QUANTUM MANY-BODY RANDOM SYSTEMS,

COMPLEXITY,

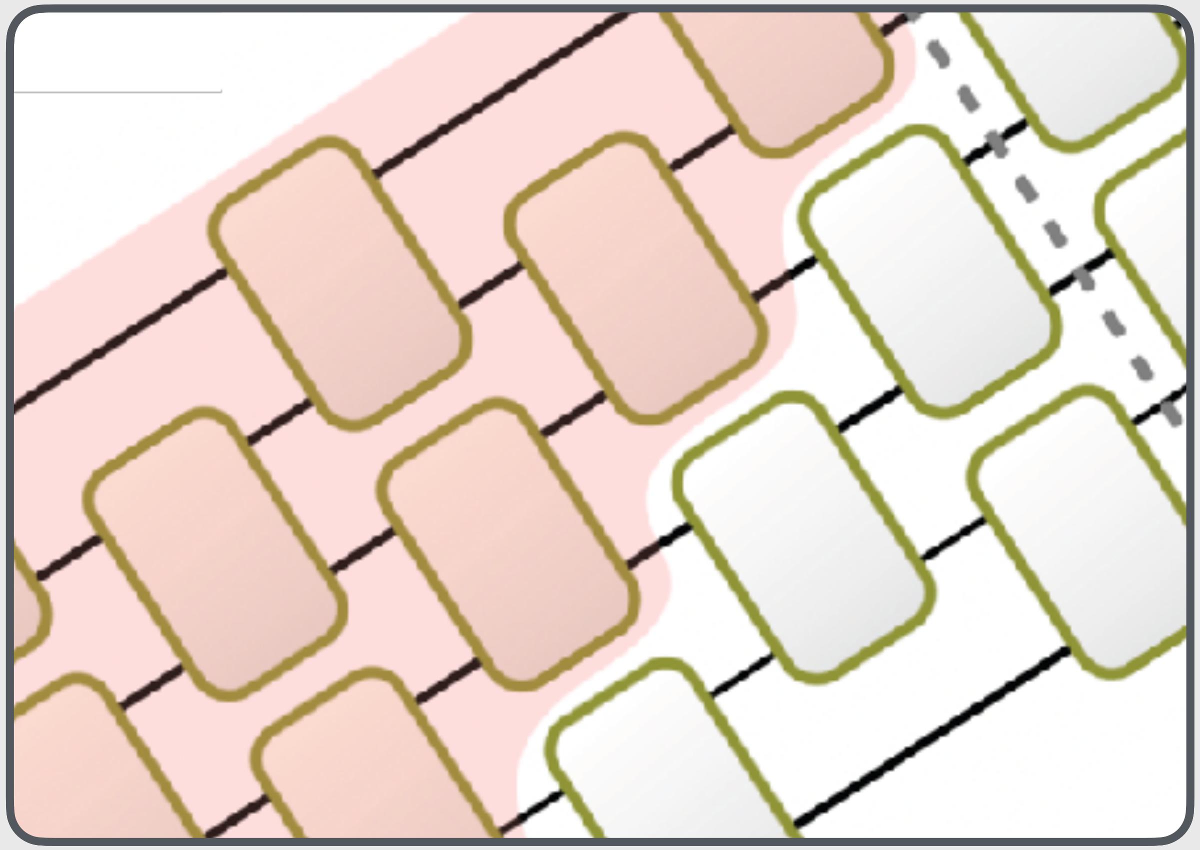
AND RANDOM QUANTUM CIRCUITS

JENS EISERT, FU BERLIN

ENTANGLE THIS: RANDOMNESS, COMPLEXITY AND QUANTUM CIRCUITS, BENASQUE 2023

MANIFESTO OF THIS TALK

- **Random objects** are proxies for complex statics and dynamics

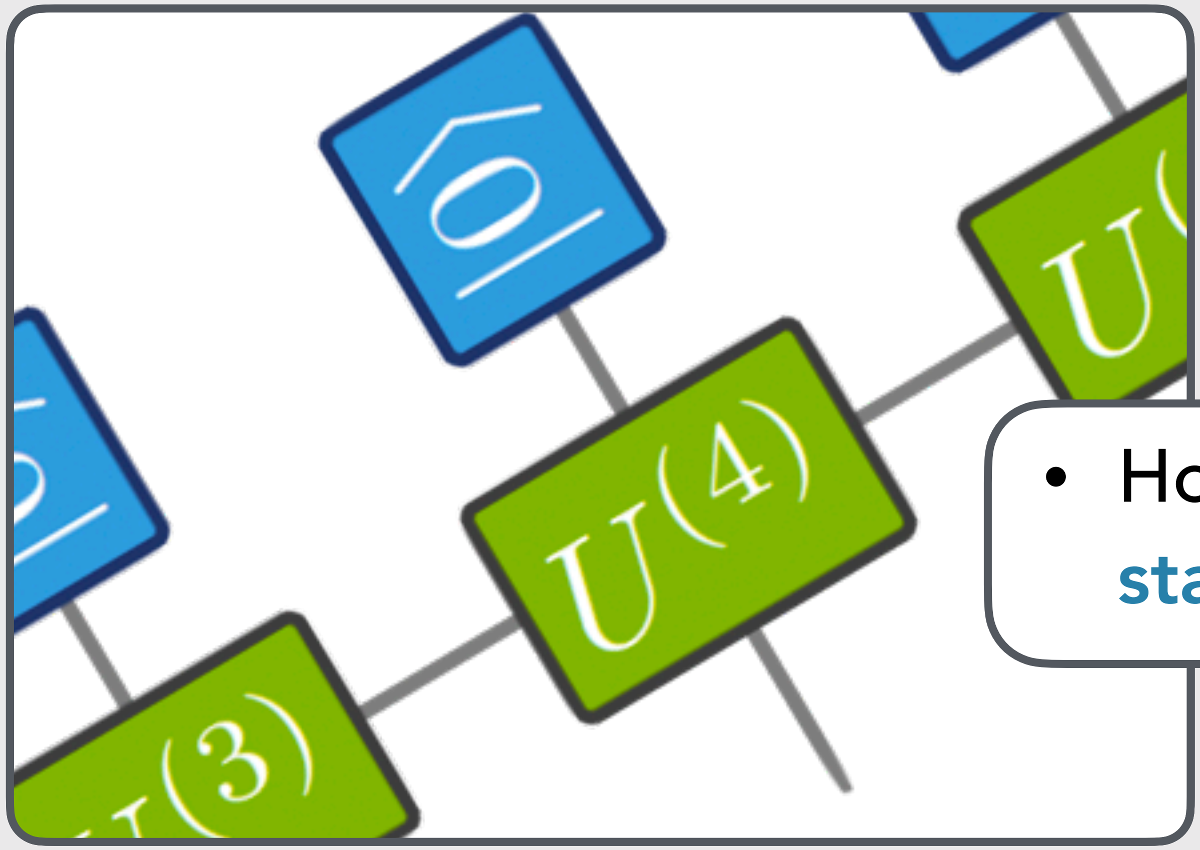


- **Random circuits** show features of *quantum chaotic dynamics* (OTOC), many-body *localization* etc
- **Random tensor networks** for *typical states in phases of matter, holographic prescriptions*
- Reminiscent of **random coding**

- **Randomness** as a powerful tool: Can **prove** statements fully out of reach otherwise

Brown, Fawzi, Commun Math Phys 340, 867-900 (2015)
Brandao, Harrow, Horodecki, Commun Math Phys 346, 397 (2016)
Sünderhauf, Pérez-García, Huse, Schuch, Cirac, Phys Rev B 98, 134204 (2018)
Bertini, Piroli, Phys Rev B 102, 064305 (2020)

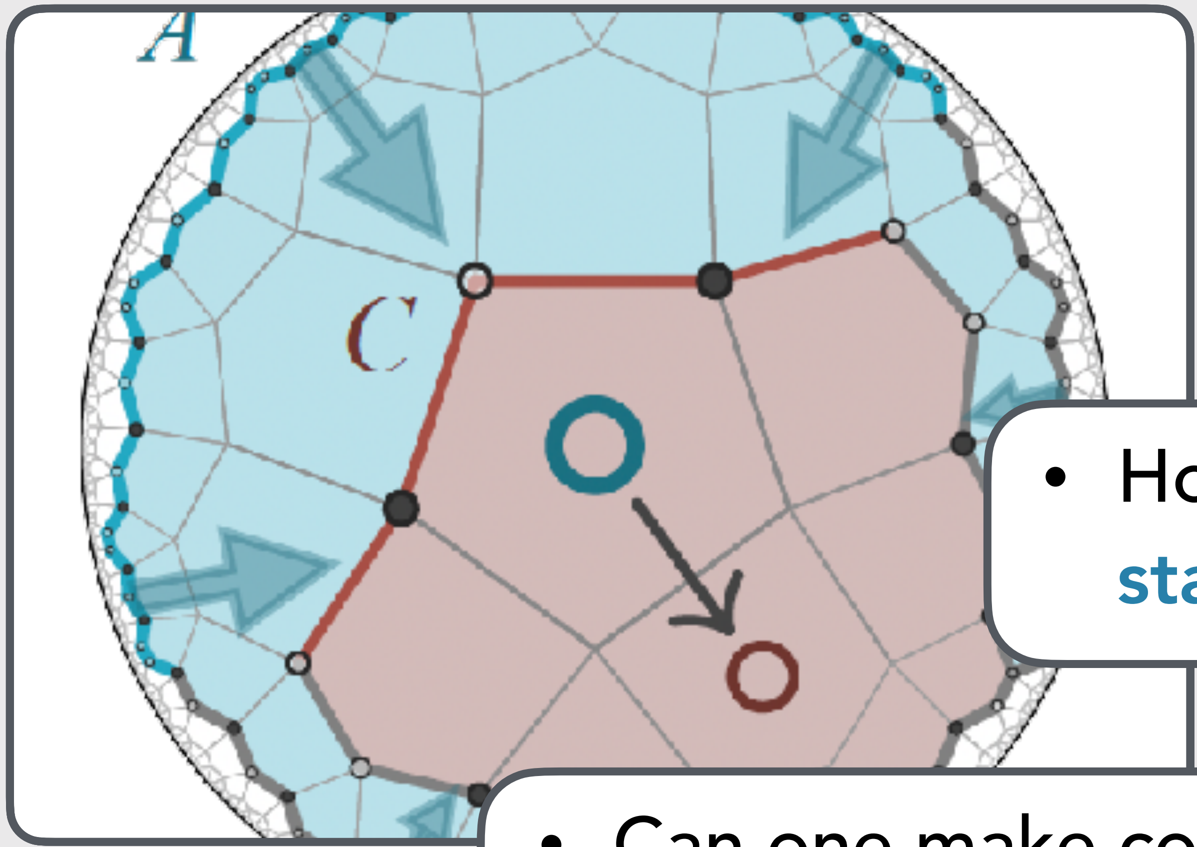
- **Random tensor networks** as typical representatives of phases of matter



- How can features of **quantum statistical mechanics** be proven?

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)

- Random tensor networks as typical representatives of phases of matter

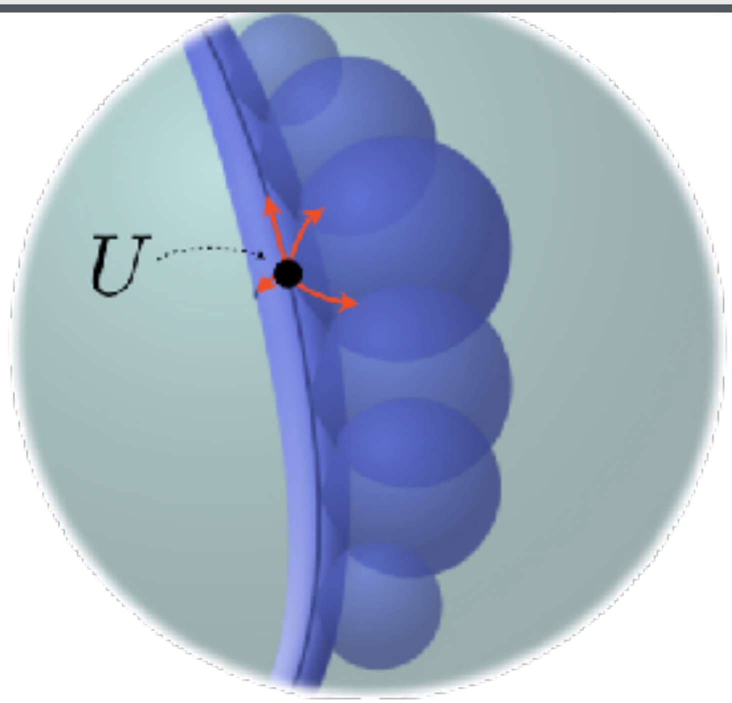


- How can features of quantum statistical mechanics be proven?

- Can one make contact with quantum field theory and holography?

Wille, Altland, Jahn, Eisert, in preparation (2023)
Jahn, Zimboras, Eisert, Quantum 6, 643 (2022)

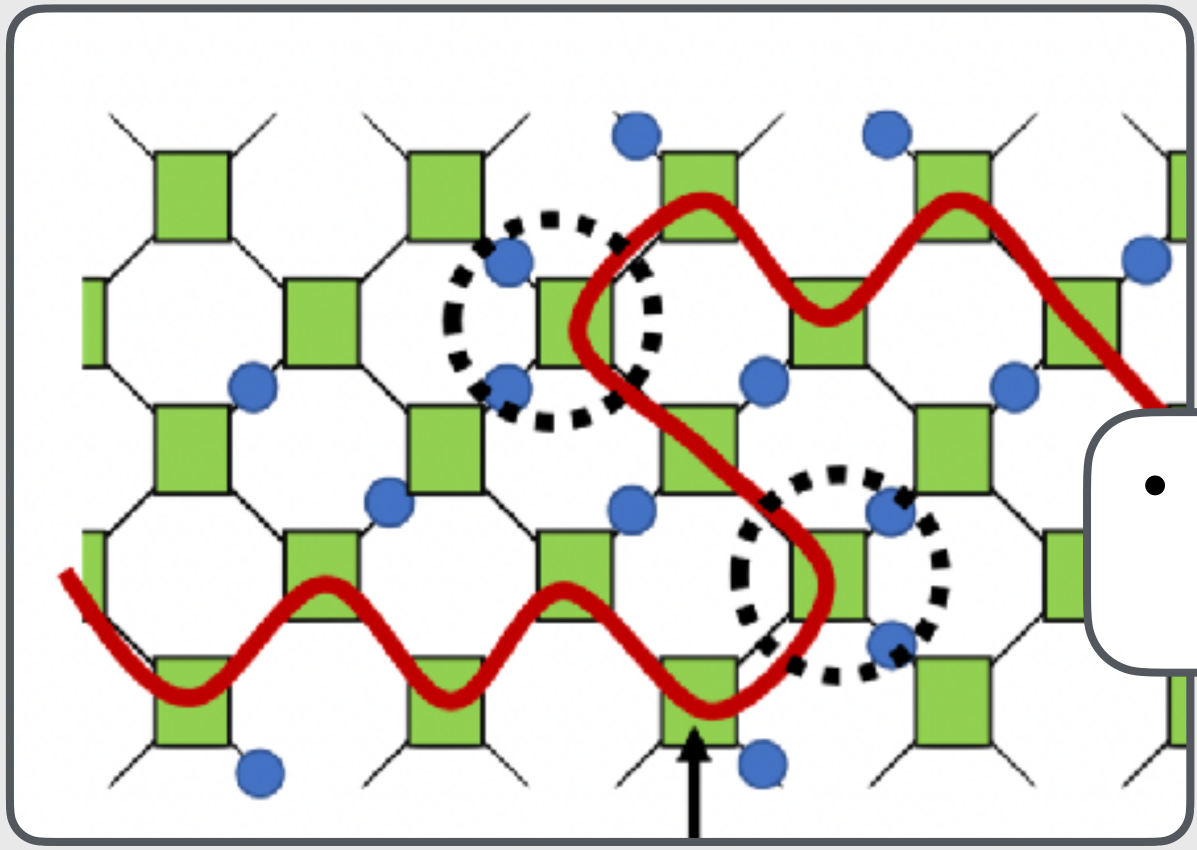
- **Brown Susskind conjecture** on linear **complexity** growth of random circuits



- The complexity is **computationally hard** to compute: How can the notorious conjecture be **proven**?

Haferkamp, Faist, Kothakonda, Eisert, Yunger-Halpern, Nature Physics 18, 528 (2022)

- **Complexity and random circuits**



- How do **entanglement** and **resource theories** come in?

- Complexity phase transitions in **monitored** quantum circuits?

