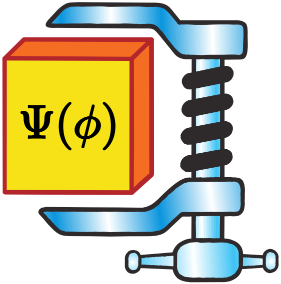
Tensor network states for relativistic quantum field theory

Entangle This V



Antoine Tilloy

June 16th, 2023

Benasque, Spain

MINES PARIS | PSL

ENS | PSL

Inria

erc

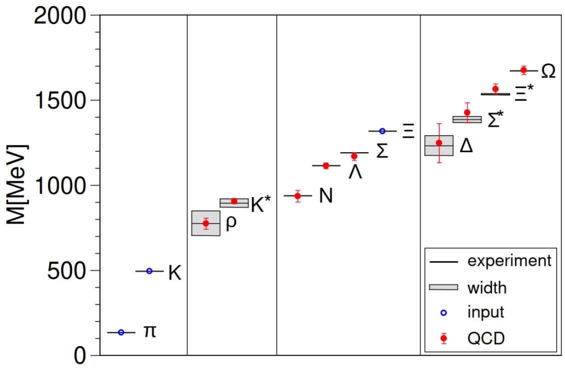
Goal: strongly coupled relativistic field theories

QCD $\equiv$ High T_c supra of HEP

Goal: strongly coupled relativistic field theories

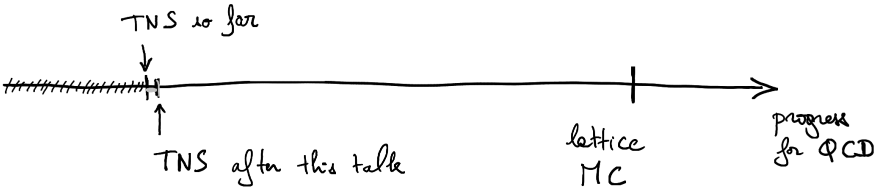
QCD $\equiv$ High T_c supra of HEP

Monte Carlo on Wick-rotated lattice-discretized $=$ only game in town



Science, 2008, BMW collaboration

With tensor network states

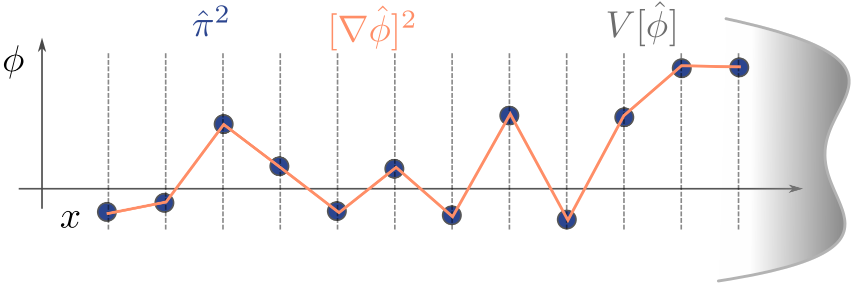


- $3+1$ dimensions
- Relativistic fermions
- Gauge fields
- Taking the continuum limit for relativistic models $\leftarrow$ **today**

Objective: understand the continuum on the simplest non-trivial model: ϕ^4_2

Relativistic field theory as a condensed matter system

Casual definition of a relativistic scalar field ϕ^4_2



Hamiltonian

A continuum of nearest neighbor coupled anharmonic oscillators

$$\hat{H} = \int_{\mathbb{R}} \mathrm{d}x \; \underbrace{\frac{\hat{\pi}(x)^2}{2}}_{\text{on-site inertia}} + \underbrace{\frac{[\nabla\hat{\phi}(x)]^2}{2}}_{\text{spatial stiffness}} + \underbrace{\frac{m^2\,\hat{\phi}^2(x)}{2} + g\,\hat{\phi}^4(x)}_{\text{on-site potential }\hat{V}}$$

with $[\hat{\phi}(x), \hat{\pi}(y)] = i\delta(x-y)\mathbb{1}$ – *i.e.* bosons / harmonic oscillators

Better definition of ϕ_2^4

Renormalized ϕ_2^4 theory

$$H = \int \mathrm{d}x \, \frac{:\pi^2:_m}{2} + \frac{:(\nabla\phi)^2:_m}{2} + \frac{m^2}{2} :\phi^2:_m + g :\phi^4:_m$$

