_1T(-u_k)$ $\qquad H = \sum_{k=0}^{M}\epsilon_k\Psi_k\Psi_{-k}$

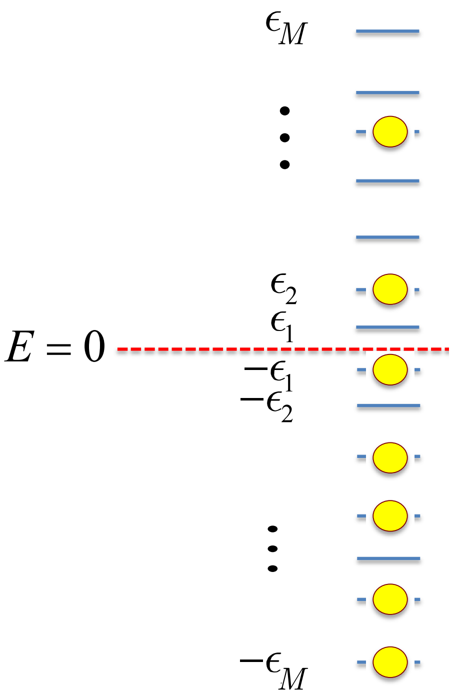
where the roots of $P(\text{-}u^2)$ are $u_k = \pm\frac{1}{\epsilon_k}$

Spectrum is that of free fermions

Even though the starting algebras are different, these raising/lowering operators in both Ising and 4-fermi models obey the same algebra

$$\left[H,\, \Psi_{\pm k}\right] = \pm 2\epsilon_k \Psi_{\pm k} \qquad \left\{\Psi_l,\, \Psi_{l'}\right\} = \delta_{l,-l'} \qquad H = \sum_{k=0}^{M} \epsilon_k \Psi_k \Psi_{-k}$$

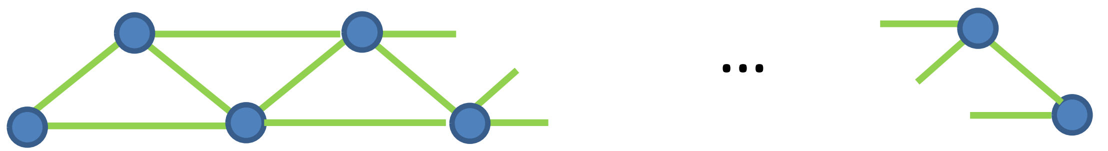
$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_M$$



Here $M = \frac{L}{3}$ because of the ``exclusion rule''.

¡ Exponentially large degeneracies for each energy level !

5. Claws



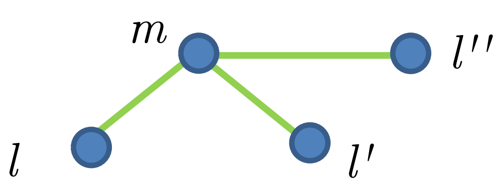
If you try this trick elsewhere, you'll be disappointed – it works only in a few instances

Alcaraz + Pimenta: extend length of interaction

Frustration graph needs to be claw-free for model to be integrable.

Elman, Chapman and Flammia, 2012.07857;
Chapman, Elman and Mann, 2305.15625

A claw is a subgraph



where the three outer vertices are not adjacent, so that

$$\{h_m,\,h_l\} = \{h_m,\,h_{l'}\} = \{h_m,\,h_{l'}\} = [h_l,\,h_{l'}] = [h_l,\,h_{l''}] = [h_{l'},\,h_{l''}] = 0$$

Why claw-free?

$$H = \sum_m h_m \quad \Longrightarrow \quad H^2 = \sum_{m=0}^{L-2} t_m^2 + 2 \sum_{m_1+2<m_2} h_{m_1} h_{m_2} \quad = \text{const} + 2Q^{(2)}$$

Have I committed fraud?

Two kinds of terms in $H^3 = \sum_{m_1, m_2, m_3} h_{m_1} h_{m_2} h_{m_3} \quad = H' + H''$

H' contains those where one pair commutes, two pairs anticommute.

H'' contains those where all three h_{m_j} mutually commute, e.g.

$$\sum_{m_1+2<m_2<m_3-2} h_{m_1} h_{m_2} h_{m_3} \quad = Q^{(3)}$$

Claw-free condition guarantees H commutes with H' and H'' individually.

$$\{h_m, h_l\} = \{h_m, h_{l'}\} = \{h_m, h_{l'}\} = [h_l, h_{l'}] = [h_l, h_{l''}] = [h_{l'}, h_{l''}] = 0$$

In general, commuting charges are given by summing over independent sets of h_{m_j}

Claw-free is not sufficient to yield free fermions

e.g. the four-fermion model with periodic boundary conditions has a claw-free frustration graph, but is not free-fermion.