Eisert, Phys Rev Lett 127, 020501 (2021)
Yunger Halpern, Kothakonda, Haferkamp, Munson, Eisert, Faist, Phys Rev A 106, 062417 (2022)
Haferkamp, Montealegre-Mora, Heinrich, Eisert, Gross, Roth, Commun Math Phys 397, 995–1041 (2023)
Suzuki, Haferkamp, Eisert, Faist, arXiv:2305.15475 (2023)

- Complexity and random circuits

- How do **entanglement** and **resource theories** come in?

- Complexity phase transitions in **monitored** quantum circuits?

- "**Quantum homeopathy**" of random quantum circuits?

Eisert, Phys Rev Lett 127, 020501 (2021)
Yunger Halpern, Kothakonda, Haferkamp, Munson, Eisert, Faist, Phys Rev A 106, 062417 (2022)
Haferkamp, Montealegre-Mora, Heinrich, Eisert, Gross, Roth, Commun Math Phys 397, 995–1041 (2023)
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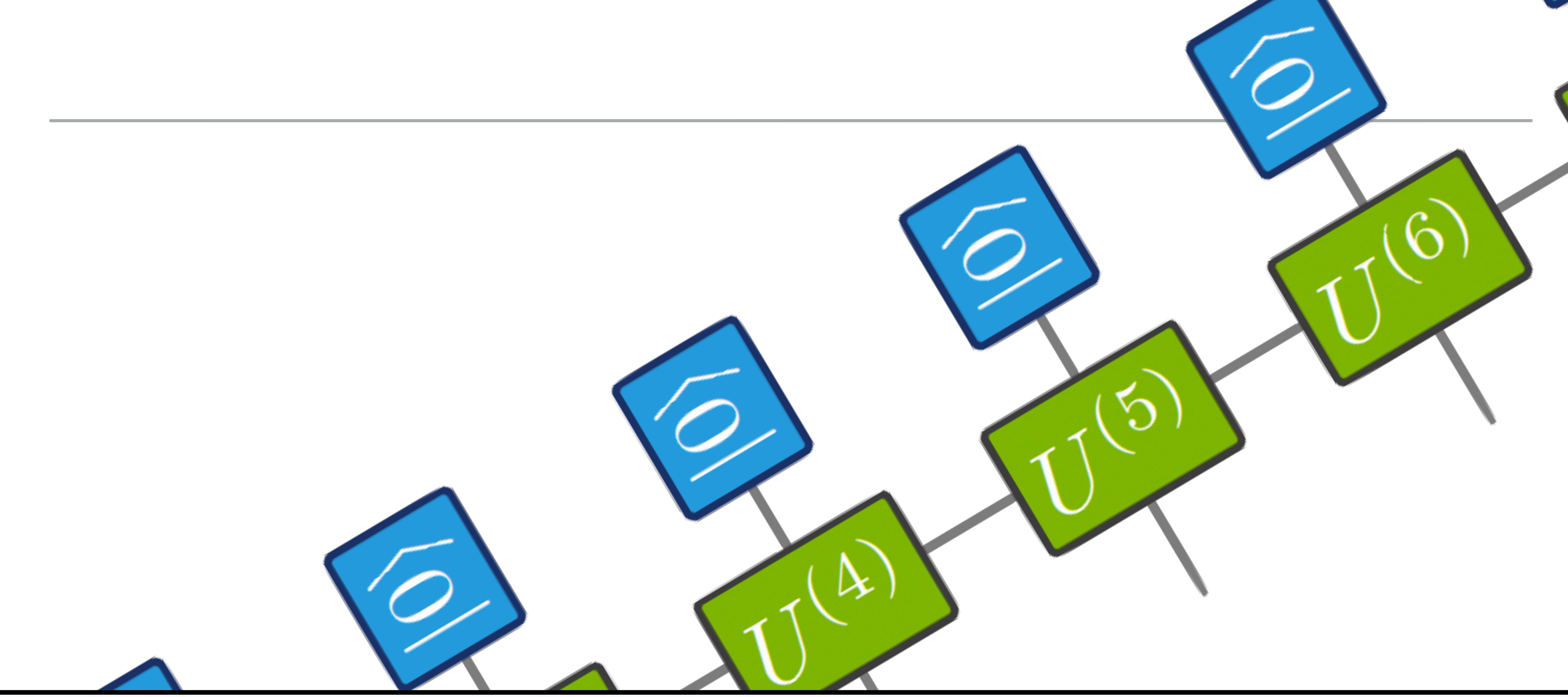


RANDOM TENSOR NETWORKS: FROM STATISTICAL MECHANICS TO HOLOGRAPHY

PRX Quantum 2, 040308 (2021)
PRX Quantum 3, 030312 (2022)
Quantum 6, 643 (2022)
In preparation (2023)

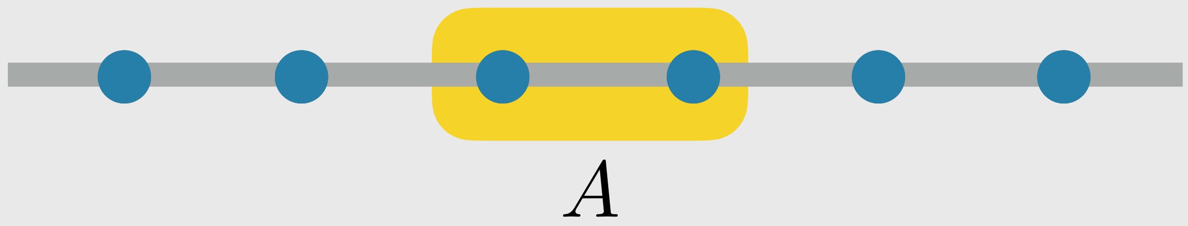
RANDOM MATRIX PRODUCT STATES

- How can **statistical physics** and **quantum dynamics** be reconciled?

von Neumann, Zeitschrift für Physik 57, 30 (1929)
Gogolin, Eisert, Rep Prog Phys 79, 056001 (2016)
Linden, Popescu, Short, Winter, Phys Rev E 79, 061103 (2009)

- Local observables are expected to **equilibrate**

$$|\psi(t)\rangle = e^{-itH}|\psi\rangle$$



- Observables A evolving under H have **time averages**

$$A_\psi^\infty := \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle\psi|A(t')|\psi\rangle \mathrm{d}t'$$

- **Fluctuations** must be small

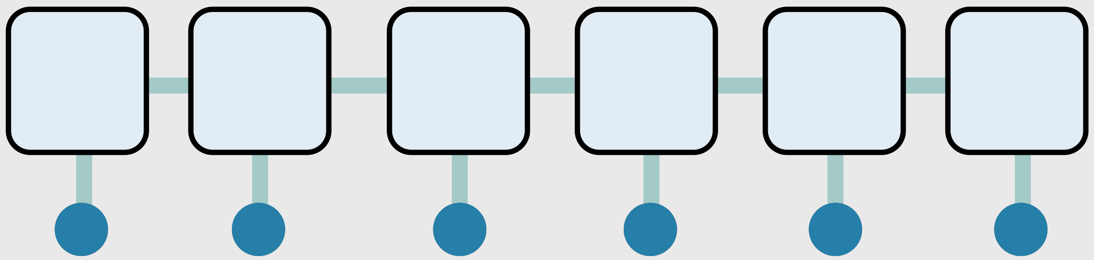
$$\Delta A_\psi^\infty := \lim_{t\to\infty} \frac{1}{t}\int_0^t |\langle\psi|A(t')|\psi\rangle - A_\psi^\infty|^2 \mathrm{d}t'$$

- How can one **judge**?

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)
Lancien, Perez-García, Ann Henri Poincare 23, 141 (2022)
Garnerone, de Oliveira, Haas, Zanardi, Phys Rev A 82, 052312 (2010)

RANDOM MATRIX PRODUCT STATES

- Random **matrix product states** as initial states*



Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)
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Garnerone, de Oliveira, Haas, Zanardi, Phys Rev A 82, 052312 (2010)

RANDOM MATRIX PRODUCT STATES

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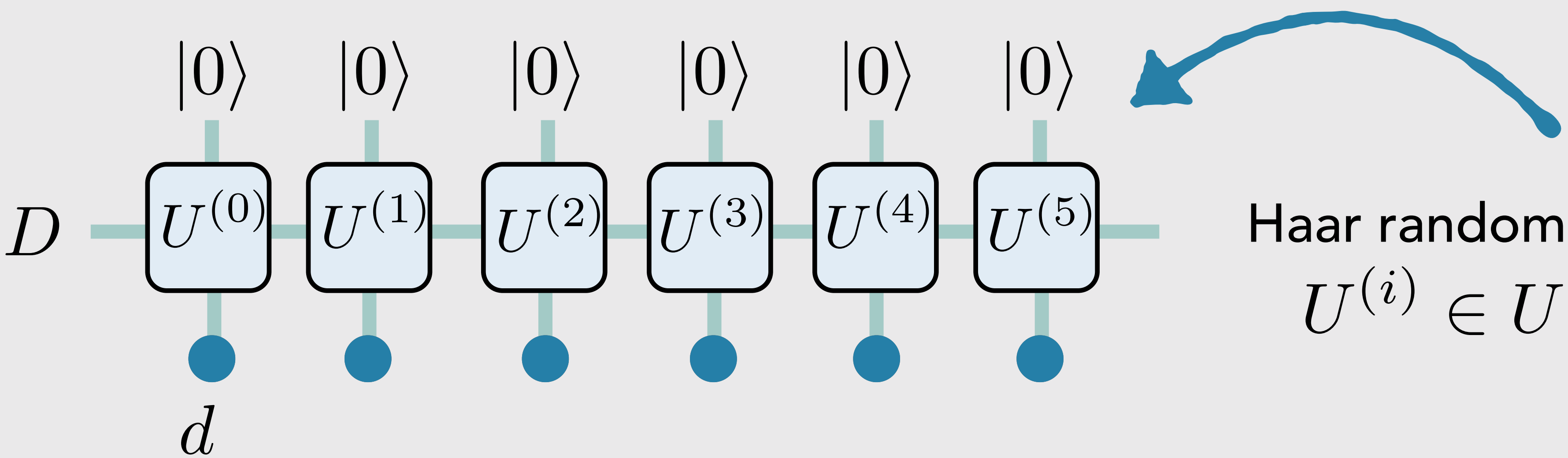
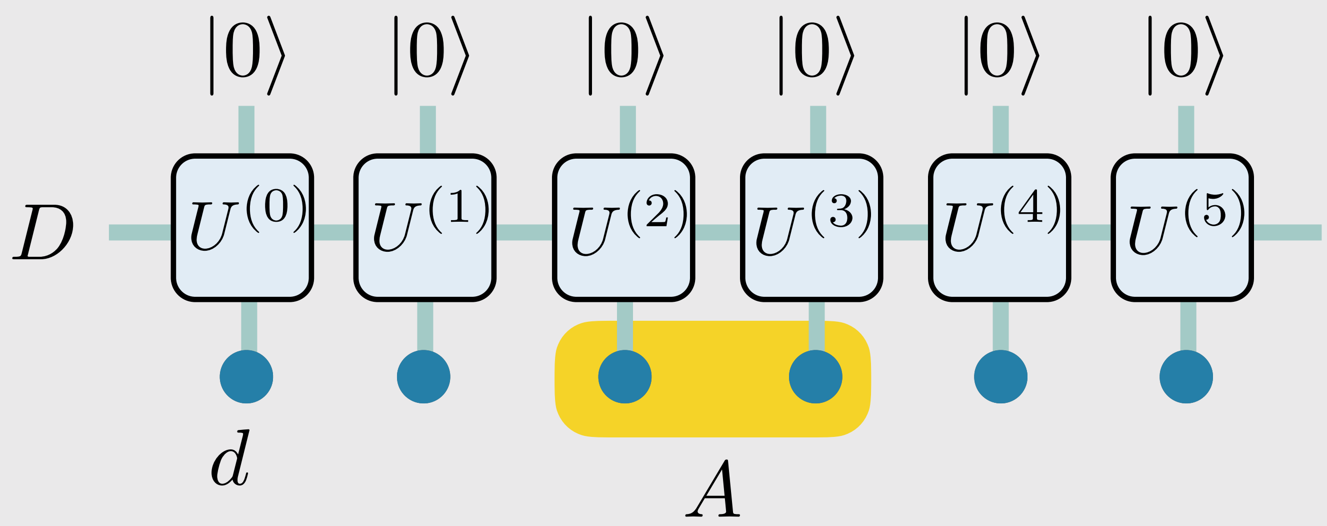


Haar random tensors
$$U^{(i)} \in U(dD)$$

- Can be seen as generic **representatives of phases of matter**
- Several **interesting properties** can be proven

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)
Lancien, Perez-García, Ann Henri Poincare 23, 141 (2022)
Garnerone, de Oliveira, Haas, Zanardi, Phys Rev A 82, 052312 (2010)

- Random **matrix product states**



- They **equilibrate exponentially** well:

$$\Pr\left(\Delta A_{\psi}^{\infty} \leq e^{-c_1 \alpha(d,D)n)}\right) \geq 1 - e^{-c_2 \alpha(d,D)n}$$

for

$$\alpha(d,D) = \log\left(\frac{d - \frac{1}{dD^2}}{(1+\frac{1}{D})(1+\frac{1}{dD})}\right)$$

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)
Compare Huang, Harrow. arXiv:1907.13392 (2019)

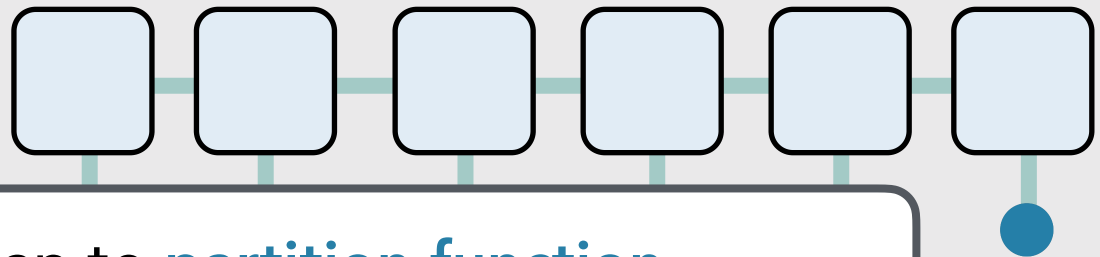
IDEA OF PROOF

- Bound "effective dimension"

$$\Delta A_\psi^\infty = O(1/D_{\text{eff}})$$

$$1/D_{\text{eff}} := \sum_j |\langle \psi | j \rangle|^4$$

as overlap of initial state with energy eigenstates



- For any state vector $|\phi\rangle$

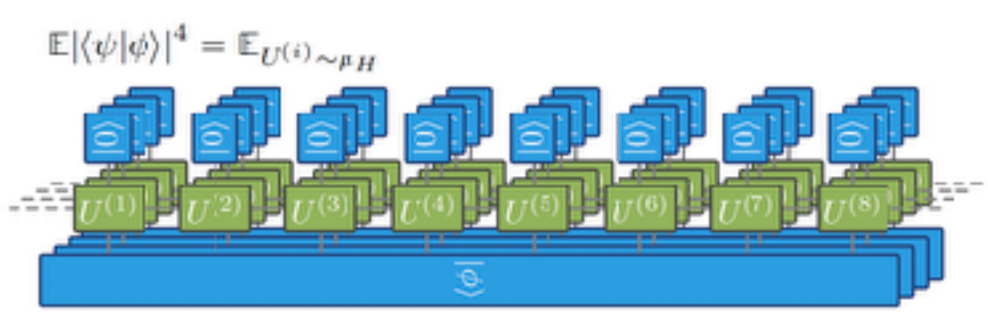
$$\mathbb{E}|\langle\psi|\phi\rangle|^4 = \langle\phi|^{\otimes 2}\mathbb{E}(|\psi\rangle\langle\psi|)^{\otimes 2}|\phi\rangle^{\otimes 2}$$