Better definition of ϕ^4_2

Renormalized ϕ^4_2 theory

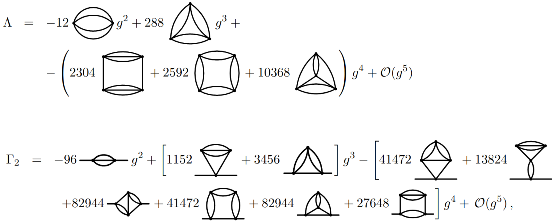
$$H = \int \mathrm{d}x \, \frac{:\pi^2:_m}{2} + \frac{:(\nabla\phi)^2:_m}{2} + \frac{m^2}{2} :\phi^2:_m + g :\phi^4:_m$$

1. Rigorously defined relativistic QFT without cutoff (Wightman QFT)
2. Vacuum energy density ε_0 finite for all g
3. Difficult to solve unless $g \ll m^2$ – not integrable
4. Phase transition around $f_c = \frac{g}{4m^2} = 11$ i.e. $g \simeq 2.7$ in mass units

Two (main) games in town

Perturbation theory

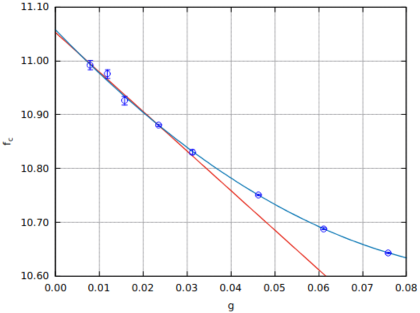
+ resummation



state of the art is $O(g^8)$

arXiv:1805.05882
Serone, Spada, Villadoro

Lattice Monte-Carlo



arXiv:1807.03381
Bronzin, De Palma, Guagnelli

Short distance troubles

Similarity between relativistic and critical models

- A critical model is scale invariant in the IR

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \underset{|x-y|\to+\infty}{\sim} \frac{1}{|x-y|^{2\Delta_{\mathcal{O}}}}$$

Similarity between relativistic and critical models

- A critical model is scale invariant in the IR

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \underset{|x-y|\to+\infty}{\sim} \frac{1}{|x-y|^{2\Delta_{\mathcal{O}}}}$$

- A relativistic QFT is scale invariant in the UV

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \underset{|x-y|\to 0}{\sim} \frac{1}{|x-y|^{2\Delta_{\mathcal{O}}}}$$

Similarity between relativistic and critical models

- A critical model is scale invariant in the IR

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \underset{|x-y|\to+\infty}{\sim} \frac{1}{|x-y|^{2\Delta_{\mathcal{O}}}}$$

- A relativistic QFT is scale invariant in the UV

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \underset{|x-y|\to 0}{\sim} \frac{1}{|x-y|^{2\Delta_{\mathcal{O}}}}$$

Consequence on entanglement

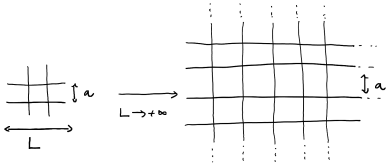
With a UV cutoff $\Lambda = 1/a$ in $1+1$ dimensions:

$$S \propto \log(\Lambda)$$

$\implies$ infinite amount of information in high frequency modes

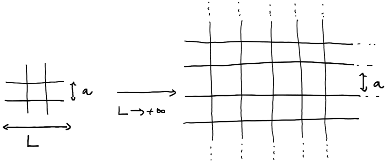
Consequence for lattice discretizations

1. **easy**: taking thermodynamic limit

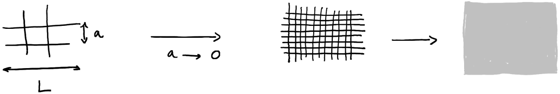


Consequence for lattice discretizations

1. **easy**: taking thermodynamic limit

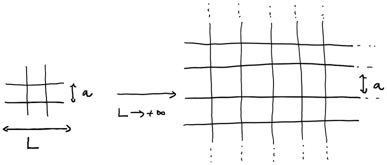


2. **hard**: taking small lattice spacing

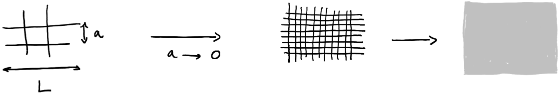


Consequence for lattice discretizations

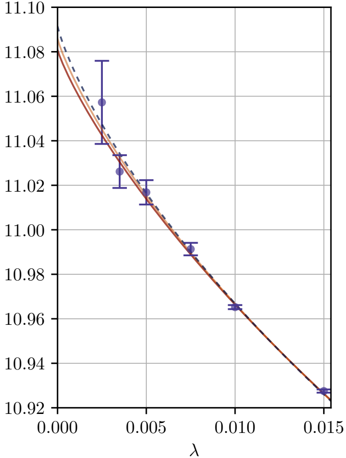
1. **easy**: taking thermodynamic limit



2. **hard**: taking small lattice spacing



A finely discretized relativistic QFT, seen as a lattice model, is almost critical.