Need to generalize the edge mode $\chi = Z_1$

One is guaranteed to exist if the frustration graph has a simplicial clique.

Chapman, Elman, Flammia, Mann

A clique is a subset of vertices all connected to each other. It is simplicial when the neighborhood of each such vertex is also simplicial. ¡Graph theorists have studied claw-free graphs with simplicial modes!

Chudnovsky et al

$$\Psi_k = T(u_k)\chi T(-u_k) \qquad H = \sum_{k=0}^{M} \epsilon_k \Psi_k \Psi_{-k}$$

and the whole procedure follows

Claw-free is not necessary to yield disguised free fermions

Work in progress with Balazs Pozsgay

Balazs found an integrable model that interpolates between my four-fermi model and another model with a free-fermion spectrum (but was thought to be another class)

Fendley and Schoutens 2006; de Gier et al 2015; Feher et al 2017

$$A_j \equiv a_{j-1}a_j\, X_{j-1}X_jZ_{j+1}, \qquad B_j = b_jb_{j+1}\, Z_{j-1}Y_jY_{j+1}, \qquad C_j = a_jb_jZ_{j-1}Z_{j+1}$$

$$H = \sum_{j=2}^{L} A_j \;+\; \sum_{j=1}^{L-1} B_j \;+\; \sum_{j=1}^{L} C_j$$

Frustration graph is not claw-free (and too nasty to draw). But nonetheless can run procedure using a modified frustration graph (it has even more edges).

The loophole is that the generators are not independent: $B_jA_{j+1} = -C_jC_{j+1}$

6. The physics of the four-fermi chain

Find that for uniform couplings $t_m = 1$, theory is critical, but not a CFT.

Instead, it has dynamical critical exponent $z=3/2$, i.e. parametrizing

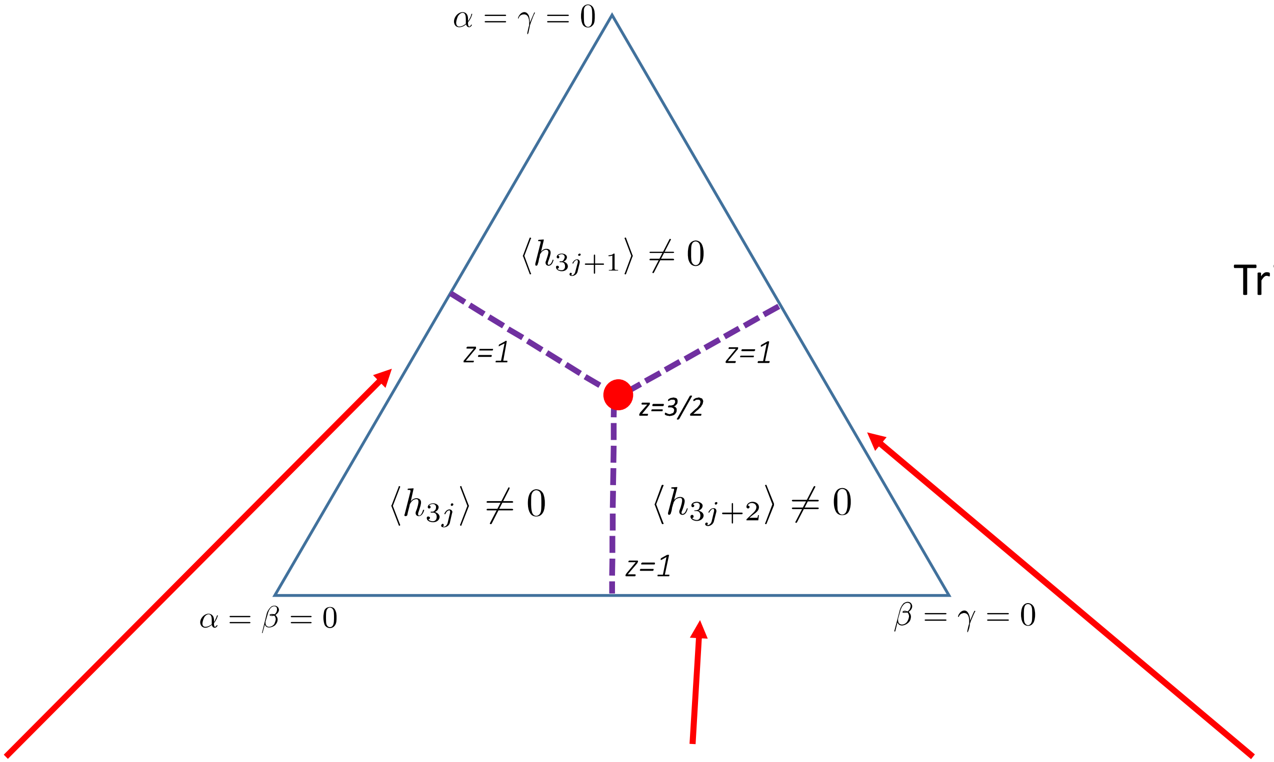
$$\epsilon^2(p) = \frac{\sin^3 p}{\sin\frac{p}{3}\sin^2\frac{2p}{3}} \qquad \text{yields} \qquad \epsilon(p) \approx \left(\frac{4}{3}\right)^{3/4} |\pi - p|^{3/2}$$