- The t-th moment operator of Haar-random unitaries

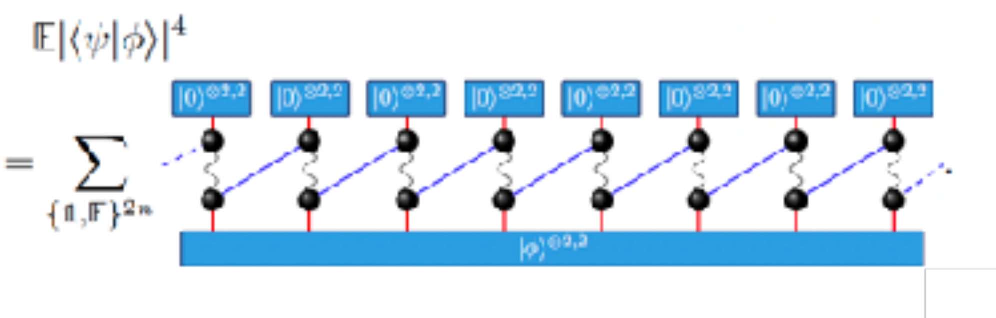
$$\mathbb{E}_{U\sim\mu_H} U^{\otimes t}\otimes \overline{U}^{\otimes t} = \sum_{\sigma,\pi\in S_t} \mathrm{Wg}(\sigma^{-1}\pi, q)|\sigma\rangle\langle\pi|$$

in Weingarten calculus

- Gives tensorial expression

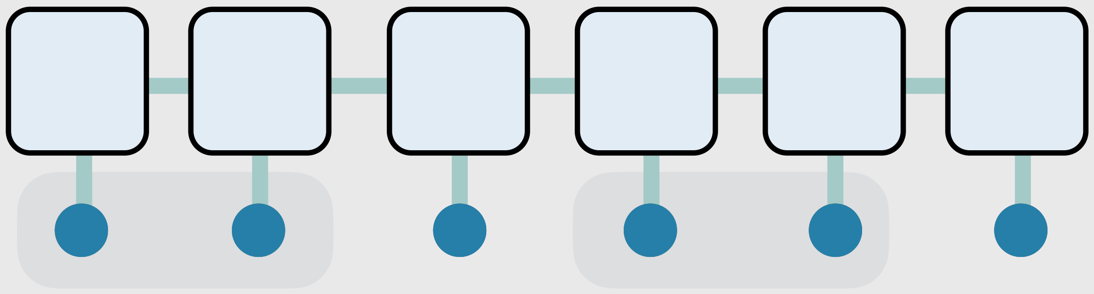


- Map to partition function



Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)
Hunter-Jones, arXiv:1905.12053 (2019)

RANDOM MATRIX PRODUCT STATES EQUILIBRATE EXPONENTIALLY WELL



- **Further results:**
  - Extensivity of 2-Renyi **entropies**
  - **Maximum entropy** for small connected subsystems
  - Ground states of **disordered parent Hamiltonians**
  - Insights into generic **phases of matter**

  - **Exponential decay of correlations** in similar (TI) model

    Lancien, Perez-García, Ann Henri Poincare 23, 141 (2022)

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)

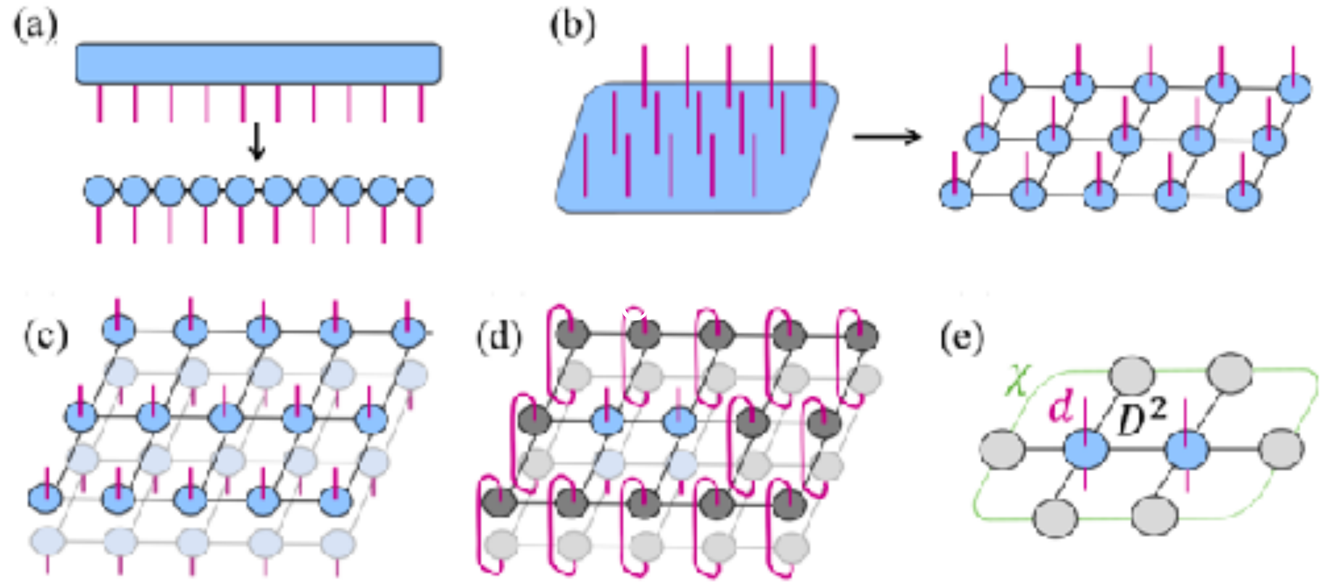
- Use random sampling to **estimate entanglement** in tensor network-states

- Resource-economically estimate **Renyi entanglement entropies**

$$E_n(A) = \frac{1}{1-n} \log \mathrm{tr}(\rho_A^n)$$

and negativity moments using **frames**, random vectors $|v\rangle \in \mathbb{C}^d$ with

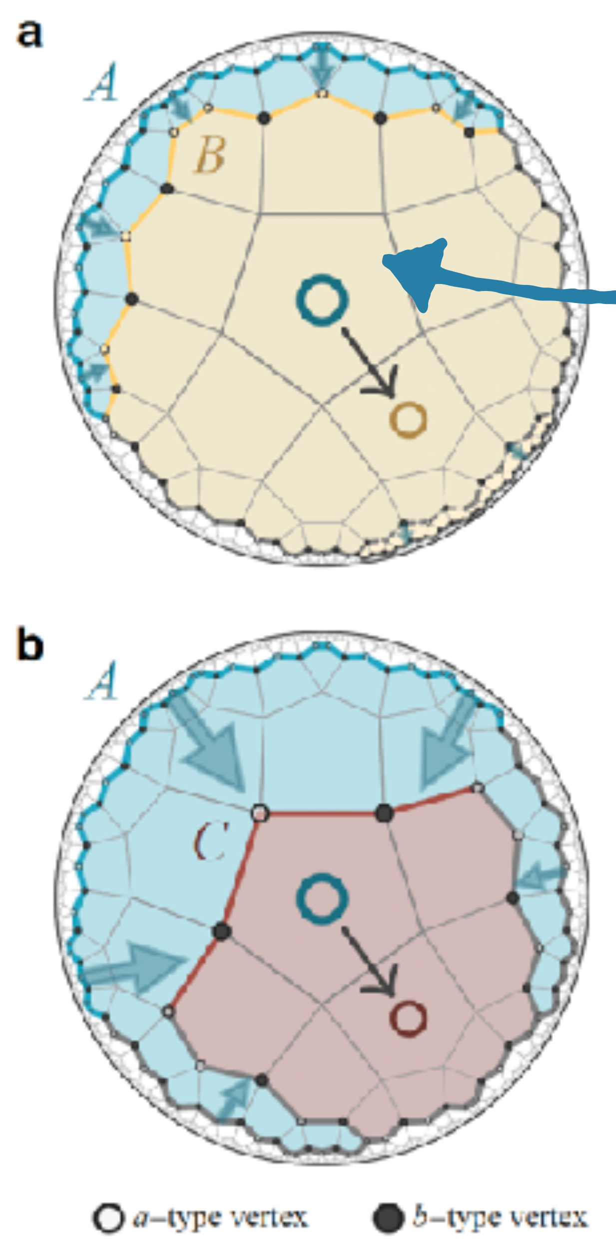
$$\mathbb{E}(|v\rangle\langle v|) = \mathbb{I}$$



Feldman, Kshetrimayum, Eisert, Goldstein, PRX Quantum 3, 030312 (2022)

HOLOGRAPHY AND MATCHGATE TENSOR NETWORKS

- Toy models of AdS-CFT can be formulated as **matchgate tensor networks** (my last talk)



- **Matchgate** per tensor

$$T_v : \{0,1\}^{\times r} \to \mathbb{C}$$

per vertex $v \in V$

- Hyperbolic **tiling** of plane
- Contraction rules via **Grassmann** integrals

Jahn, Gluza, Pastawski, Eisert, Science Advances 5, eaaw0092 (2019)
Jahn, Zimboras, Eisert, Quantum 6, 643 (2022)

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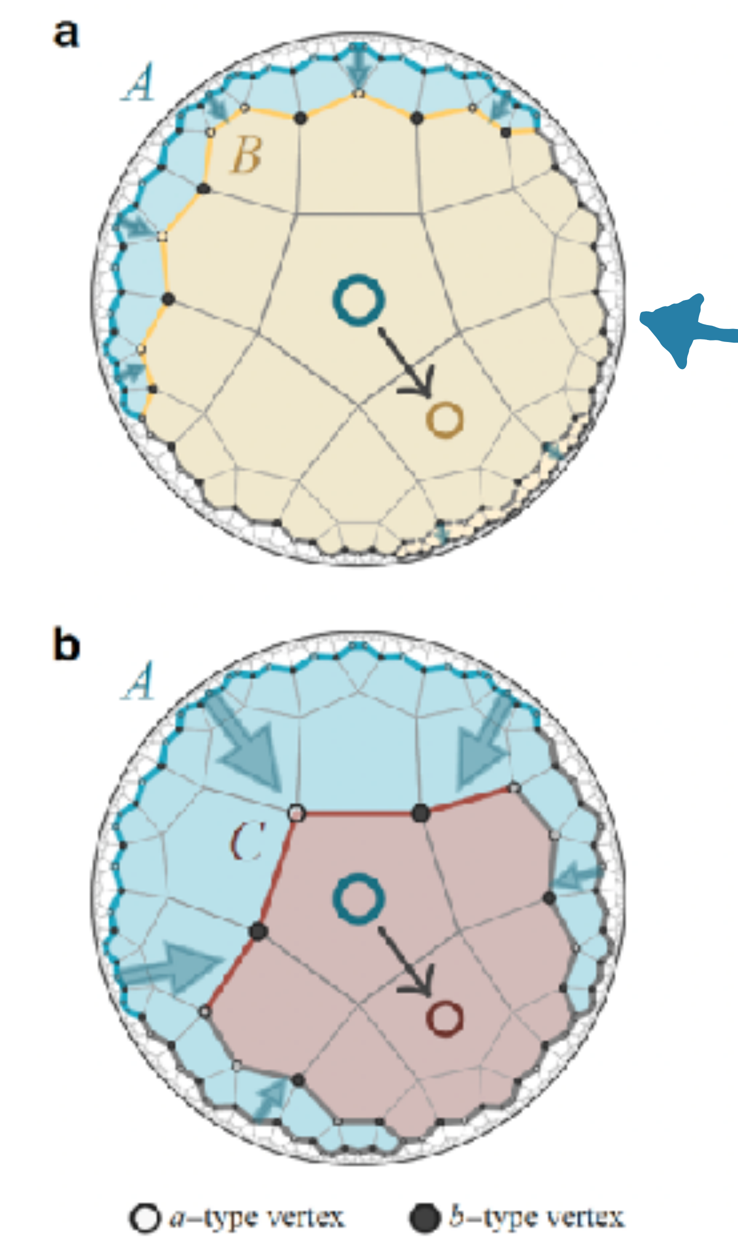
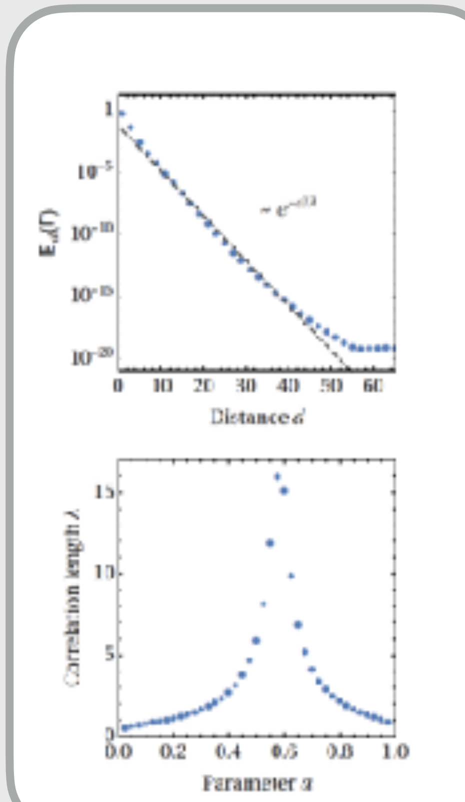
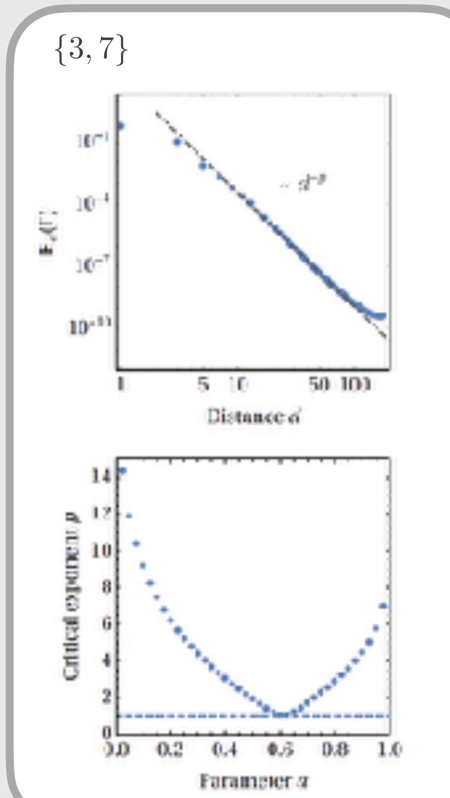


- Get **critical** or **gapped behavior**





dependent on **bulk curvature**

Jahn, Gluza, Pastawski, Eisert, Science Advances 5, eaaw0092 (2019)
Jahn, Zimboras, Eisert, Quantum 6, 643 (2022)

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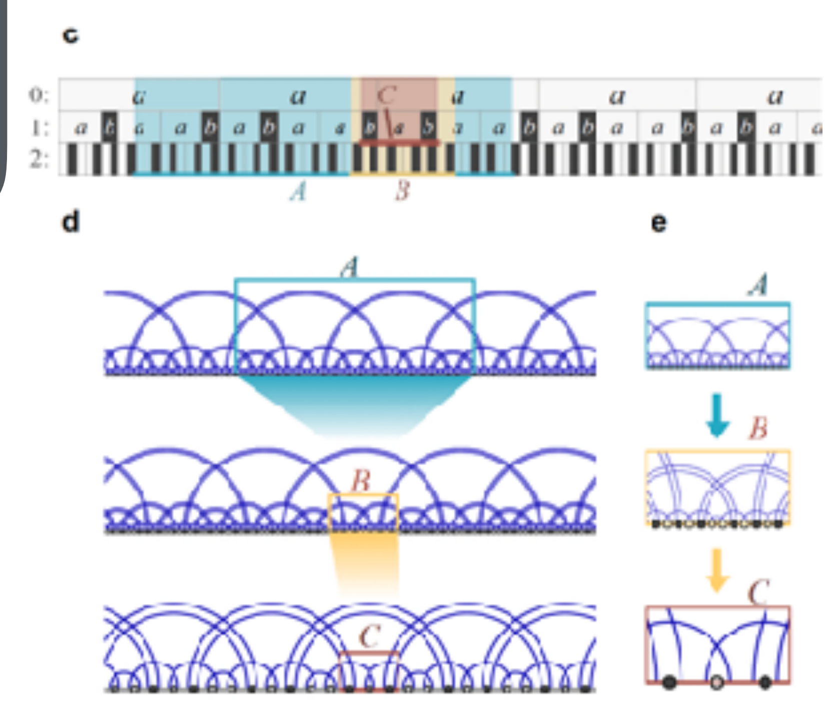
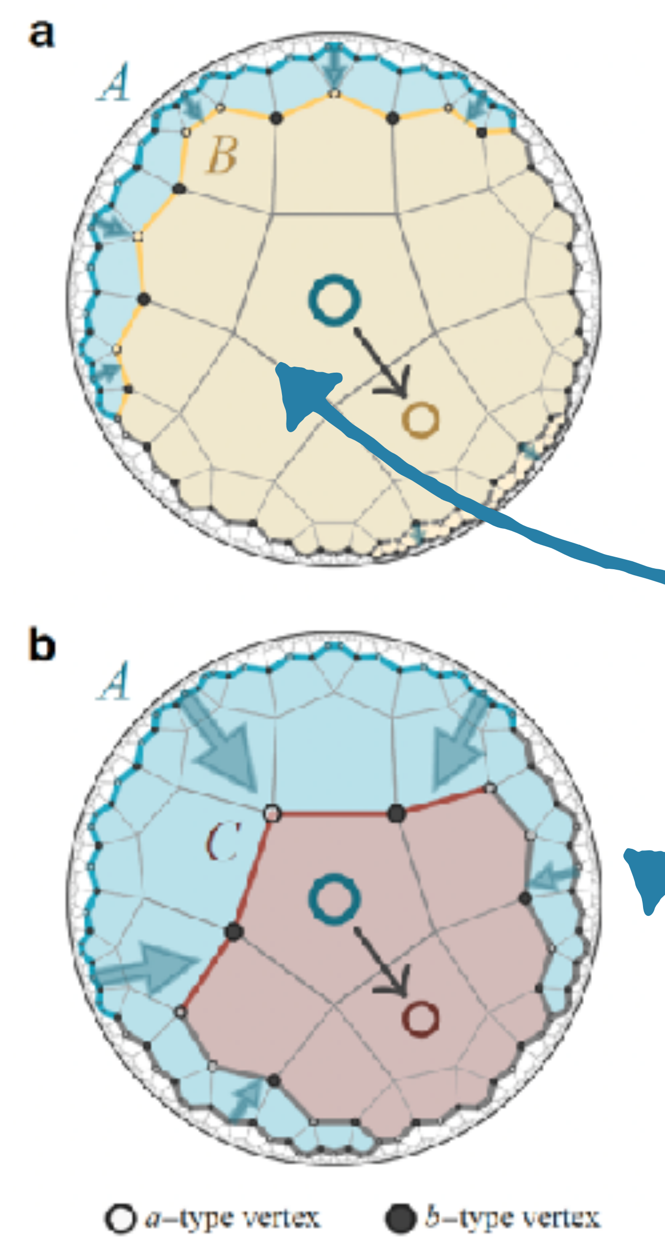




- **Inflation rules** to go from one layer to the next
  - **Critical theory** on boundary with effective central charges depending on tiling, e.g.