f_c estimate continuum extrapolation with GILT-TNR
Clément Delcamp, AT, 2020

UV “criticality” is usually milder than IR criticality

UV CFT tend to be kind

For QFT that are either

1. **super renormalizable** or
2. **asymptotically free**

the critical behavior at short distance is **free**

UV "criticality" is usually milder than IR criticality

UV CFT tend to be kind

For QFT that are either

1. **super renormalizable** or
2. **asymptotically free**

the critical behavior at short distance is **free**

E.g. for ϕ_2^4 at short distances

$$H \longrightarrow H_0 = \int \mathrm{d}x \, \frac{:\pi^2:_m}{2} + \frac{:(\nabla\phi)^2:_m}{2} + \frac{m^2}{2} :\phi^2:_m$$

which is exactly solvable

Objective

Stop wasting parameters on short distance criticality

1. Disentangle the trivial UV behavior
2. Put some tensor network on top to deal with the IR

Gaussian disentangling

Disentangle short distance criticality

1 – Bogoliubov transform

Define modes $a(p), a^\dagger(p)$ as

$$a(p) = \frac{1}{\sqrt{2}}\left(\sqrt{\omega_p}\,\phi(p) + i\frac{\pi(p)}{\sqrt{\omega_p}}\right) \quad \text{with} \quad \omega_p = \sqrt{p^2+m^2}$$

which verify $[a(p), a^\dagger(q)] = 2\pi\,\delta(p-q)$ and yield

$$H_0 = \int_{\mathbb{R}} \mathrm{d}p\,\omega_p\; a_p^\dagger a_p$$

The ground state of H_0 is the Fock vacuum, *i.e.* $|\mathrm{GS}\rangle = |0\rangle$ with $\forall p,\; a_p|0\rangle = 0$

Disentangle short distance criticality

2 – Go back to real space

Fourier transform the modes a_p

$$a(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}p \; e^{ipx} \; a_p$$

which enforces $[a(x), a^\dagger(y)] = \delta(x - y)$

Disentangle short distance criticality

2 – Go back to real space

Fourier transform the modes a_p

$$a(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}p \; e^{ipx} \; a_p$$

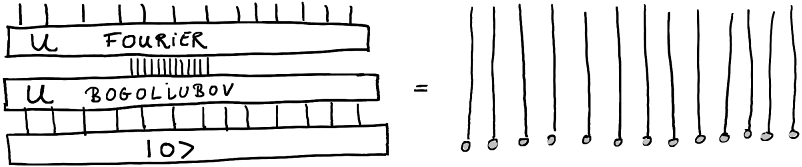
which enforces $[a(x), a^\dagger(y)] = \delta(x-y)$

Note

1. We integrate with $\mathrm{d}p$ not $\omega_p^{-1/2}\mathrm{d}p$
2. ϕ is *not* a local function of $a, a^\dagger$

$$\phi(x) = \int_{\mathbb{R}} \mathrm{d}y \, J(x-y) \left[a(y) + a^\dagger(y)\right] \quad \text{with} \quad J(x) = \int_{\mathbb{R}} \frac{\mathrm{d}p}{\sqrt{2\omega_p}} \, e^{ipx}$$

Tensor network intuition

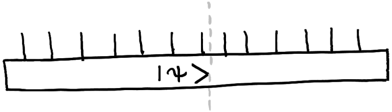


Free particle entanglement entropy

We now have two possible ways to split $\mathscr{H} = \mathscr{H}_- \otimes \mathscr{H}_+$

1. Standard one, yielding $S \propto \log \Lambda$

$$\mathscr{H}_+ = \text{span}\left\{ \phi(x_1) \cdots \phi(x_n) |\Omega_+\rangle \right\} \quad \text{for} \quad x \geqslant 0 \}$$