for $|p-\pi|$ small

Imposing periodic b.c. breaks degeneracies and gives rise to a distinct z. Even though it's integrable, not free-fermion, and too difficult to extract answer analytically.

Hello numerical experts?

Staggering on every third site $t_{3j-2} = \alpha,\ t_{3j-1} = \beta,\ t_{3j} = \gamma$



Triality!

Free fermions not in disguise when every third coupling vanishes – equivalent to Ising.

Combining Ising and four-fermi chains

$$H_{\text{Ising}} - gH_{4-\text{fermi}}$$

O'Brien and Fendley 2017

- Combination is not only not free-fermion, it's not even integrable.
- Find a non-trivial critical point (tricritical Ising) with only one-parameter tuning and without changing the Hilbert space. Thus ideally suited for testing numerical methods.
- Along self-dual line, interesting properties such as supersymmetry and order-disorder coexistence.

Procedure generalizes to free parafermions!

Anticommutation relations generalise to

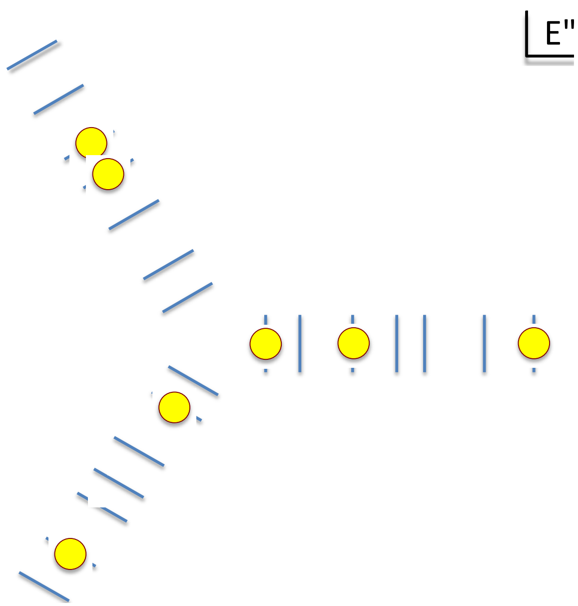
$$(\epsilon_k - \omega\epsilon_l)\Psi_k\Psi_l = (\epsilon_l - \omega\epsilon_k)\Psi_l\Psi_k$$

Baxter's non-Hermitian $\mathbb{Z}_n$ chains have spectrum

$$E = \omega^{m_1}\epsilon_1 \; + \; \omega^{m_2}\epsilon_2 \; + \; \cdots \; + \; \omega^{m_L}\epsilon_L$$

where $\omega = e^{2\pi i/n}$

and each $m_j = 0, 1, \ldots n-1$



Baxter 1989 2014 Fendley 2013 unpub Au-Yang/Perk 2014, 2016

Lots more to do

- Exact edge zero modes?
- Field theory? Connection to usual Bethe ansatz?
- (Superintegrable) chiral Potts transfer matrix is related to free parafermions
- Connection to experiment both in Ising+4-fermion and in chiral Potts

  Aasen et al 2020 Rydberg blockade
- Close connection to integrable Bazhanov-Baxter models in 3d
- Connection to chiral CFTs?