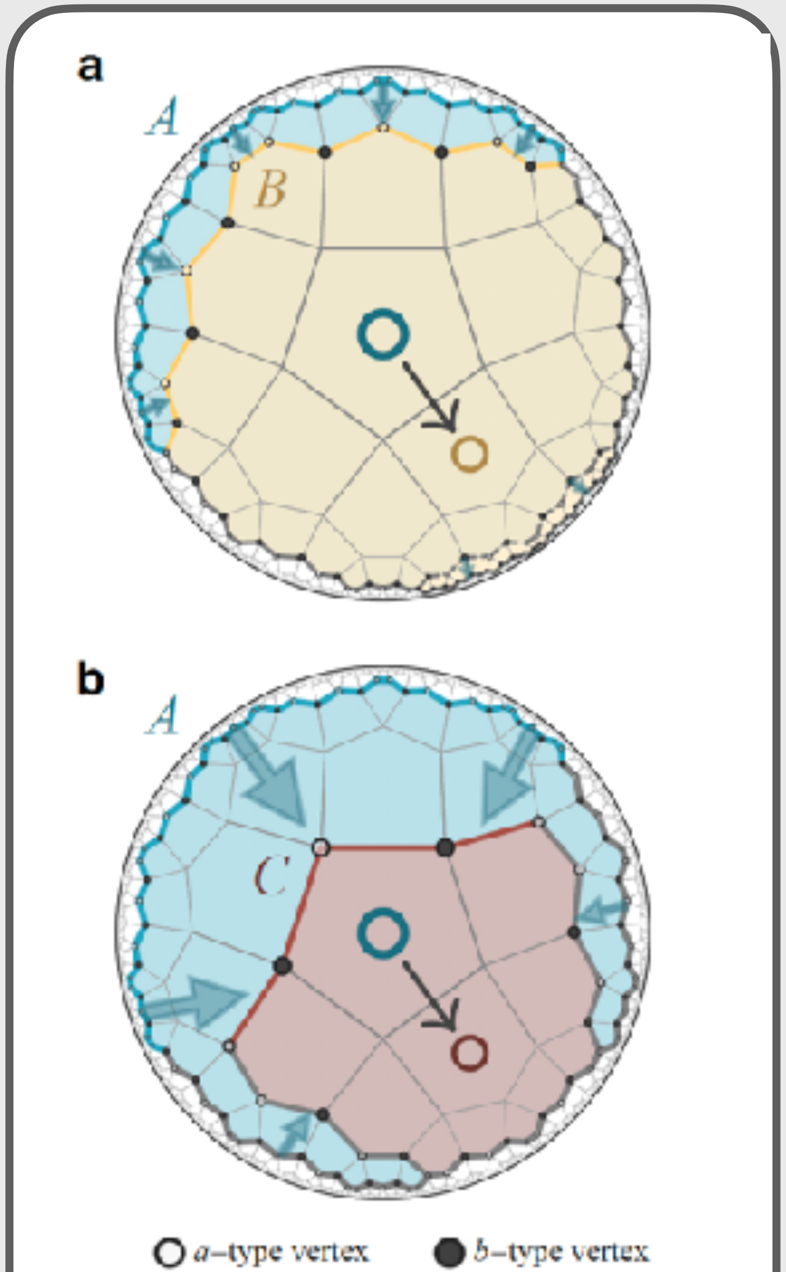
$$c_{\{5,4\}} \approx 4.74$$

- Get **actual CFT** (up to quasi-crystalline symmetry)

Jahn, Zimboras, Eisert, Quantum 6, 643 (2022)
Jahn, Eisert, Quant Sc Tech 6, 033002 (2021)

CONTINUUM MODELS

- **Ongoing: Continuum limit of random matchgates** to arrive at quantum field theories

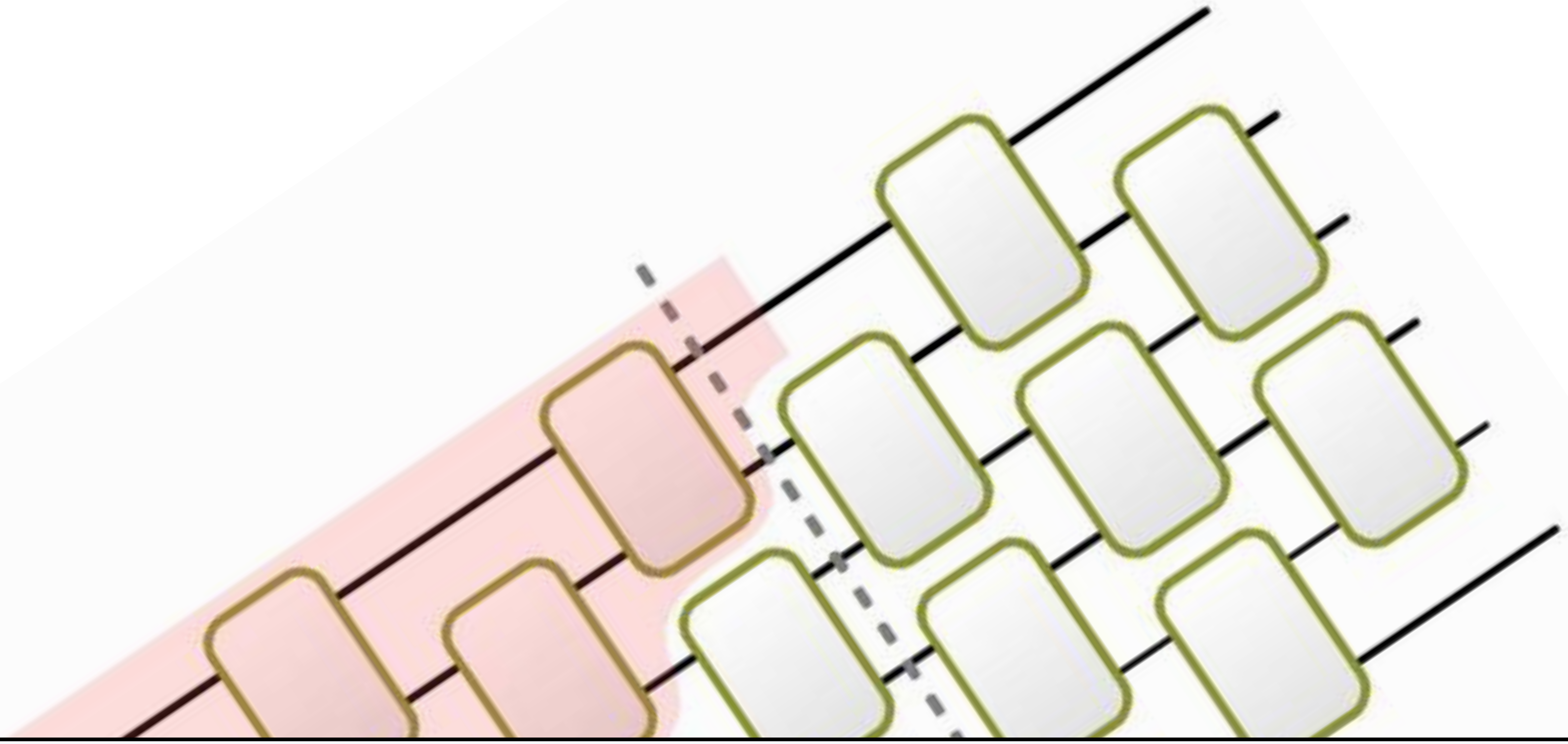


- Contraction of tensor network ~ free fermion partition function
- Can analytically evaluate static and Gaussian random contribution in disorder average
- In stationary phase approximation get smooth continuum limit expressions
- Bulk can be seen as class D superconductor

Wille, Altland, Jahn, Eisert, in preparation (2023)

LESSON

LESSON: RANDOM TENSOR NETWORKS ARE A FUN PLAYGROUND FOR ANALYTICAL STUDIES

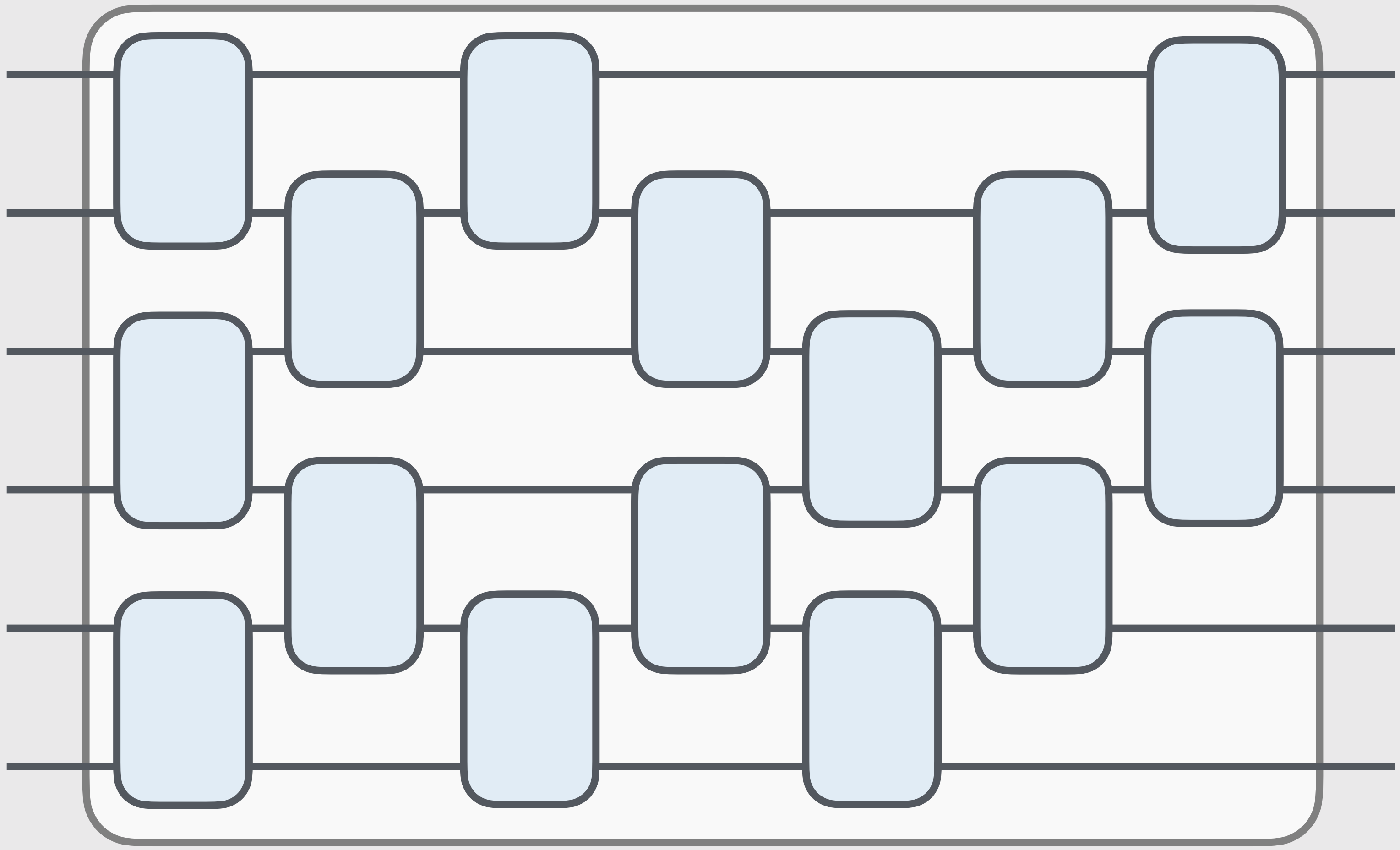


COMPLEXITY IN RANDOM QUANTUM CIRCUITS

Nature Physics 18, 528 (2022)
Phys Rev Lett 127, 020501 (2021)
Phys Rev A 106, 062417 (2022)
arXiv:2305.15475 (2023)
Commun Math Phys 397, 995–1041 (2023)

COMPLEXITY GROWTH IN RANDOM CIRCUITS

- **Circuit complexity**: Smallest number of quantum gates from gate set to generate a **given unitary** (similar, circuit complexity)



- Separates problems into **'easy'** and **'hard'**
- In quantum setting relevant for **phases of matter**

Gosset, Gosset, Kliuchnikov, Mosca, Russo, Quant Inf Comp 14, 1277 (2014)
Aaronson, Gottesman, Phys Rev A 70, 02328 (2004)

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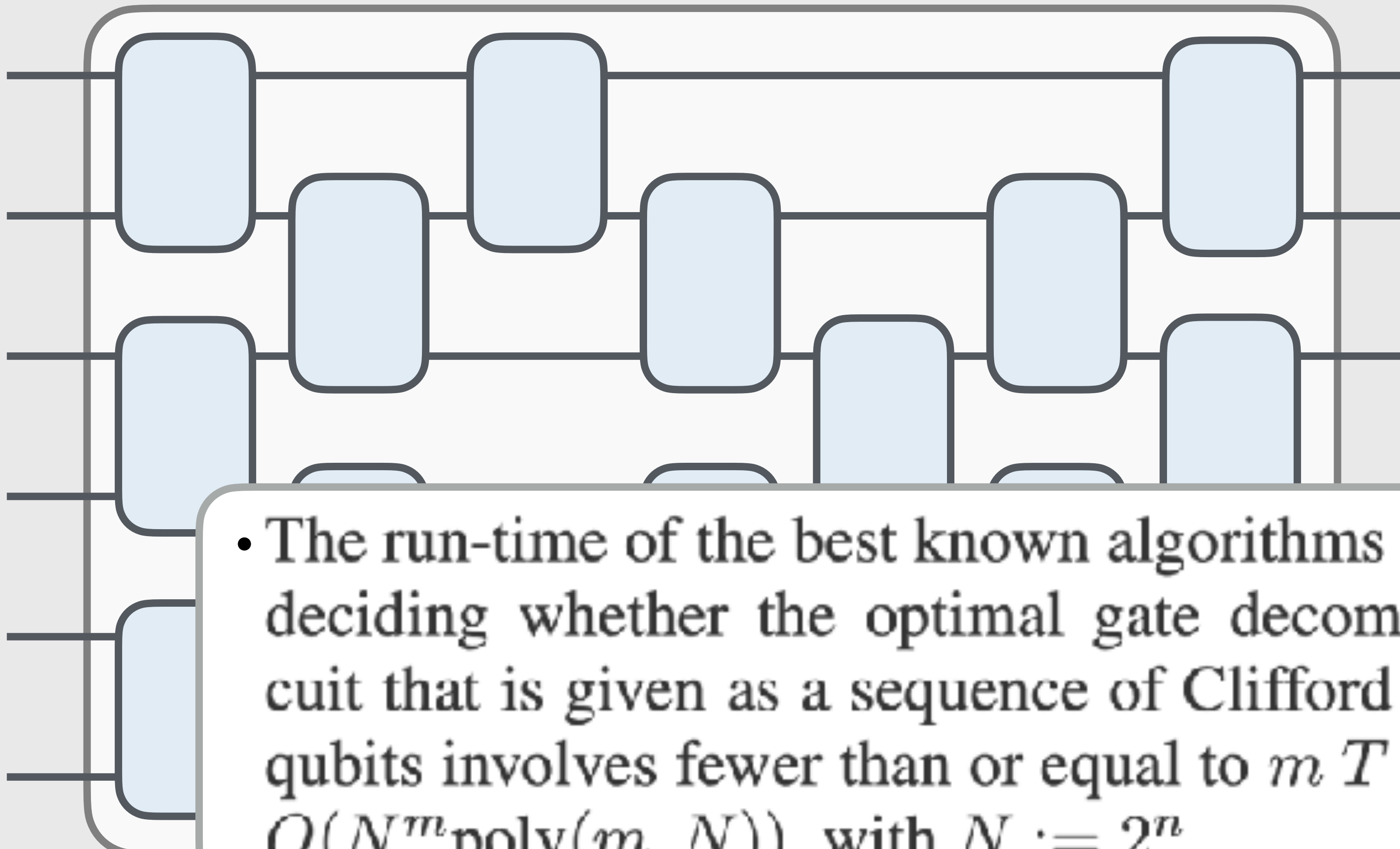


- The run-time of the best known algorithms for the T-count—deciding whether the optimal gate decomposition of a circuit that is given as a sequence of Clifford and T gates on n qubits involves fewer than or equal to m T gates or more—is $O(N^m \text{poly}(m, N))$, with $N := 2^n$.