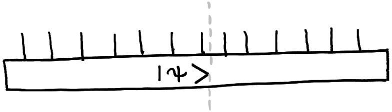


Free particle entanglement entropy

We now have two possible ways to split $\mathscr{H} = \mathscr{H}_- \otimes \mathscr{H}_+$

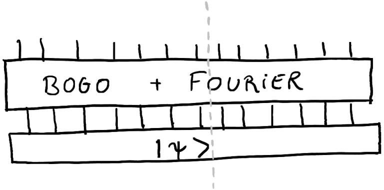
1. Standard one, yielding $S \propto \log \Lambda$

$$\mathscr{H}_+ = \mathrm{span}\{\phi(x_1)\cdots\phi(x_n)|\Omega_+\rangle\} \quad \text{for} \quad x \geqslant 0\}$$



2. The free particle one S_{free}

$$\mathscr{H}_+ = \mathrm{span}\{a^\dagger(x_1)\cdots a^\dagger(x_n)|0\rangle \quad \text{for} \quad x \geqslant 0\}$$



Free particle entanglement entropy

Super-renormalizability $\implies$ Gaussian disentangling kills the divergent part of S:

Conjecture

For any bosonic QFT with strongly relevant interaction $V(\phi)$ in $1+1$d, the free particle entanglement entropy S_{free} is **finite** in the ground state

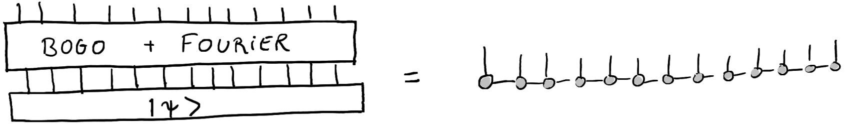
Free particle entanglement entropy

Super-renormalizability $\Longrightarrow$ Gaussian disentangling kills the divergent part of S:

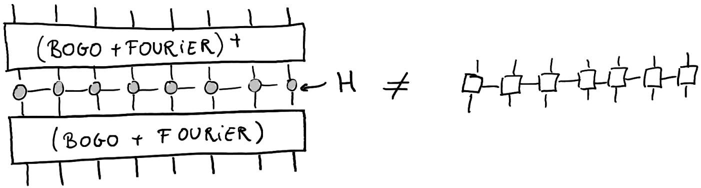
Conjecture

For any bosonic QFT with strongly relevant interaction $V(\phi)$ in $1+1$d, the free particle entanglement entropy S_{free} is **finite** in the ground state

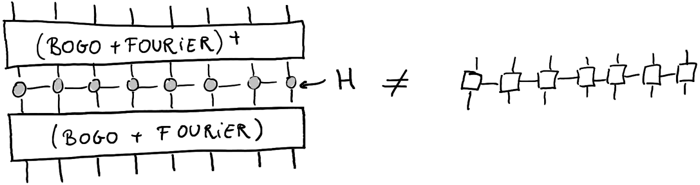
Hence the ground state has an efficient (continuous) MPS representation:



Trading entanglement for (mild) non-locality

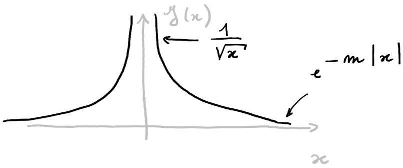


Trading entanglement for (mild) non-locality



H local in $\phi(x)$ hence mildly non-local in $a(x)$, e.g.

$$\int \mathrm{d}x \phi(x)^2 = \int \mathrm{d}x \int \mathrm{d}x_1\, \mathrm{d}x_2\, J(x_1 - x) J(x_2 - x)(a(x_1) + a^\dagger(x_1))(a(x_2) + a^\dagger(x_2))$$



1. UV singular
   $J(x) \underset{0}{\sim} \frac{1}{\sqrt{|x|}}$
2. IR nice
   $J(x) \underset{+\infty}{\sim} e^{-m|x|}$

Remarks on Gaussian disentanglement

Idea used the lattice, in Quantum chemistry, for impurity models *e.g.*

- Krumnow, Veis, Legeza, and Eisert 2016
- Wu, Fishman, Pixley, Stoudenmire 2022

Here minor differences

1. The disentangler is not optimized (not needed)
2. The disentangler does not have a simple local representation
3. The disentangler makes the optimization well defined $\rightarrow$ kills divergence

Relativistic continuous matrix product states

Relativistic continuous matrix product states

aka continuous matrix product states (CMPS) [Verstraete and Cirac 2010] on Gaussian disentanglement steroids