- **Computationally hard:** Notorious cancellations

Gosset, Gosset, Kliuchnikov, Mosca, Russo, Quant Inf Comp 14, 1277 (2014)
Aaronson, Gottesman, Phys Rev A 70, 02328 (2004)

COMPLEXITY GROWTH IN RANDOM CIRCUITS

- Circuit and state complexities organize unitaries and quantum states

Topological phases

Thermalization

Long time quantum chaotic dynamics

$O(1)$ $O(n)$ $\mathrm{poly}(n)$ $\exp(n)$

Quantum states

Product states

Matrix product states, GHZ states

Multi-scale renormalization

Unitaries

Variational quantum circuits, QAOA

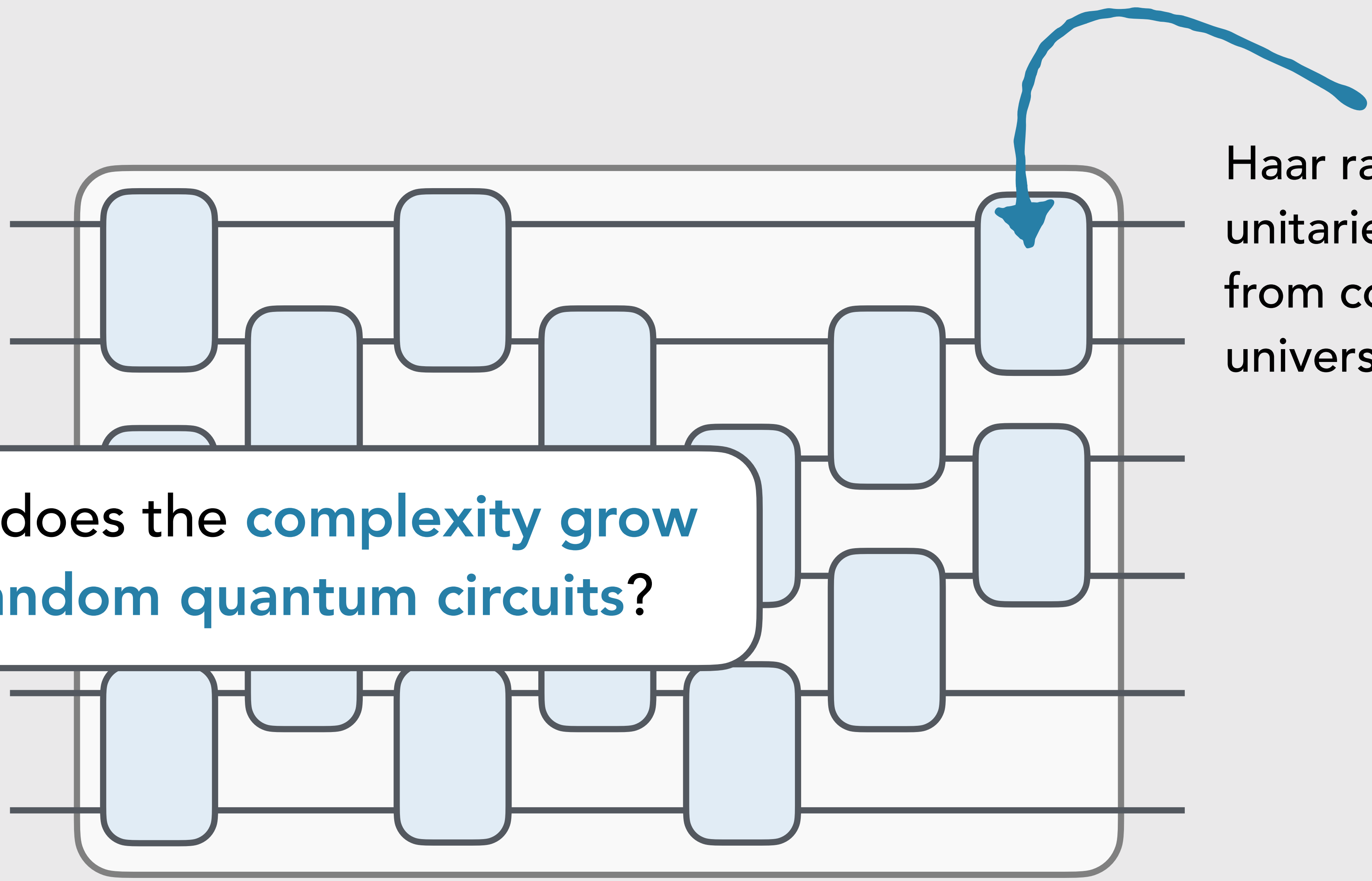
Circuits for classical shadows

Unitary k-designs

Polynomial random circuits

Haar-random unitaries

COMPLEXITY GROWTH IN RANDOM CIRCUITS

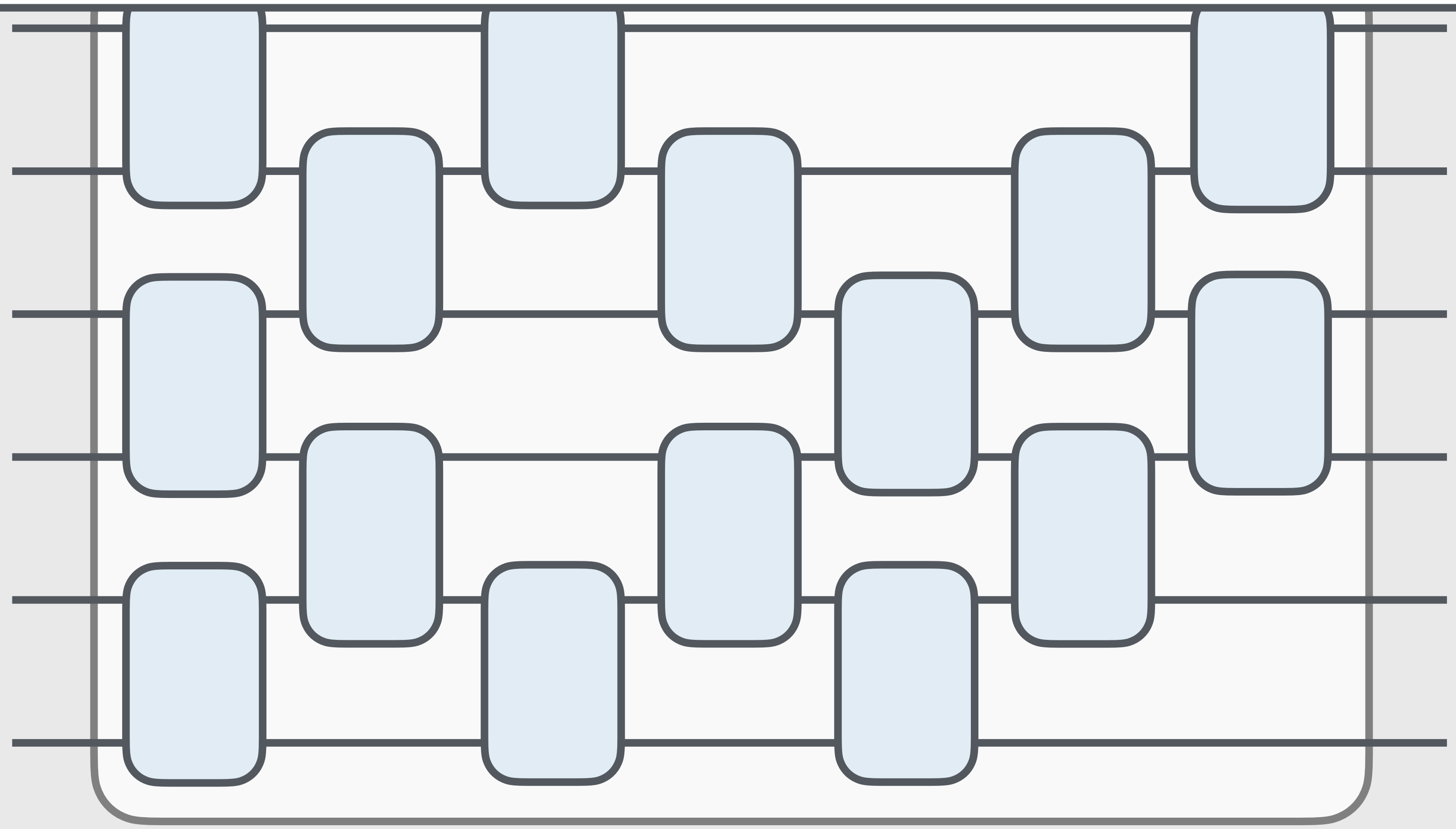


Haar random unitaries, or gates from continuous universal gate set

- How does the **complexity grow** for **random quantum circuits**?

Gosset, Gosset, Kliuchnikov, Mosca, Russo, Quant Inf Comp 14, 1277 (2014)
Aaronson, Gottesman, Phys Rev A 70, 02328 (2004)

- Has risen to prominence as **Brown-Susskind** conjecture

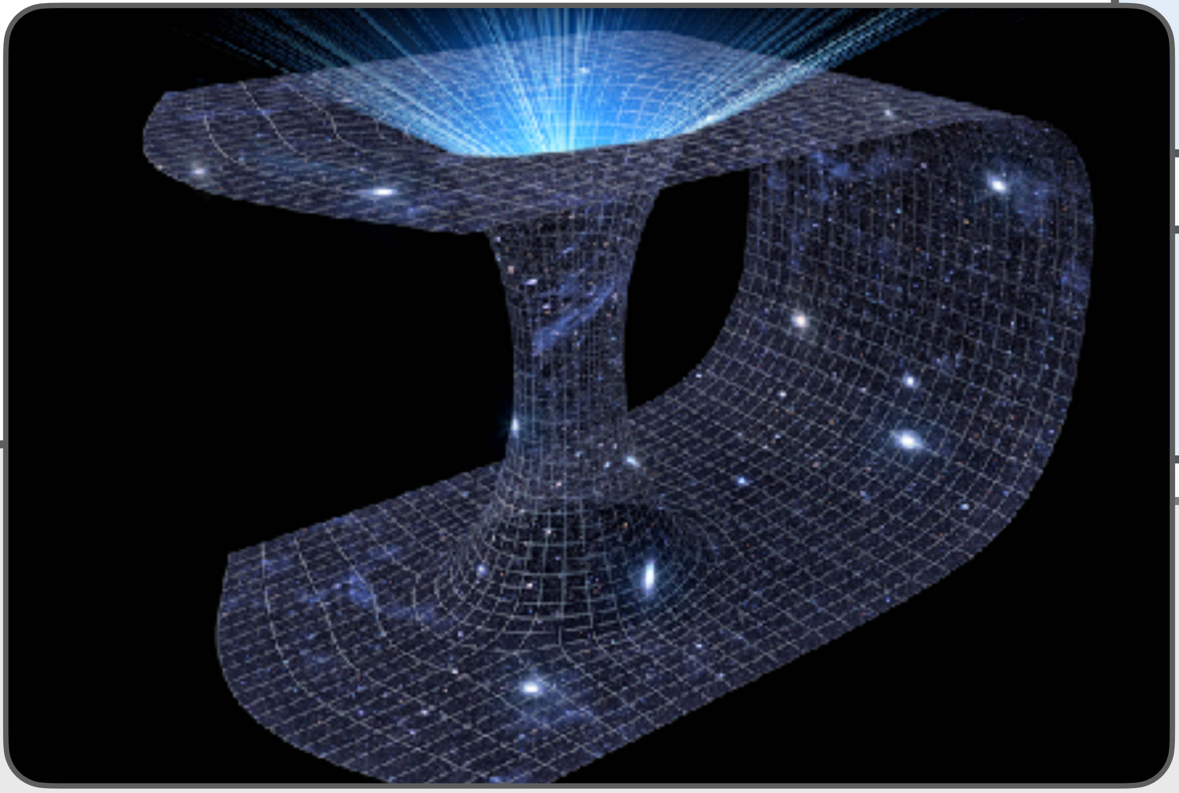


Brown, Roberts, Susskind, Swingle, Zhao, Phys Rev Lett 116, 191301 (2016)
Brown, Roberts, Susskind, Swingle, Zhao, Phys Rev D 93, 086006 (2016)
Chapman, Eisert, Hackl, Heller, Jefferson, Marrochio, Myers, SciPost Phys 6, 034 (2019)
Brown, Susskind, Phys Rev D 97, 086015 (2018)

BROWN-SUSSKIND CONJECTURE

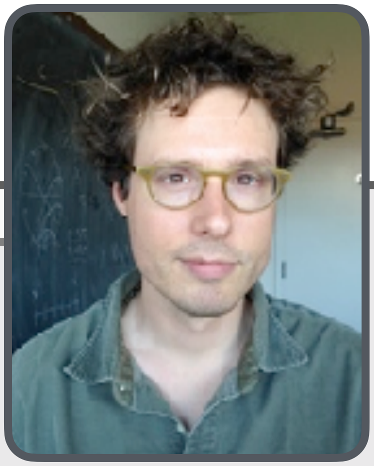
- Has risen to prominence as Brown-Susskind conjecture

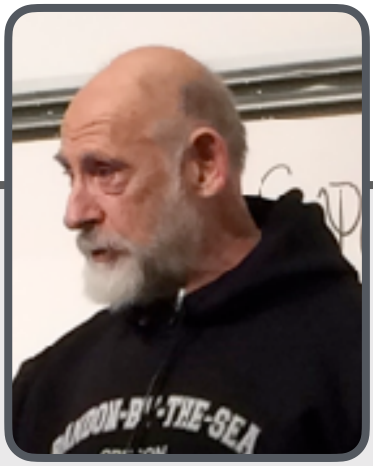
- **AdS:** Volume grows for exponentially long time



- **CFT:** Local observables equilibrating?

$|\psi\rangle$





Brown, Roberts, Susskind, Swingle, Zhao, Phys Rev Lett 116, 191301 (2016)
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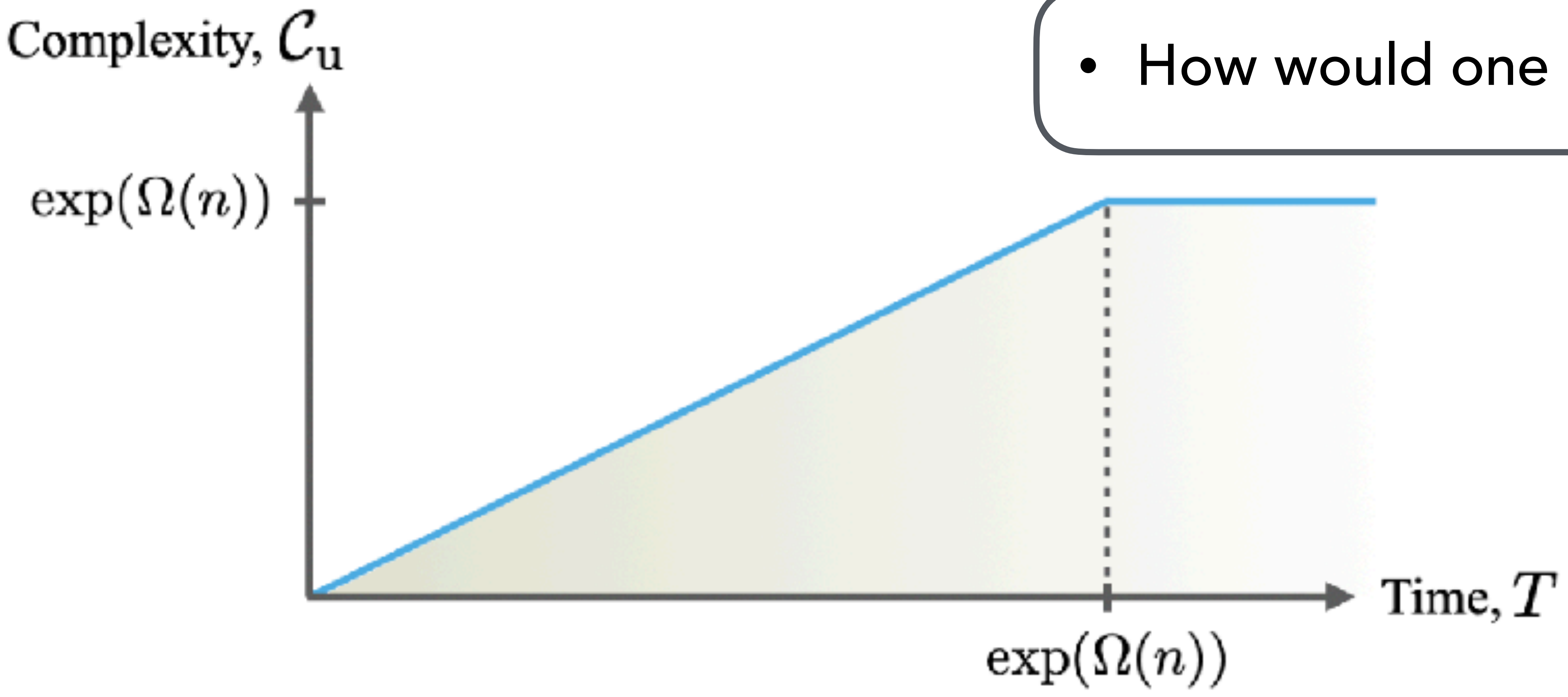


- How would one know?

Brown, Susskind, Phys Rev D 97, 086015 (2018)

LINEAR COMPLEXITY GROWTH

- Indeed, the linear growth conjecture (until exponential times) is provably **true**!

Haferkamp, Faist, Kothakonda, Eisert, Yunger-Halpern, Nature Physics 18, 528 (2022)

- How can this be judged?

IDEA OF PROOF

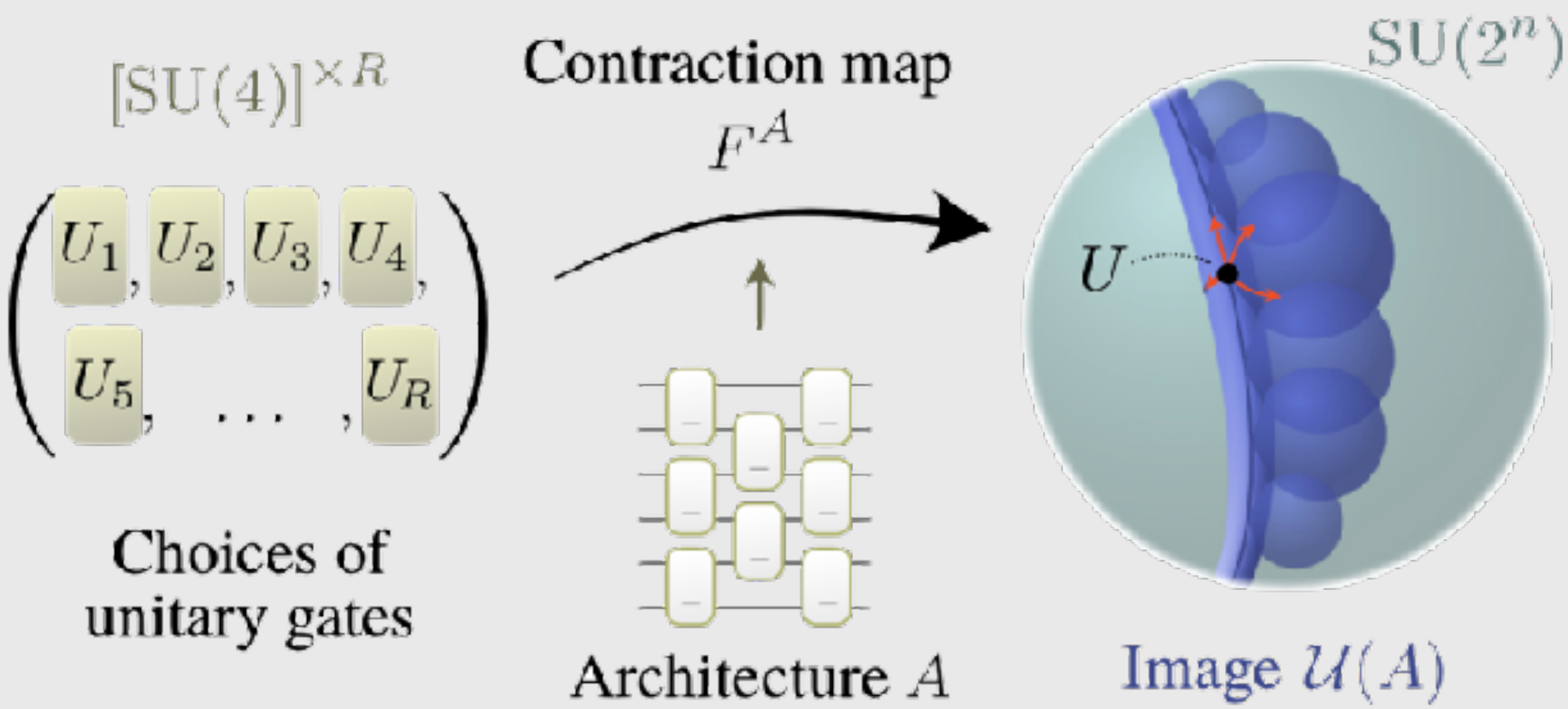
- **Contraction map**

  $F^A : \mathrm{SU}(4)^{\times R} \to \mathrm{SU}(2^n)$

- **Quasialgebraic set**: Polynomial equalities and inequalities

- **Tarski-Seidenberg** principle

- **Quasialgebraic set**



- **Accessible dimension*** is almost always the same throughout the domain

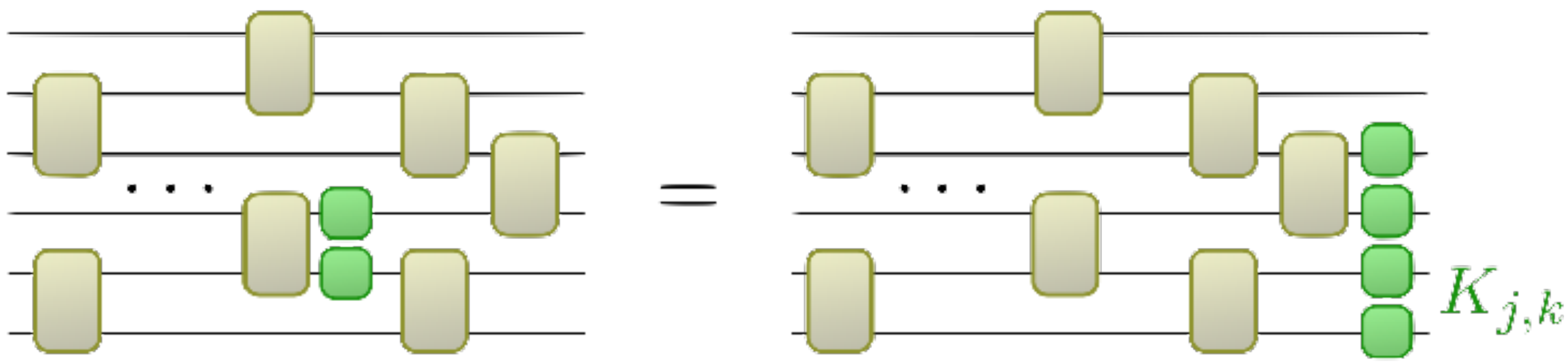
  $$d_A = \dim(\mathcal{U}(A))$$

* Set is no manifold

Haferkamp, Faist, Kothakonda, Eisert, Yunger-Halpern, Nature Physics 18, 528 (2022)

IDEA OF PROOF

- Demonstrate the point's existence by perturbing **Clifford circuits,** 'appending infinitesimal unitaries', 'count independent directions'



- Counting

- Identify a point where dimension **grows linearly** with circuit depth

- **Accessible dimension*** is almost always the same throughout the domain

$$d_A = \dim(\mathcal{U}(A))$$

Haferkamp, Faist, Kothakonda, Eisert, Yunger-Halpern, Nature Physics 18, 528 (2022)
Li, arXiv:2205.05668 (2022)

- **Theorem:** The linear growth conjecture (until exponential times) is provably **true**

$$\mathcal{C}_{\mathrm{u}}(U) \geq \frac{R}{9L} - \frac{n}{3} \quad \text{until } T \geq 4^n - 1$$

- R blocks involving L gates, $T = R/L$

SO THERE IS A LINEAR COMPLEXITY GROWTH: FAIR RESOLUTION OF A READING OF THE BROWN-SUSSKIND CONJECTURE

Haferkamp, Faist, Kothakonda, Eisert, Yunger-Halpern, Nature Physics 18, 528 (2022)

APPROXIMATE NOTIONS AND UNITARY DESIGNS

- **Approximate notions**: Approximate in $\|.\|_\infty$-norm
  Haferkamp, arXiv:2303.16944 (2023)
- **Unitary designs:** The generation of unitary t-designs at a depth $O(nt)$ implies the approximate Brown-Susskind conjecture

- How can **unitary designs** be implemented?

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  Haferkamp, arXiv:2303.16944 (2023)
- **Unitary designs:** The generation of unitary t-designs at a depth $O(nt)$ implies the approximate Brown-Susskind conjecture

- Random **Clifford circuits** are unitary 3-designs
- T-gates uplift then to arbitrary order designs

Haferkamp, Montaelegre-Mora, Heinrich, Eisert, Gross, Roth, Comm Math Phys 397, 995-1041 (2023)

APPROXIMATE NOTIONS AND UNITARY DESIGNS

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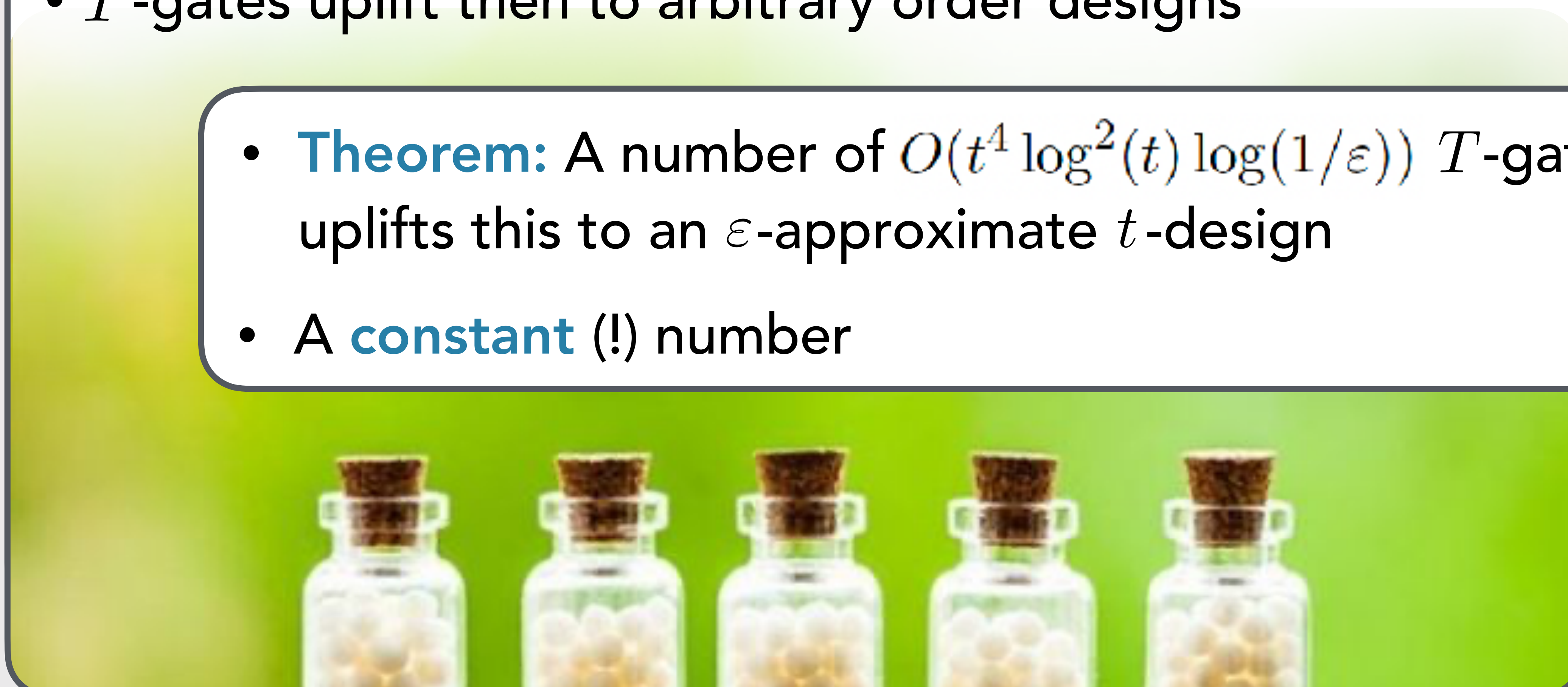
- **Unitary designs:** The generation of unitary t-designs at a depth $O(nt)$ implies the approximate Brown-Susskind conjecture

  - Random **Clifford circuits** are unitary 3-designs
  - T-gates uplift then to arbitrary order designs

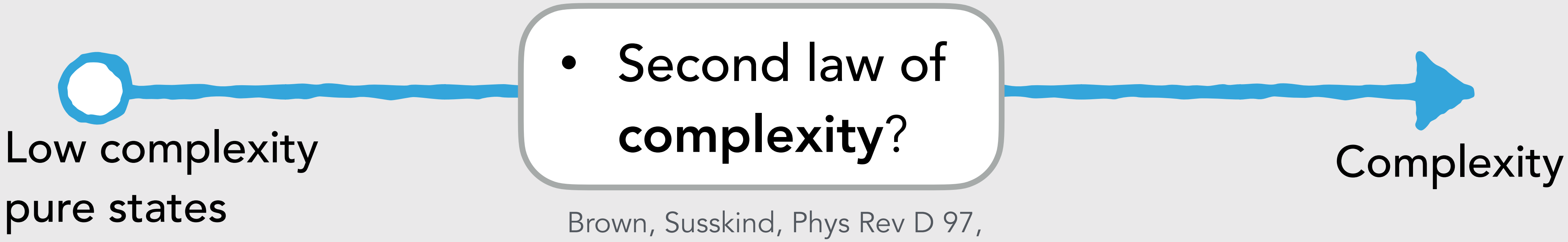
    - **Theorem:** A number of $O(t^4 \log^2(t) \log(1/\varepsilon))$ T-gates uplifts this to an ε-approximate t-design
    - A **constant** (!) number



Haferkamp, Montaelegre-Mora, Heinrich, Eisert, Gross, Roth, Comm Math Phys 397, 995-1041 (2023)

- That settles one Brown-Susskind conjecture - can one think of a **resource theory** of “uncomplexity”?

RESOURCE THEORY OF UNCOMPLEXITY?



Brown, Susskind, Phys Rev D 97, 086015 (2018)

RESOURCE THEORY OF UNCOMPLEXITY?

Entropy

- What processes can be performed by a macroscopic observer?

- Second law of **thermodynamics**

?

- How can computational resources to carry out a process be captured?

Low complexity pure states

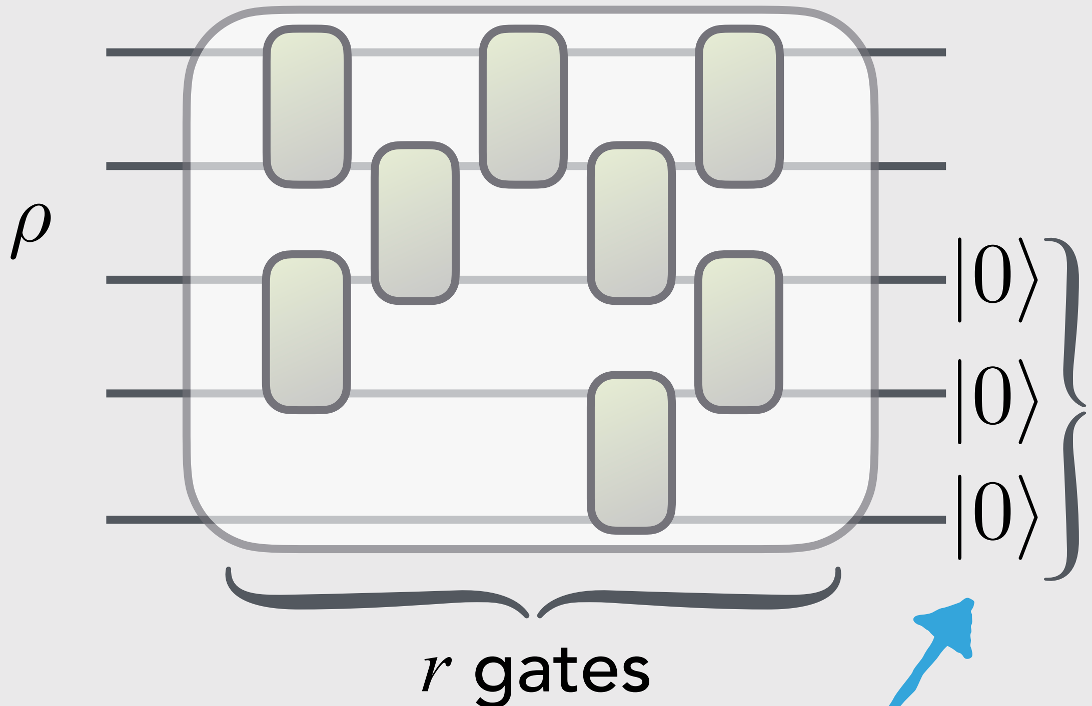
- Second law of **complexity**?