Definition

RCMPSs are a manifold of states parameterized by 2 $(D \times D)$ matrices Q, R

$$|Q, R\rangle = \mathrm{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q \otimes \mathbb{1} + R \otimes a^\dagger(x)\right]\right\}|0\rangle$$

with

- $|0\rangle$ is the Fock vacuum of the free model H_0
- trace taken over $\mathbb{C}^D$
- $\mathcal{P}$ path-ordering exponential

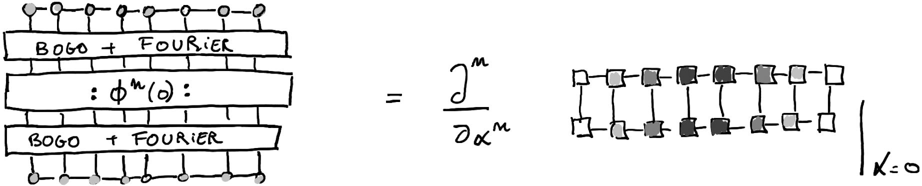
Basic properties of RCMPS

$$|Q,R\rangle = \text{tr}\left\{\mathcal{P}\exp\left[\int dx\, Q\otimes \mathbb{1} + R\otimes a^\dagger(x)\right]\right\}|0\rangle_a$$

Checklist:

1. **Extensive** because of $\mathcal{P}\exp\int$
2. Observables **computable** at cost D^3 (non trivial!)
   requires $[a(x), a^\dagger(y)] = \delta(x-y)$
3. **No UV problems**
   $|0,0\rangle = |0\rangle$ is the ground state of H_0 hence exact CFT UV fixed point
   $\langle Q,R| : P(\phi) : |Q,R\rangle$ is finite for all Q, R (not trivial!)

Tensor network intuition



In the **continuum limit** contracting a non-uniform ladder is numerically exact with high order Runge-Kutta.

The variational algorithm

Optimization

Compute $e_0 = \langle Q, R|h|Q, R\rangle$ and $\nabla_{Q,R} e_0$
Minimize e_0 with (geometric improvements of) gradient descent

The variational algorithm

Optimization

Compute $e_0 = \langle Q, R|h|Q, R\rangle$ and $\nabla_{Q,R} e_0$
Minimize e_0 with (geometric improvements of) gradient descent

Computations of e_0 and ∇e_0 in a nutshell:

1. $V_b = \langle :e^{b\phi(x)}: \rangle_{QR}$ computable by solving an ODE with cost $\propto D^3$
2. $\langle :\phi^n: \rangle_{QR}$ computable doing $\partial_b^n V_b \Big|_{b=0} \quad \rightarrow \quad \propto D^3$
3. $e_0 = \langle h \rangle_{QR}$ computable by summing such terms at cost $D^3 \quad \rightarrow \quad \propto D^3$
4. ∇e_0 computable by solving the adjoint ODE (backpropagation) $\quad \rightarrow \quad \propto D^3$

The variational algorithm

Optimization

Compute $e_0 = \langle Q, R|h|Q, R\rangle$ and $\nabla_{Q,R} e_0$
Minimize e_0 with (geometric improvements of) gradient descent

Computations of e_0 and ∇e_0 in a nutshell:

1. $V_b = \langle :e^{b\phi(x)}: \rangle_{QR}$ computable by solving an ODE with cost $\propto D^3$
2. $\langle :\phi^n: \rangle_{QR}$ computable doing $\partial_b^n V_b \Big|_{b=0} \quad \rightarrow \quad \propto D^3$
3. $e_0 = \langle h \rangle_{QR}$ computable by summing such terms at cost $D^3 \quad \rightarrow \quad \propto D^3$
4. ∇e_0 computable by solving the adjoint ODE (backpropagation) $\quad \rightarrow \quad \propto D^3$

Functioning `Julia` implementation. `OptimKit.jl` to solve the Riemannian minimization, `KrylovKit.jl` to solve fixed point equations, `DifferentialEquations.jl` (`Vern7` solver) to solve ODE. Soon `Rcmps.jl`?

Using the optimized state

After optimization: $|Q, R\rangle \simeq |0\rangle_{\text{int.}}$ with $\langle Q, R| \hat{h} |Q, R\rangle = e_0 + \varepsilon$

This gives:

- All equal-time N-point functions

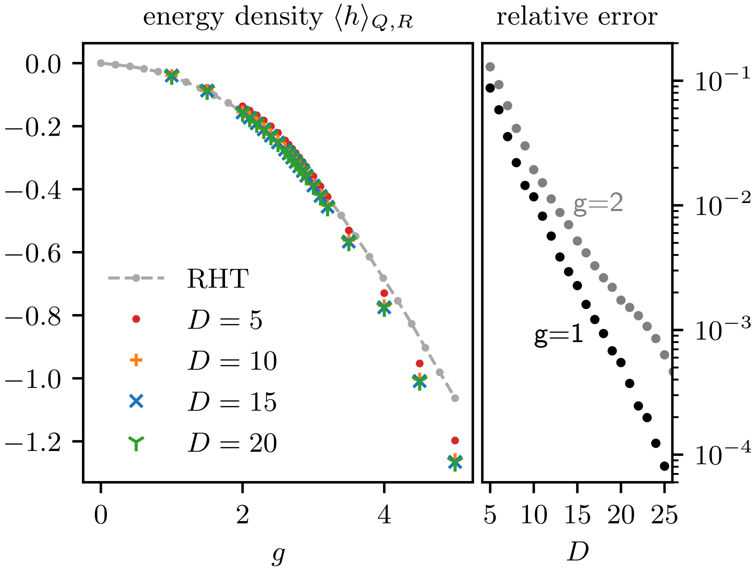
$$\langle \phi(x_1)\,\phi(x_2)\cdots\phi(x_n)\rangle \simeq \langle Q, R|\,\phi(x_1)\,\phi(x_2)\cdots\phi(x_n)\,|Q, R\rangle$$

  at cost D^3 by solving coupled linear ODEs

- In particular *all* Euclidean 2-point functions $\implies$ spectral function

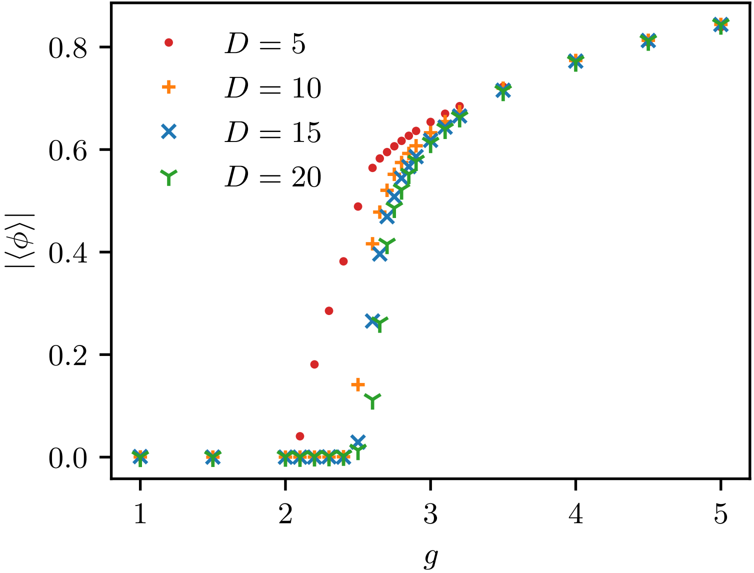
$$\langle \phi(x)\phi(0)\rangle = \int_0^{+\infty} \mathrm{d}\mu\, \mu\rho(\mu) K_0(\mu x)$$

Results: ϕ_2^4 energy density



New: D can now be pushed to 32 or even 64 with some effort

Results: ϕ_2^4 – field expectation value $\langle\phi\rangle$



New: the mass can be fitted from 2-point function and agrees with RHT to 10^{-3}

Todo-list for continuous tensor networks

In $1+1$ dimensions

- Solve Fermion / Gauge theories
- Go beyond strongly renormalizable interactions
- Do general CFT perturbations
- Compute more observables (masses, spectra, c-function...)

And of course the grand goal: do higher dimensions!

Come work on it in Paris with Edo and Karan!

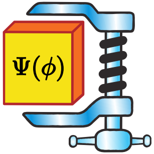
Summary

Problem

- Relativistic QFT have infinite entanglement at short distance

Solution in $1+1$d

$$|Q,R\rangle = \text{tr}\left\{\mathcal{P}\exp\left[\int \mathrm{d}x\, Q\otimes \mathbb{1} + R\otimes a^\dagger(x)\right]\right\}|0\rangle$$



1. Ansatz for $1+1$ relativistic QFT
2. The $\phi(x)\to a(x)$ trick disentangles the divergent UV
3. The CMPS on top solves the rest
4. Efficient (cost poly D, error plausibly 1/superpoly D)