Complexity

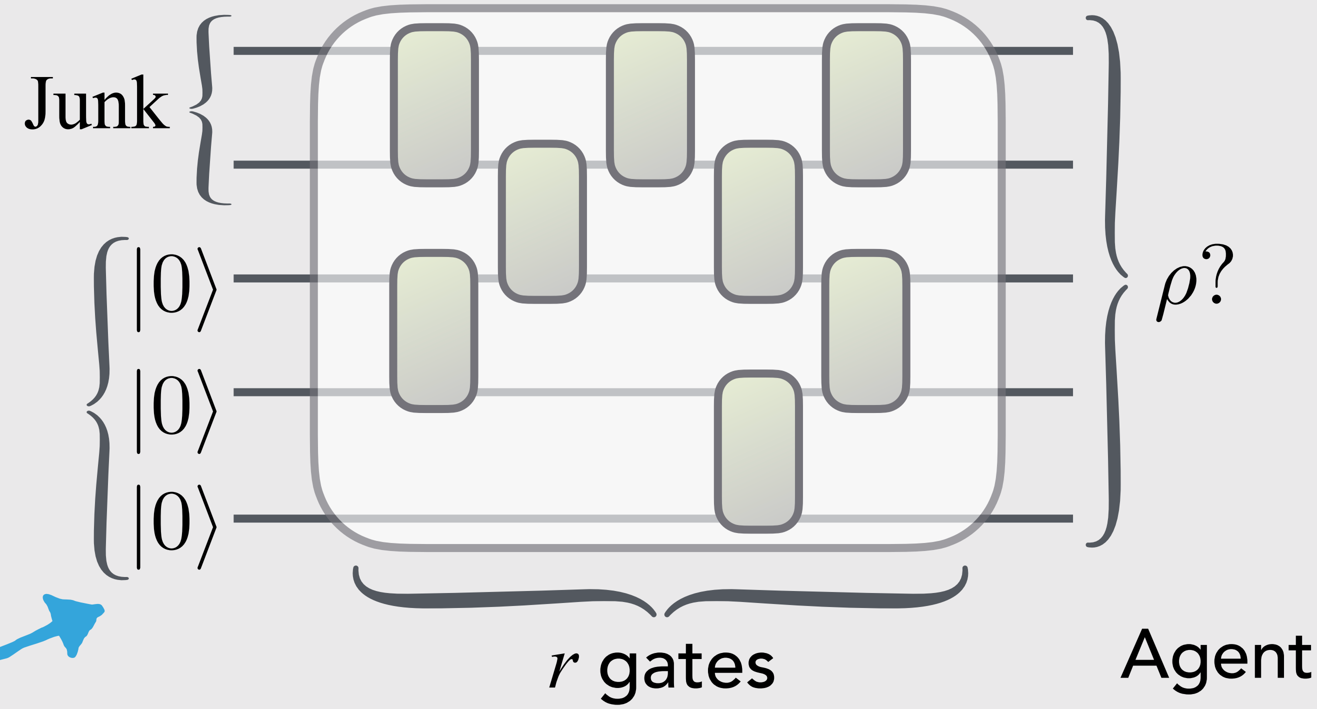
Brown, Susskind, Phys Rev D 97, 086015 (2018)

RESOURCE THEORY OF UNCOMPLEXITY?

- **Uncomplexity extraction:**
  Distills pure qubits from a state



- **Uncomplexity expenditure:**
  Imitates a state



"Uncomplexity"

- **Complexity entropy,** as variant of hypothesis testing entropy

$$H_h^{r,\eta}(\rho) = \log \min_{\substack{\mathrm{tr}(Q\rho)\geq\eta \\ Q\in M_r}} \mathrm{tr}(Q)$$

Yunger Halpern, Kothakonda, Haferkamp, Munson, Eisert, Faist, arXiv:2110.11371 (2021)
Munson, Yunger Halpern, Haferkamp, Kothakonda, Eisert, Faist, in preparation (2023)
Brandao, Datta, IEEE Trans Inf Th 57,1754 (2011)

RESOURCE THEORY OF UNCOMPLEXITY?

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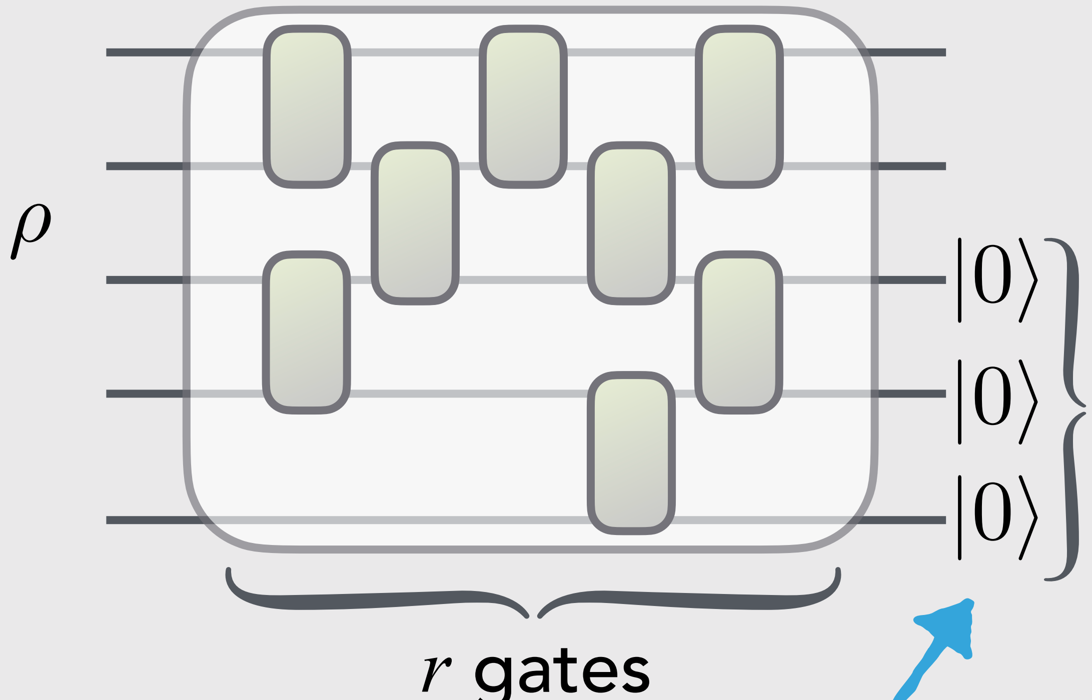

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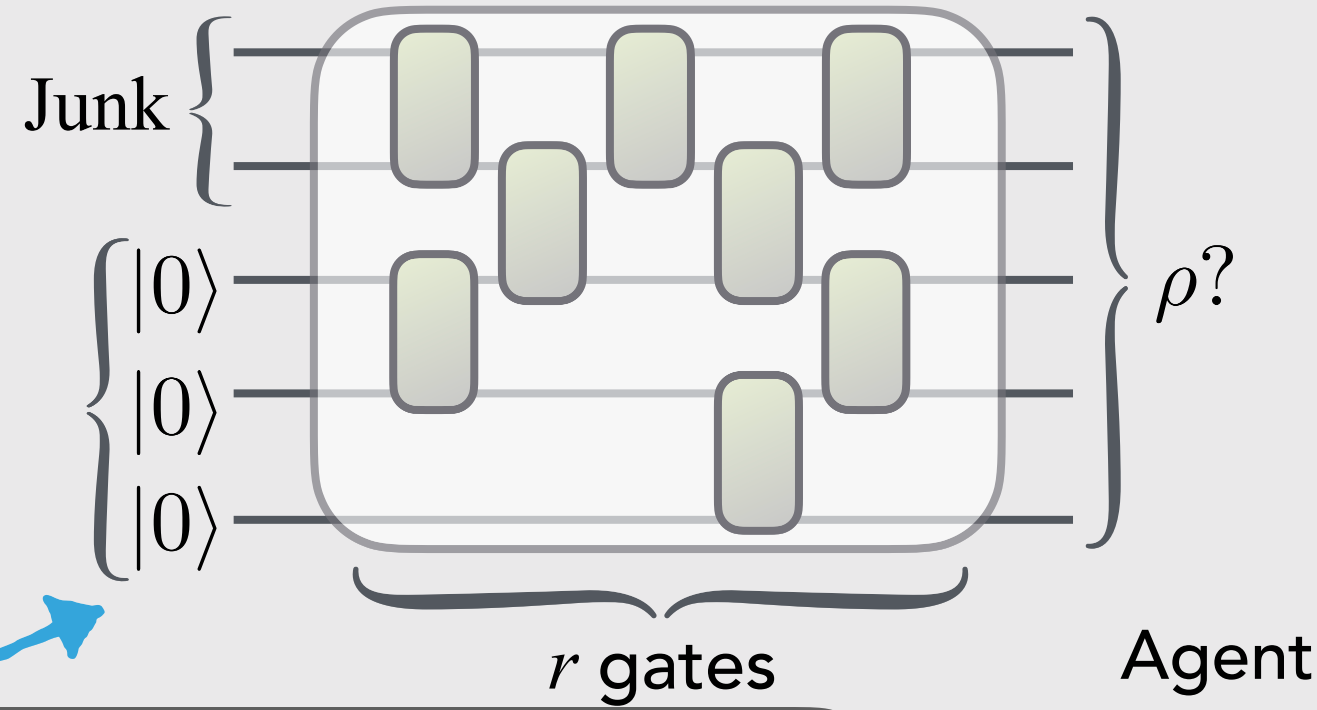

CAN THINK OF A RESOURCE THEORY OF UNCOMPLEXITY

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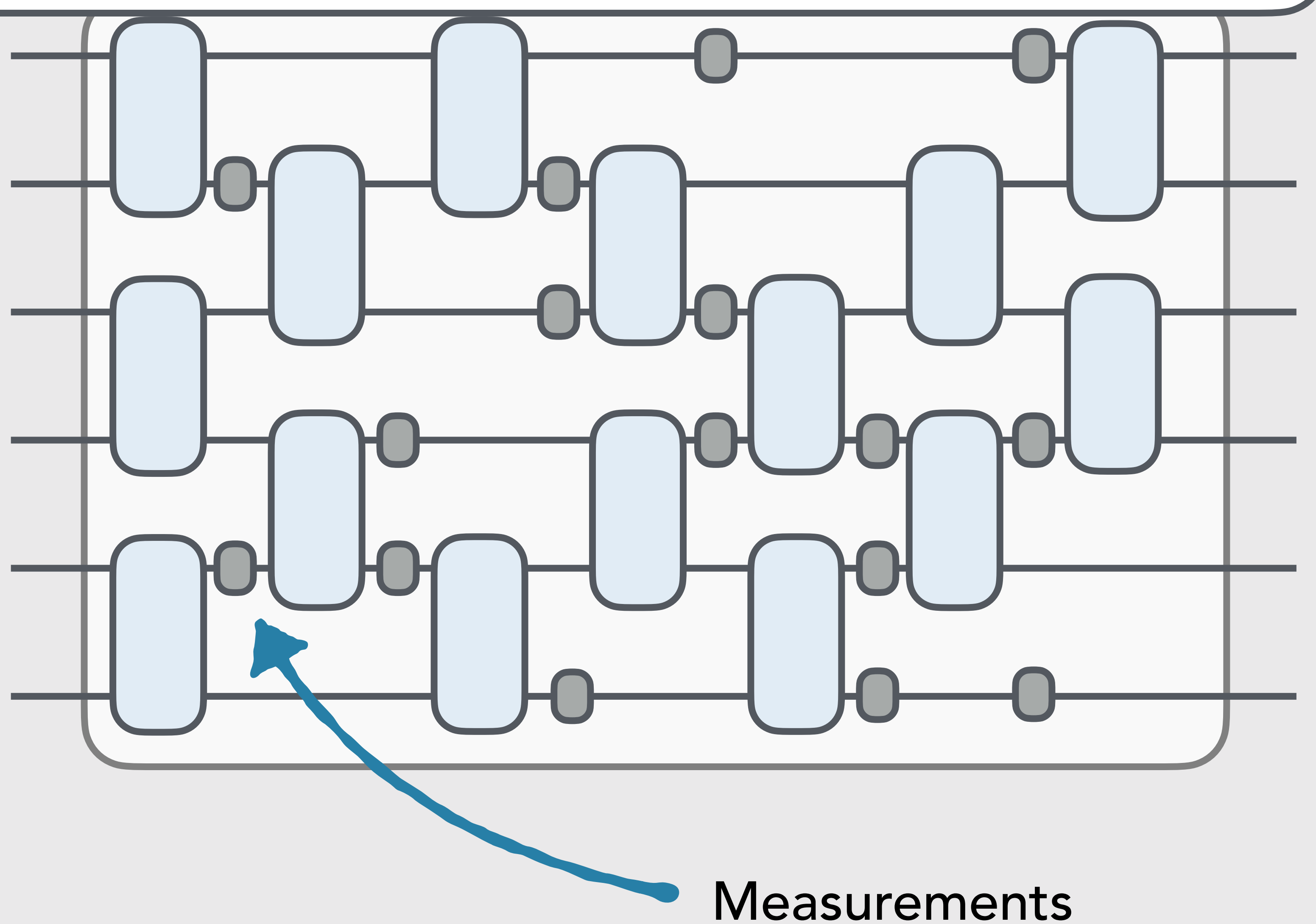
MONITORED QUANTUM CIRCUITS

- How do **monitored quantum circuits** come into play?



Skinner, Ruhman, Nahum, Phys Rev X 9, 031009 (2019)
Bao, Choi, Altman, Phys Rev B 101, 104301 (2020)
Li, Chen, Fisher, Phys Rev B 98, 205136 (2018)
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Li, Chen, Fisher, Phys Rev B 100, 134306 (2019)

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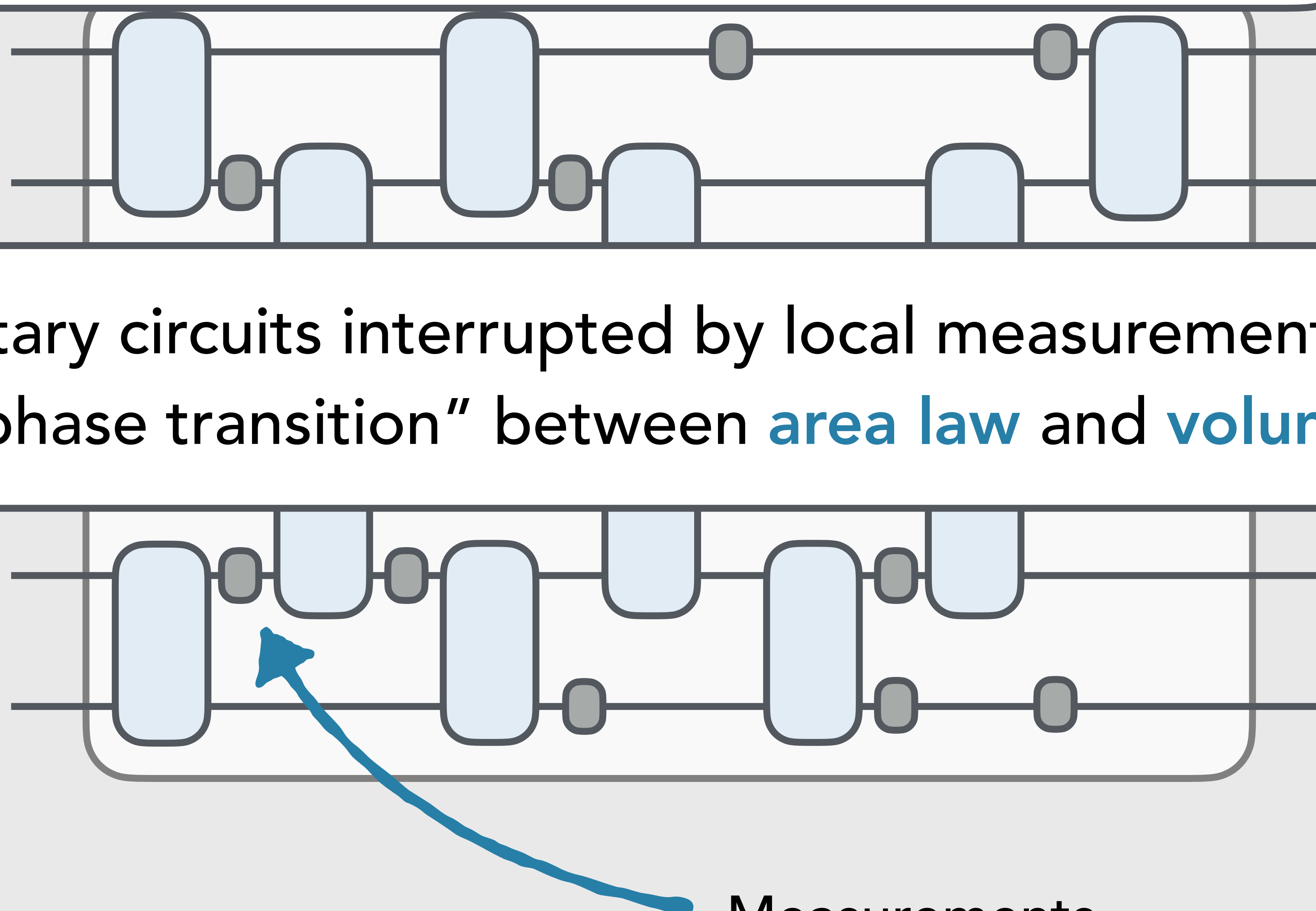

- Unitary circuits interrupted by local measurements at rate p undergo a "phase transition" between **area law** and **volume law** phase

Skinner, Ruhman, Nahum, Phys Rev X 9, 031009 (2019)
Bao, Choi, Altman, Phys Rev B 101, 104301 (2020)
Li, Chen, Fisher, Phys Rev B 98, 205136 (2018)
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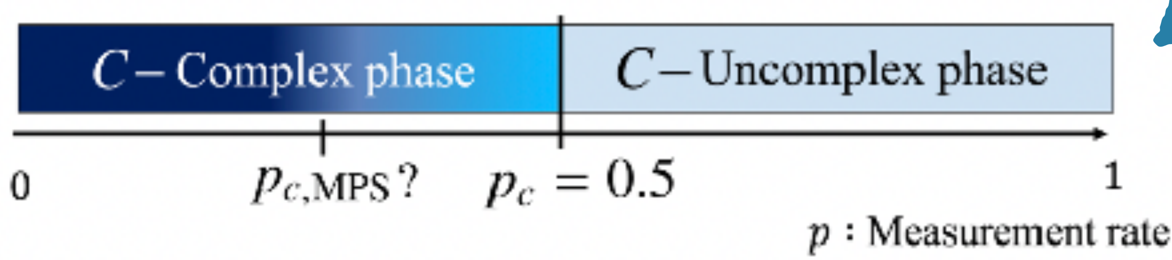
- **Theorem:** There are also two phases of
  - a complexity upper bounded by a **constant** and
  - **linearly growing complexity**



Percolation theory

Semi-algebraic geometry tools

Suzuki, Haferkamp, Eisert, Faist, arXiv:2305.15475 (2023)

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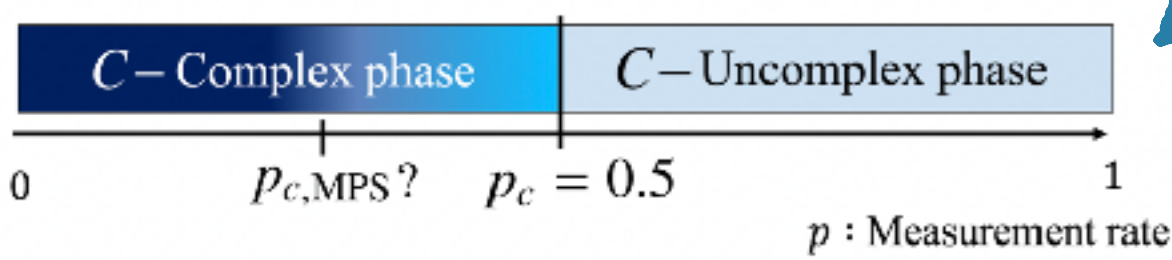

Percolation theory

Semi-algebraic geometry tools

THERE IS A "PHASE TRANSITION" OF COMPLEXITY

Suzuki, Haferkamp, Eisert, Faist, arXiv:2305.15475 (2023)

OUTLOOK

- Quantum many-body random systems, complexity, and random quantum circuits

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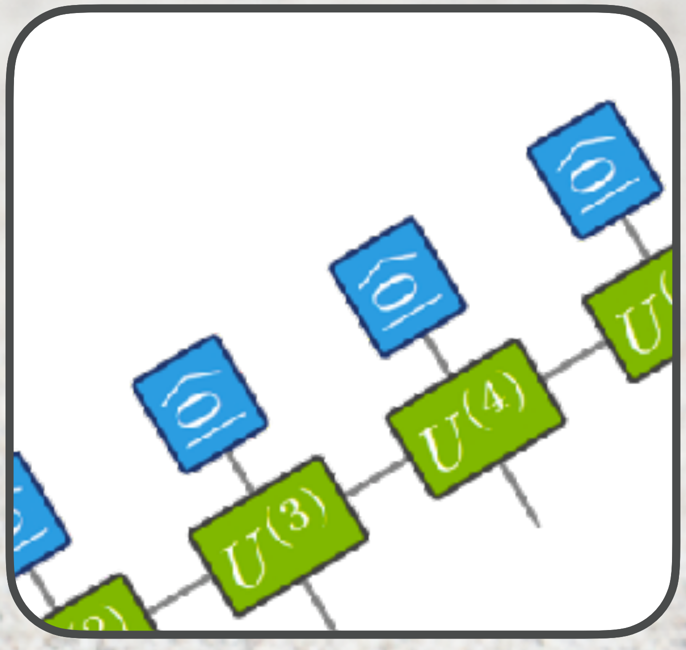
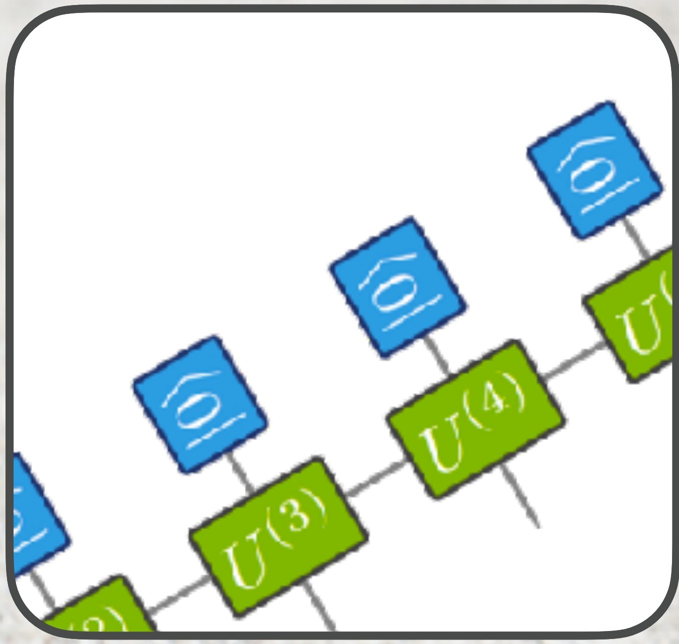


- **Random tensor networks**
  Analytical insights
  out of reach otherwise

- Quantum many-body random systems, complexity, and random quantum circuits



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- **Open:** Would like to see more “physical” random tensor network models: Interacting continuum models
  Wille, Altland, Jahn, Eisert, in preparation (2023)

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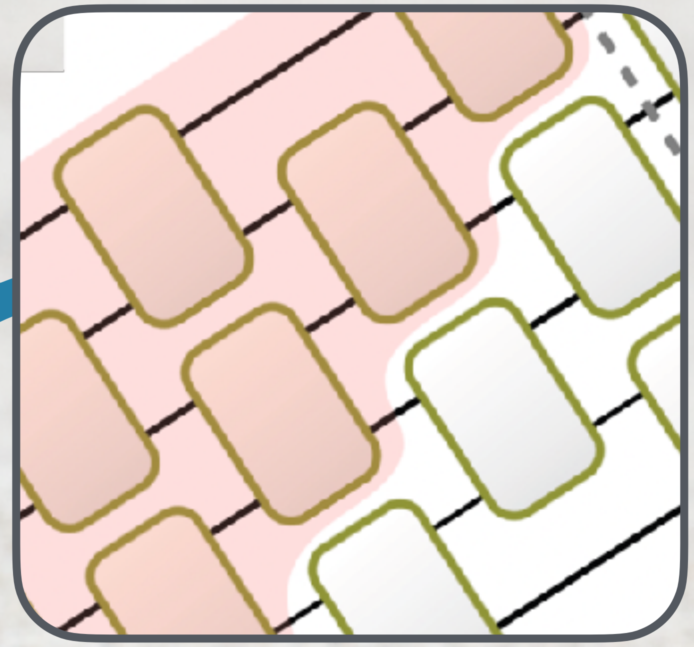

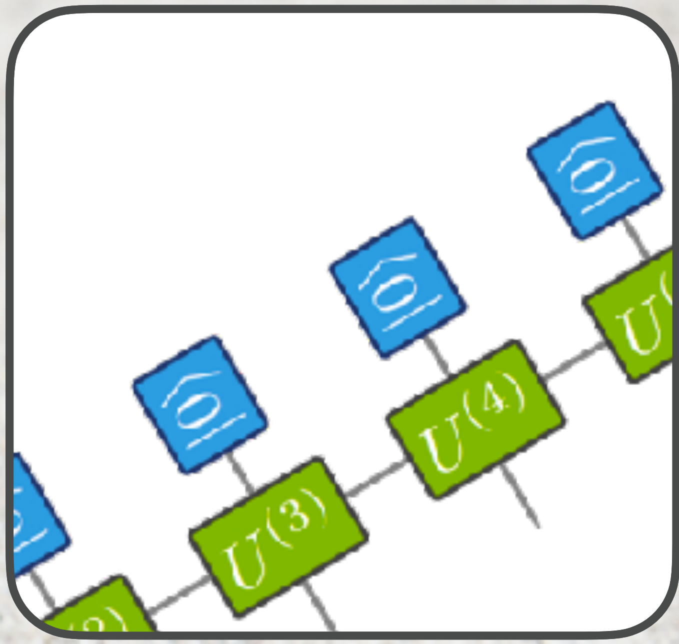

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  Solution of reading of Brown-Susskind conjecture, designs, resource theories, and measurements

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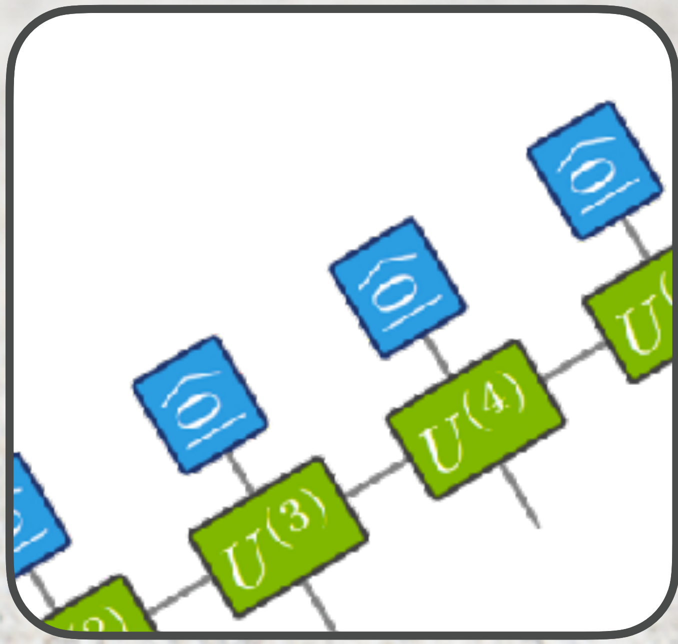

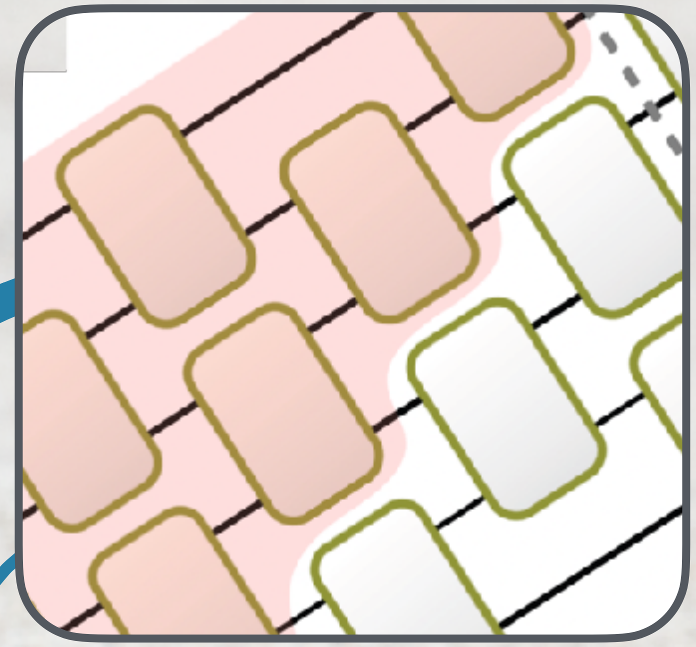

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  Solution of reading of Brown-Susskind conjecture, designs, resource theories, and measurements

  - **Open:** Connection to computational complexity, learnability of output distributions, quantum advantages, power of measurements etc

    Hinsche, Ioannou, Nietner, Haferkamp, Quek, Hangleiter, Seifert, Eisert, Sweke, Phys Rev Lett 130, 240602 (2023)

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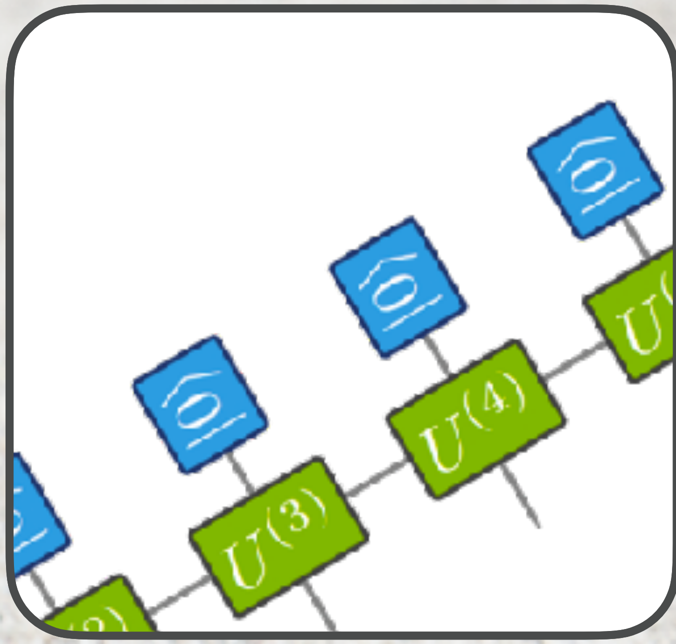

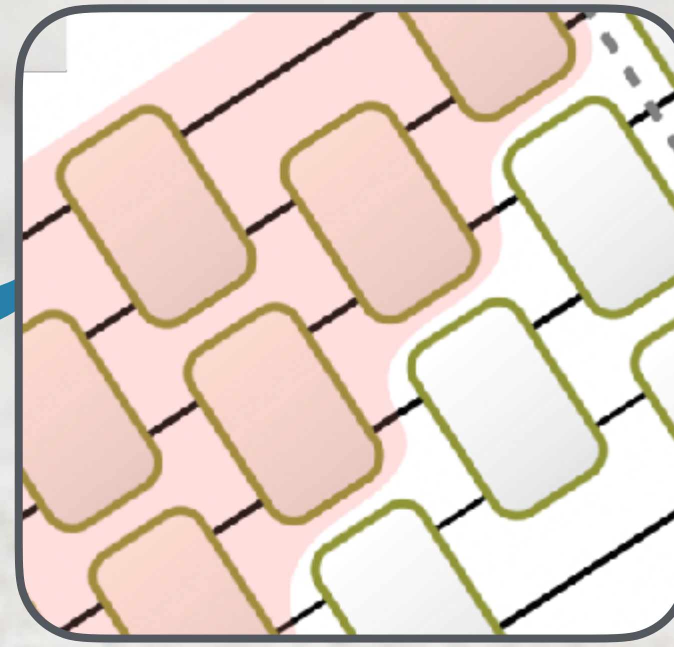

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THANKS FOR YOUR ATTENTION